Midterm for Linear Algebra

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Generic Rules:

1. The due date is Nov 2nd.
2. Each sub-problem worths 2 pt. So you can get as much as 52 pts in total. However, we use 50 pts as full credit. If you get more than 50 pts, then your score is 50. (So you can safely skip one subproblem, or did one subproblem completely wrong.) We will give partial credits for partial solutions.
3. Feel free to use calculators or softwares to help you, or collaborate with your classmates. But in your submitted midterm, you should always express your ideas in your own words. **Write down the names of your collaborators if you collaborated with someone.** (Also, if you collaborated with someone, then each of you should hand in your own midterm answer in your own words. Do NOT give me a joint answer or copied answer, as it would be treated as plagiarism.)
4. If I suspect plagiarism, I might call you into my office, and ask you to do the problem in front of me to see if you actually understand what you have written down. You should be able to explain your own written answers. Failure to do so will be a confirmation of plagiarism, and will be punished accordingly.
5. Write your answers in English. You should always explain any statement you made, and always show some process of calculation (which may help you earn some partial credit). This is a mid-term after all, so please be more formal and careful in your answers.

Problem 1 (Game Character Creation)**.** *In a video game, you are trying to create a warrior character. To create a warrior, you have some points to distribute on three attributes: strength, dexterity and constitution. These attributes will determine your attack, defense and HP in the following mannor:* $\textit{Attack} = 2\textit{Strength} + \textit{Dexterity}$, $\textit{Defence} = 2\textit{Dexterity} + \textit{Constitution}$, $\textit{HP} = 5\textit{Constitution} + \textit{Strength}$.

When two warriors fight, each turn each warrior will deal damage to the other's HP equal to the attack of the attacker minus the defense of the defender. (So your HP might go up if your defense is higher than your opposition's attack.) The damage will happen simultaneously. Round after round, whoever has HP first reduced to 0 or below will lose. It is possible for two warriors to lose simultaneously, or to have the fight continue indefinitely.

Let V *be the warrior space. Let* $\mathcal{B}$ *be the basis such that each warrior* $\boldsymbol{w}$ *will have coordinates* $\begin{bmatrix} \textit{Stength}(\boldsymbol{w}) \\ \textit{Dexterity}(\boldsymbol{w}) \\ \textit{Constitution}(\boldsymbol{w}) \end{bmatrix}$.

Let $\mathcal{C}$ *be the basis such that each warrior* $\boldsymbol{w}$ *will have coordinates* $\begin{bmatrix} \textit{Attack}(\boldsymbol{w}) \\ \textit{Defense}(\boldsymbol{w}) \\ \textit{HP}(\boldsymbol{w}) \end{bmatrix}$.

1. *If the first vector in the basis* $\mathcal{B}$ *fight with the third vector in basis* $\mathcal{B}$*, who will win?*
2. *Find the change of coordinate matrix from* $\mathcal{B}$ *to* $\mathcal{C}$*, and from* $\mathcal{C}$ *to* $\mathcal{B}$*.*
3. *Suppose we have a constraint that* $\textit{Strength} + \textit{Dexterity} + \textit{Constitution} = 10$*. What constraint will this imply on attack, defense and HP of a warrior? If all warriors must satisfy this constraint, then can you*

build an undefeatable warrior? (Note that you just need to never lose, but you don't need to actually win.)

4. *Let W be the space of pairs of warriors. We pick basis $\mathcal{D}$ such that each pair of warriors $(\boldsymbol{v}, \boldsymbol{w})$ will have coordinates* $\begin{bmatrix} \text{Attack}(\boldsymbol{v}) \\ \text{Defense}(\boldsymbol{v}) \\ \text{HP}(\boldsymbol{v}) \\ \text{Attack}(\boldsymbol{w}) \\ \text{Defense}(\boldsymbol{w}) \\ \text{HP}(\boldsymbol{w}) \end{bmatrix}$. *After a round, the HP of the two warriors will change. So we have a map $R : W \to W$ that sends the status of the two warriors $(\boldsymbol{v}, \boldsymbol{w})$ to the new status of them after a round (so their HP will be lower after R). Find the matrix $R_{\mathcal{D}\to\mathcal{D}}$. What is the row operation represented by this matrix? What is the column operation represented by this matrix?*

Problem 2 (Discrete Lanchester's Law)**.** *Lanchester's Law is a mathematical formula invented during world war one. It is a continuous model described by differential equations. Here I revised it some what to make it into a discrete model, so that we can compute better.*

Suppose two armies are at war with one another. The first army started with x_0 soldiers, and the second army started with y_0 soldiers. Each round, they fire at each other, killing the opposition. Say for each natural number t, we have $x_{t+1} = x_t - by_t$, and $y_{t+1} = y_t - ax_t$. Here a, b are positive real numbers describing the fire power of the two armies. Let $\boldsymbol{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$, *and let* $\boldsymbol{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$.

1. *Find matrices D, E such that $D\boldsymbol{x} = -bE\boldsymbol{y} + x_0\boldsymbol{e}_1$, and $D\boldsymbol{y} = -aE\boldsymbol{x} + y_0\boldsymbol{e}_1$.*

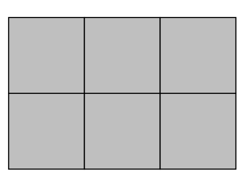
2. *Find a $(2n+2) \times (2n+2)$ block matrix M in terms of D and E, and a vector $\boldsymbol{b} \in \mathbb{R}^{2n+2}$ such that* $M\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}$.

3. *Write $D, D^{-1}, ED^{-1}E$ as polynomials of the matrix E. Also calculate E^{n+1} explicitly.*

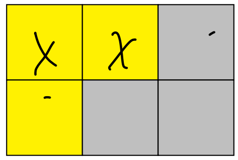
4. *Find all the entries of the Schur complement of the upper left block of M in the above subproblems. Show that M is invertible. (Recall: if* $M = \begin{bmatrix} A & B \\ X & Y \end{bmatrix}$, *then the Schur complement of A is $Y - XA^{-1}B$.) (In calculus, this corresponds to the fact that given initial conditions, the solution to the corresponding differential equations is unique.)*

5. *For a block matrix* $\begin{bmatrix} A & B \\ X & Y \end{bmatrix}$, *if B is invertible and I want to kill the lower right block by a block row operation, what matrix E should I multiply to* $\begin{bmatrix} A & B \\ X & Y \end{bmatrix}$? *Should it be* $E\begin{bmatrix} A & B \\ X & Y \end{bmatrix}$ *or* $\begin{bmatrix} A & B \\ X & Y \end{bmatrix}E$? *If B is $n \times n$ and X is $m \times m$, describe the size of each block of E, and also calculate the matrix after the block row operation.*

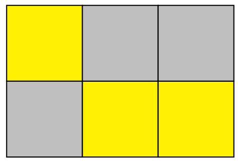
Problem 3 (Light Out Puzzle)**.** *The Light Out puzzle is a famous game. Here we do the 2×3 version of it. Suppose we have six tiles arranged as below. Each tile is a light, and the light might be on or off.*



If we press a tile, then this tile and all adjacent tiles will change status (from on to off, or from off to on). For example, in the status above, if I press the upper left tile, then I have:



If I then press the middle tile in the second row, then I have:

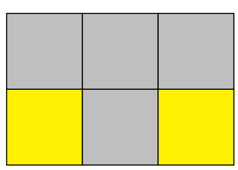


The goal of this puzzle is this: starting from some configuration, which tiles should you push, in order to turn all the lights off?

For this question, we shall set the field of scalars as $\mathbb{F}_2 = \{0, 1\}$*, such that* $0 + 1 = 1 + 0 = 1 \times 1 = 1$*, and* $1 + 1 = 0 + 0 = 1 \times 0 = 0 \times 1 = 0 \times 0 = 0$*. All vectors spaces in this problem is over* $\mathbb{F}_2$*. We order the six tiles from left to right, first row and then second row. Then given a "button-pressing" vector* $\boldsymbol{v} \in \mathbb{F}_2^6$*, say* $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$*, then it means we are going to press the corresponding tiles, i.e., the middle tile in the first row and the lower left tile in the second row.*

We define the button-pressing matrix to be $A = \begin{bmatrix} D & I \\ I & D \end{bmatrix}$ *where* $D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ *and* I *is the identity matrix as usual. As you can verify yourself, if we press buttons according to some* $\boldsymbol{v} \in \mathbb{F}_2^6$*, then the change in light configuration is exactly* $A\boldsymbol{v}$*.*

1. *Find a vector* $\boldsymbol{b} \in \mathbb{F}_2^6$ *representing the following configuration, and solve* $A\boldsymbol{x} = \boldsymbol{b}$ *by Gaussian elimination. Then tell me which tiles you should push to solve this puzzle.*



2. *Find the inverse of* D*, and then use block row elimination on* $A = \begin{bmatrix} D & I \\ I & D \end{bmatrix}$ *to kill the lower left block, and then multiply the first row by* D^{-1}*. What is the resulting matrix in block form? What are the entries of the resulting matrix? What is the rank of* A*?*

3. *Find a basis to the kernel of* A*. Then find four "button-pressing" vectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_2, \boldsymbol{v}_4 \in \mathbb{F}_2^6$ *that does absolutely nothing to any given configuration.*

$\vdots$

4. *Prove that the range of A and the kernel of A are "orthogonal complements" by the usual dot product. We recall that $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}_2^6$ are orthogonal if $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$. Then find a basis for the range of A. (Hint: A is symmetric, and you may use the "fundamental theorem of linear algebra" without proof.)*

5. *Give an unsolvable configuration, and prove that it is not solvable. How many unsolvable configurations are there?*

Problem 4 (3D rotations)**.** *We study rotations in $\mathbb{R}^3$. As you can imagine, a 3D rotation must have an "axis of rotation", i.e., a fixed direction to rotate around. Given a non-zero vector $\boldsymbol{u}$, let us define $R_{\boldsymbol{u}}^{\theta}$ to be the rotation around the direction $\boldsymbol{u}$. Imagine that $\boldsymbol{u}$ is pointing to your nose, then we want this rotation to be counter-clockwise by θ. Since this is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$, we see that $R_{\boldsymbol{u}}^{\theta}$ should be a 3×3 matrix.*

Geometrically, it is easy to see that for all real number $a \in \mathbb{R}$ and integer $b \in \mathbb{Z}$, we have $(R_{\boldsymbol{u}}^{a})^b = R_{\boldsymbol{u}}^{ab}$. For all real numbers $a, b \in \mathbb{R}$, we also have $R_{\boldsymbol{u}}^{a} R_{\boldsymbol{u}}^{b} = R_{\boldsymbol{u}}^{a+b}$. This is the reason why we write θ as an exponent.

1. *Find the matrix $R_{\boldsymbol{a}}^{\frac{2\pi}{3}}$ where $\boldsymbol{a} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$.*

2. *Find two 3D rotations R, R' such that $RR' \neq R'R$. Show me why. (In $\mathbb{R}^2$, all rotations must commute. But as you can see here, this is no longer true in higher dimensions.)*

3. *Prove that for any non-zero $\boldsymbol{u}$ and any θ, the columns of $R_{\boldsymbol{u}}^{\theta}$ must all be unit vectors, and they must all be mutually orthogonal to each other.*

4. *For any non-zero vector $\boldsymbol{v}$, let $\boldsymbol{u}$ be any unit vector perpendicular to it. Then let $\boldsymbol{w} = R_{\boldsymbol{v}}^{\frac{\pi}{2}} \boldsymbol{u}$. Show that $(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w})$ is a basis.*

5. *Continue the last subproblem. For a rotation $R_{\boldsymbol{v}}^{\theta}$, find the vectors $R_{\boldsymbol{v}}^{\theta}\boldsymbol{v}, R_{\boldsymbol{v}}^{\theta}\boldsymbol{u}, R_{\boldsymbol{v}}^{\theta}\boldsymbol{w}$ in terms of linear combinations of $\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}$. Find the matrix M of $R_{\boldsymbol{v}}^{\theta}$ under the basis $(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w})$ for both domain and codomain, and find a matrix X such that $R_{\boldsymbol{v}}^{\theta} = XMX^{-1}$. (Hint: How would $R_{\boldsymbol{v}}^{\theta}$ act on the subspace spanned by $\boldsymbol{u}$ and $\boldsymbol{w}$? What is the nature of this abstract linear map? Also recall the relation between block diagonal matrix and "submaps" of a map.)*

6. *For any rotations M and $R_{\boldsymbol{v}}^{\theta}$, show that $MR_{\boldsymbol{v}}^{\theta}M^{-1} = R_{M\boldsymbol{v}}^{\theta}$. (Hint: use what we have learnt about rotation matrices in the last subproblem.)*

7. *Show that if $R_{\boldsymbol{v}}^{\theta} R_{\boldsymbol{u}}^{\phi} = R_{\boldsymbol{u}}^{\phi} R_{\boldsymbol{v}}^{\theta}$ where ϕ, θ are not multiples of 2π, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel. (Hint: use what we have learnt about rotation matrices in the last subproblem.)*

Problem 5 (Rotations in Higher Dimensions and Rotationary Gaussian Eliminations)**.** *In this problem, we always assume that $n \geq 2$. On $\mathbb{R}^n$, a Givens rotation is a matrix of the form*

$$R_{ij}^{\theta} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \cos\theta & & -\sin\theta & & \\ & & & \ddots & & & \\ & & \sin\theta & & \cos\theta & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},$$

i.e., it is mostly the same with the identity matrix, except that the $(i,i), (i,j), (j,i), (j,j)$ entries are changed to the entries of the 2D rotation matrix. We define a rotation on $\mathbb{R}^n$ to be the product of many Givens rotations. In particular, product of rotations must be a rotation, and inverse of a rotation must be a rotation.

1. *Given a vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \\ 2 \end{bmatrix}$, find three Givens rotations R_1, R_2, R_3 such that $R_1\boldsymbol{v} = \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \\ 2 \end{bmatrix}$, $R_2R_1\boldsymbol{v} =$*

$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, *and* $R_3 R_2 R_1 \boldsymbol{v} = \begin{bmatrix} 2\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *(This is how we use rotations to "eliminate" lower entries.)*

2. *For any unit vector* $\boldsymbol{u} \in \mathbb{R}^n$, *show that we can find a rotation* R *such that* $R\boldsymbol{u} = \boldsymbol{e}_1$. *(Hint: mathematical induction, starting with* $n = 2$.*)*

3. *For any matrix* A, *show that we can find a rotation* R *such that* $A = RU$ *for an upper triangular matrix* U. *(Hint: Rotate the first column of* A, *then look at the* $(n-1) \times (n-1)$ *lower right block and use mathematical induction.)*

4. *Suppose* H *has unit vectors as columns, and they are all mutually orthogonal. The show that for any rotation* R, RH *will still have unit vectors as columns, and they are all mutually orthogonal. (Hint: First consider the case that* R *is a single Givens rotation. Note that* $R^T R = I$ *for Givens rotation* R, *and* $||\boldsymbol{v}||^2 = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$, *and we have* $\boldsymbol{v} \perp \boldsymbol{w}$ *if and only if* $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.*)*

5. *Suppose* H *has unit vectors as columns, and they are all mutually orthogonal. Show that we can find a rotation* R *such that* RH *is a diagonal matrix with* ± 1 *on the diagonal. (Hint: Show that* $RH = U$ *where* U *is upper triangular, yet it has unit vectors as columns, and they are all mutually orthogonal. Then show that* U *is the desired diagonal matrix.)*

6. *(Read only) Note that for a diagonal matrix with* ± 1 *on the diagonal, if it has an even number of* -1 *on the diagonal, then it is still a product of many Givens rotations. So in the end, whenever* H *has unit vectors as columns, and they are all mutually orthogonal, then either* H *is a rotation, or* $H = R \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ *where* R *is a rotation and* $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ *is a reflection.*