第9讲：串联RLC分压分析

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串联RLC分压分析-时域分析 内容

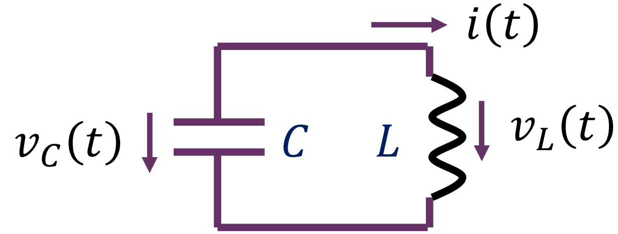
- 同属性元件分压分流电路
  - 纯阻串联分压、并联分流电路
  - 纯容串联分压、并联分流电路
  - 纯感串联分压、并联分流电路
- 阻容分压电路与分流电路
  - 直流激励、阶跃激励、正弦激励、冲激激励、方波激励
  - 阻感分压分流同理，均属一阶动态系统分析
- 阻容感分压电路和分流电路：二阶动态系统分析
  - 串联RLC分压分析：对偶的并联RLC分流分析对偶表述即可
    - 时域分析：一般性分析
    - 时频分析：二阶滤波器

串联RLC分压分析---时域分析

- 二阶LTI系统的系统参量
  - 自由振荡频率、阻尼系数
- 状态方程和微分方程
  - 特征方程和特征根
- 解的形态
  - 待定系数法
  - 五要素法
    - 欠阻尼：减幅正弦振荡
      - 无阻尼：等幅正弦振荡
      - 过阻尼：指数衰减
- 作业选讲
  - 稳态响应

LC谐振腔的自由振荡

正弦振荡



$$\frac{1}{C}\int(-i(t))\,dt=\frac{1}{C}\int i_C(t)\,dt=v_C(t)=v_L(t)=L\frac{di_L(t)}{dt}=L\frac{di(t)}{dt}$$

KCL GOL KVL GOL KCL

$$-\frac{i(t)}{C}=L\frac{d^2i(t)}{dt^2}\qquad\qquad \frac{d^2i(t)}{dt^2}+\frac{i(t)}{LC}=0$$

由电容初始电压，电感初始电流确定，同学自行练习确定：假设$\boldsymbol{i_L(0)}$，$\boldsymbol{v_c(0)}$已知

电感电流 $$i(t)=I_0\cos\left(\frac{1}{\sqrt{LC}}t+\varphi_0\right)=I_0\cos(\omega_0 t+\varphi_0)$$

电容电压 $$v(t)=-\omega_0 L I_0\sin(\omega_0 t+\varphi_0)=-Z_0 I_0\sin(\omega_0 t+\varphi_0)$$

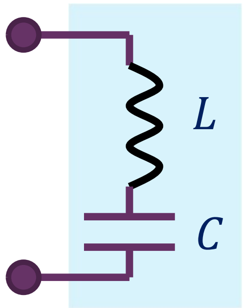
$$\omega_0=\frac{1}{\sqrt{LC}}$$ **LC**谐振腔的自由振荡频率 rad/s

$$Z_0=\omega_0 L=\sqrt{\frac{L}{C}}=\frac{1}{\omega_0 C}$$ **LC**谐振腔的特征阻抗 Ω

$$E_C(t)+E_L(t)=\frac{1}{2}Cv^2(t)+\frac{1}{2}Li^2(t)=\frac{1}{2}Cv^2(0)+\frac{1}{2}Li^2(0)$$

纯**LC**谐振腔内无能量损耗，电容电能和电感磁能之间的相互转换以正弦振荡的形态维持

对LC串联谐振腔和并联谐振腔的描述



$$Z = Z_L + Z_C = j\omega L + \frac{1}{j\omega C}$$

$$= j\left(\omega L - \frac{1}{\omega C}\right) = j\sqrt{\frac{L}{C}}\left(\omega\sqrt{LC} - \frac{1}{\omega\sqrt{LC}}\right)$$

$$= jZ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \overset{\omega=\omega_0}{\triangleq} 0$$

谐振频点上，串联**LC**短路

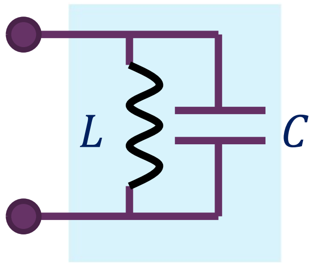
$$\omega_0 = \frac{1}{\sqrt{LC}} \quad rad/s$$

自由振荡频率
谐振频率

$$Z_0 = \sqrt{\frac{L}{C}} = \omega_0 L = \frac{1}{\omega_0 C}$$

Ω

特征阻抗：恰好是自由振荡频点的电感或电容电抗值



$$Y = Y_C + Y_L = j\omega C + \frac{1}{j\omega L}$$

$$= j\left(\omega C - \frac{1}{\omega L}\right) = j\sqrt{\frac{C}{L}}\left(\omega\sqrt{LC} - \frac{1}{\omega\sqrt{LC}}\right)$$

$$= jY_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \overset{\omega=\omega_0}{\triangleq} 0$$

谐振频点上，并联**LC**开路

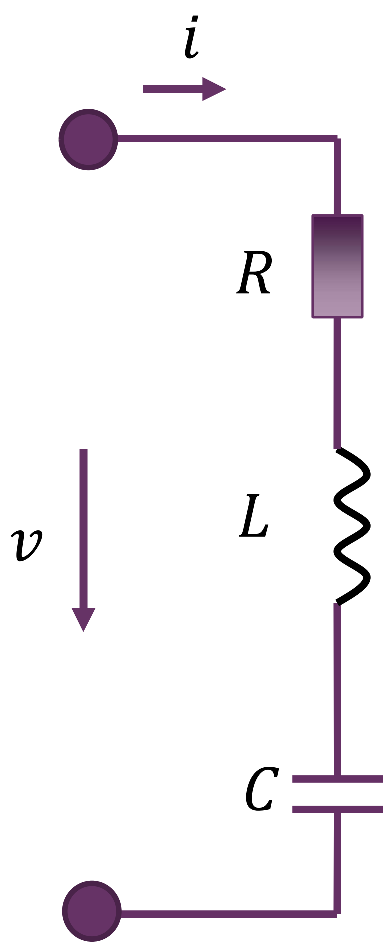
$$\omega_0 = \frac{1}{\sqrt{LC}} \quad rad/s$$

$$Y_0 = \sqrt{\frac{C}{L}} = \omega_0 C = \frac{1}{\omega_0 L}$$

S

LC谐振腔特征导纳

串联RLC系统参量



$$Z(j\omega) = Z_R + Z_L + Z_C = R + j\omega L + \frac{1}{j\omega C} = R + jZ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$= R\left(1 + j\frac{Z_0}{R}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right) = R\left(1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)$$

自由振荡频率
谐振频率 $\omega_0 = \frac{1}{\sqrt{LC}}$ rad/s

特征阻抗 $Z_0 = \sqrt{\frac{L}{C}} = \omega_0 L = \frac{1}{\omega_0 C}$ Ω

品质因数 $Q = \frac{Z_0}{R} = \frac{\text{虚功}}{\text{实功}}$

$s = j\omega$

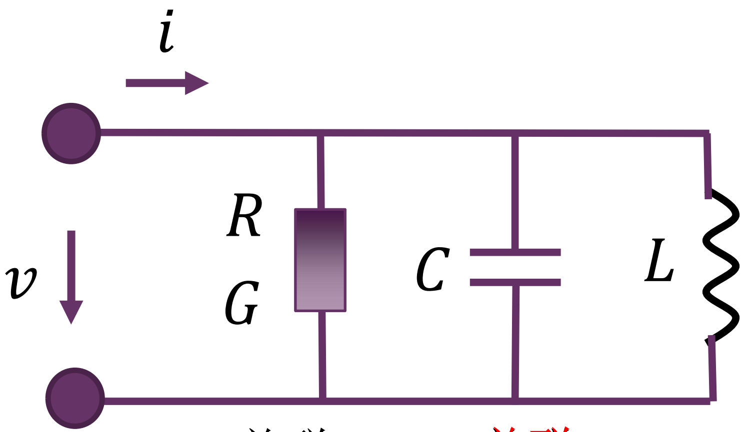
$$Z(s) = R + sL + \frac{1}{sC} = \frac{s^2LC + sRC + 1}{sC} = \frac{s^2 + s\frac{R}{L} + \frac{1}{LC}}{\frac{s}{L}}$$

$$= R\frac{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}{\frac{\omega_0}{Q}s} = R\frac{s^2 + 2\xi\omega_0 s + \omega_0^2}{2\xi\omega_0 s}$$

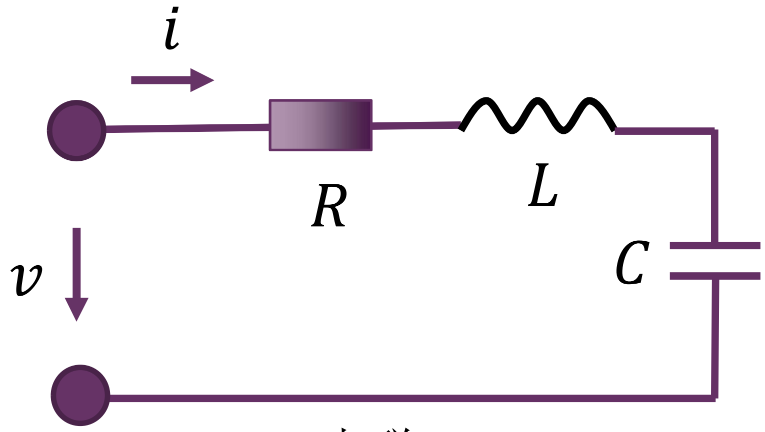
$$\xi = \frac{1}{2Q} = \frac{R}{2Z_0}$$

阻尼系数

并联RLC对偶串联RLC



并联GCL：并联RLC



串联RLC

$$\omega_0=\frac{1}{\sqrt{CL}}=\frac{1}{\sqrt{LC}}\quad rad/s$$

$$\omega_0=\frac{1}{\sqrt{LC}}\quad rad/s$$

$$Y_0=\sqrt{\frac{C}{L}}=\omega_0 C=\frac{1}{\omega_0 L}\quad S$$

$$Z_0=\sqrt{\frac{L}{C}}=\omega_0 L=\frac{1}{\omega_0 C}\quad \Omega$$

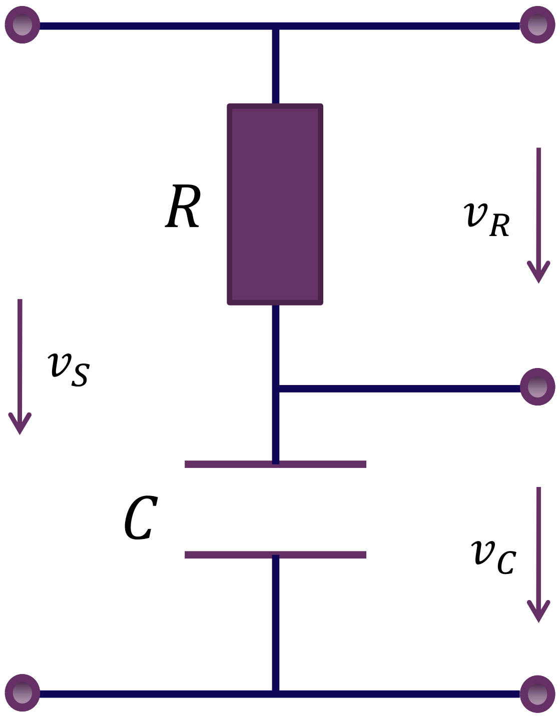
$$Q=\frac{Y_0}{G}=\frac{虚功}{实功}=\frac{电纳}{电导}=\frac{\omega_0 C}{G}$$

$$Q=\frac{Z_0}{R}=\frac{虚功}{实功}=\frac{电抗}{电阻}=\frac{\omega_0 L}{R}$$

$$\xi=\frac{1}{2Q}=\frac{G}{2Y_0}$$

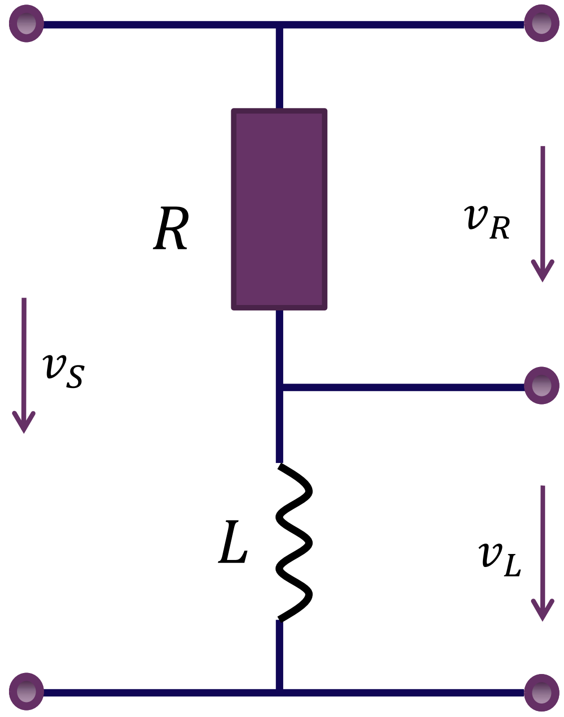
$$\xi=\frac{1}{2Q}=\frac{R}{2Z_0}$$

回顾：一阶RC分压、RL分压分析



串联**RC**：电容低频开路，分压系数为**1**，电容高频短路，分压系数为**0**：说明电容分压为输入信号中的低频分量，电容分压系数具有低通滤波特性；电阻分压系数在高频为**1**，低频为**0**，说明电阻分压为输入信号中的高频分量，电阻分压系数具有高通滤波器特性

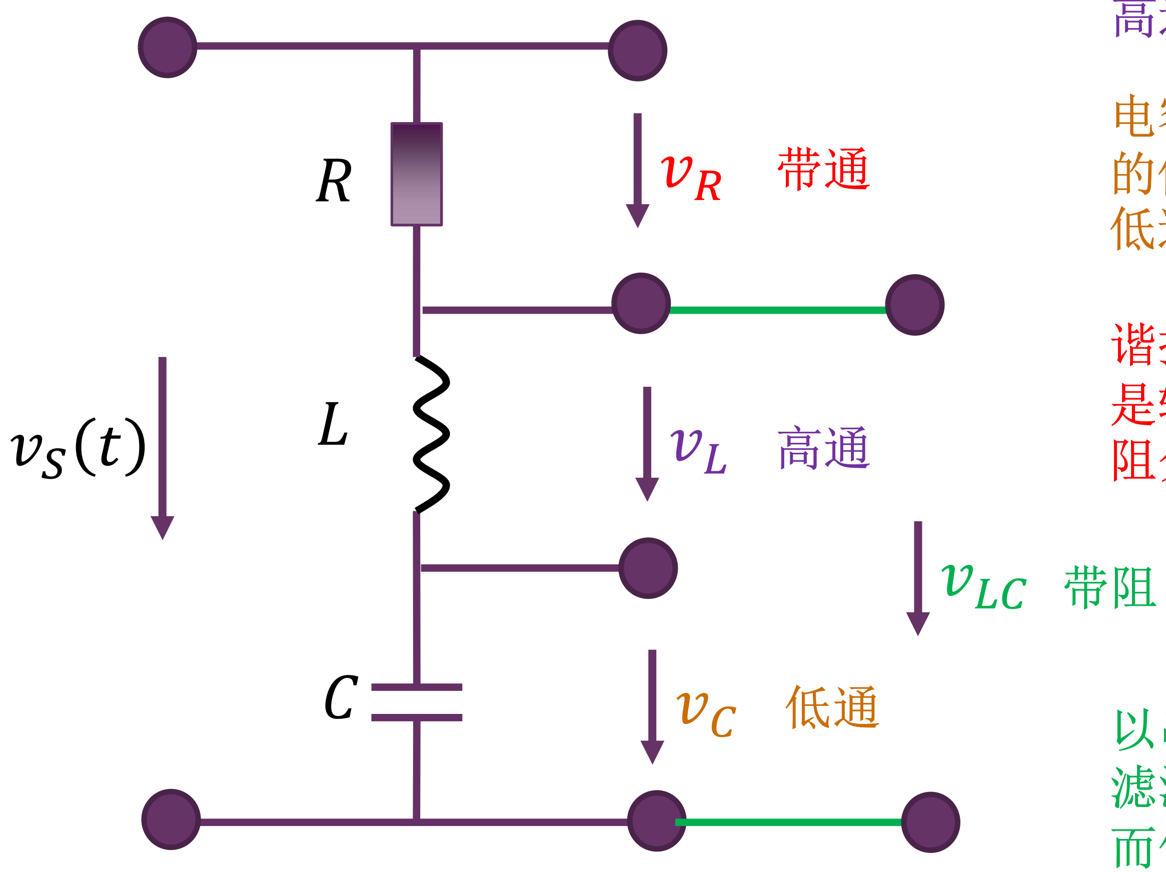
串联**RL**：电阻分压系数具有低通滤波特性；电感分压系数具有高通滤波特性



$$H_{LP1}(j\omega) = H_0 \frac{1}{1 + j\omega\tau} = H_0 \frac{\omega_0}{s + \omega_0} \qquad H_0 = 1 \qquad \omega_0 = \frac{1}{\tau} \qquad BW_{3dB} = \frac{1}{2\pi\tau}$$

$$H_{HP1}(j\omega) = H_0 \frac{j\omega\tau}{1 + j\omega\tau} = H_0 \frac{s}{s + \omega_0} \qquad \tau = RC, GL \qquad s = j\omega \qquad f_{3dB} = \frac{1}{2\pi\tau}$$

串联RLC分压分析：典型的二阶滤波特性



电感高频开路，分担的是输入电压中的高频分量，以电感分压为输出则具高通滤波特性

电容低频开路，分担的是输入电压中的低频分量，以电容分压为输出则具低通滤波特性

谐振频点串联**LC**短路，电阻分担的是输入电源中的中间频率分量，以电阻分压为输出则具带通滤波特性

以串联**LC**分压为输出者，则具带阻滤波特性：谐振频点**LC**短路无输出，而低频电容开路、高频电感开路使得串联**LC**均获得全部分压

二阶滤波器传递函数典型形态

$$H_{LP2}=H_{v_C}=\frac{\frac{1}{j\omega C}}{R+j\omega L+\frac{1}{j\omega C}}=\frac{1}{s^2LC+sRC+1}=\frac{\frac{1}{LC}}{s^2+s\frac{R}{L}+\frac{1}{LC}}=\frac{\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2}$$

$$H_{LP2}=H_{v_C}=\frac{\frac{1}{j\omega C}}{R+j\omega L+\frac{1}{j\omega C}}=\cdots H_0\frac{\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2}$$

$$\omega_0=\frac{1}{\sqrt{LC}}$$

二阶系统自由振荡频率

$$H_{HP2}=H_{v_L}=\frac{j\omega L}{R+j\omega L+\frac{1}{j\omega C}}=\cdots H_0\frac{s^2}{s^2+2\xi\omega_0 s+\omega_0^2}$$

$$\xi=\frac{R}{2Z_0}=\frac{1}{2}R\sqrt{\frac{C}{L}}$$

二阶系统阻尼系数

$$H_{BP2}=H_{v_R}=\frac{R}{R+j\omega L+\frac{1}{j\omega C}}=\cdots H_0\frac{2\xi\omega_0 s}{s^2+2\xi\omega_0 s+\omega_0^2}$$

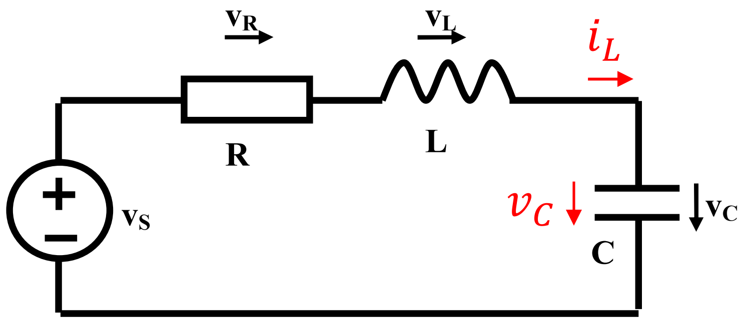
$$s=j\omega$$

$$H_{BS2}=H_{v_{LC}}=\frac{j\omega L+\frac{1}{j\omega C}}{R+j\omega L+\frac{1}{j\omega C}}=\cdots H_0\frac{s^2+\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2}$$

H_0

滤波器中心频点传递系数

RLC串联电路一般性分析：状态方程法



状态方程：就是以电路中的**n**个独立状态变量为未知量列写的**n**个一阶微分方程组，方程左侧为状态变量的一阶微分形式，方程右侧为状态变量和激励变量的代数方程形式

$$v_S = v_R + v_L + v_C = i_L R + L\frac{d}{dt}i_L + v_C \qquad i_L = i_C = C\frac{d}{dt}v_C$$

$$\frac{d}{dt}v_C = \frac{1}{C}i_L$$

$$\frac{d}{dt}i_L = -\frac{R}{L}i_L - \frac{1}{L}v_C + \frac{1}{L}v_S$$

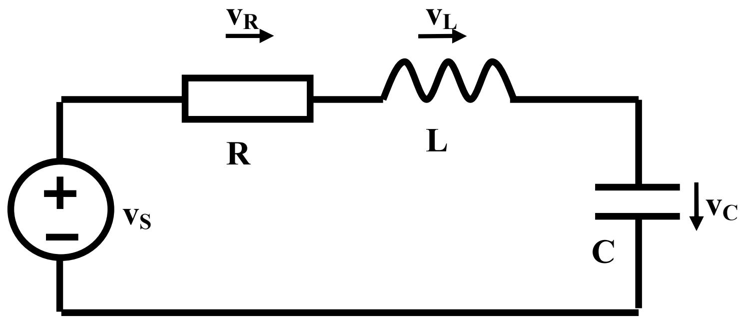
$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L}v_S \end{bmatrix}$$

$$\begin{bmatrix} v_{R,out} \\ v_{L,out} \\ v_{C,out} \end{bmatrix} = \begin{bmatrix} 0 & R \\ -1 & -R \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_S$$

状态方程法的好处是，一次性求出状态变量后，它们作为系统内蕴的源（或者由替代定理，用恒压源$\boldsymbol{v_C(t)}$替代电容**C**，用恒流源$\boldsymbol{i_L(t)}$替代电感**L**），与外加激励源共同决定系统内的任何电量：由叠加定理可知，系统中的任何电量均可表述为状态变量和外加激励源的叠加形式

LTI系统状态方程的一般形式

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} v_S \end{bmatrix}$$



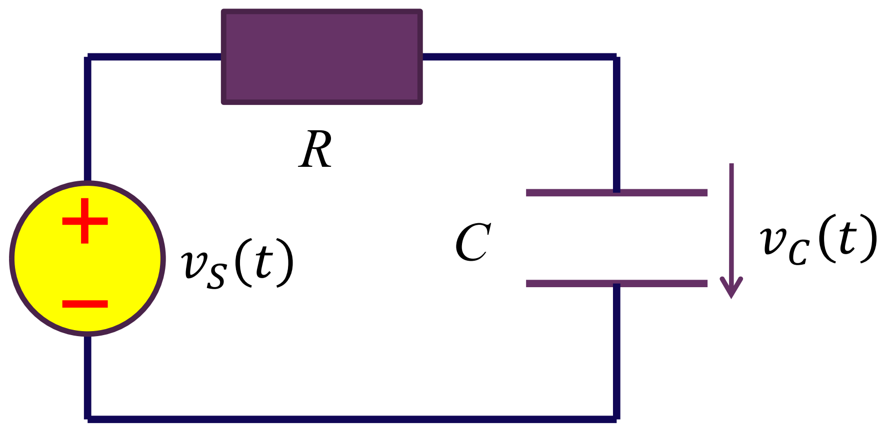
LTI系统状态方程的一般形式

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{s}(t)$$

状态变量列向量

状态矩阵
电路参量矩阵
常数矩阵

电路中独立源的作用



其形式同样适用一阶系统

$$\frac{d}{dt}v_C(t) = -\frac{1}{RC}v_C(t) + \frac{1}{RC}v_S(t)$$

LTI状态方程求解的一般过程

一阶**LTI**系统状态方程求解过程

$$\frac{d}{dt}x = ax + s$$

$$\frac{d}{dt}\left(e^{-at}x\right) = e^{-at}s$$

$$e^{-at}x(t)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{-a\lambda}s(\lambda)d\lambda$$

$$x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\lambda)}s(\lambda)d\lambda$$

零输入响应　　零状态响应

$$x(t) = x_\infty(t) + \left(x(t_0) - x_\infty(t_0)\right)e^{a(t-t_0)}$$

稳态响应　　瞬态响应

$$\left(t \ge t_0\right)$$

直接推广到高阶**LTI**系统

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{s}$$

$$\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}\right) = e^{-\mathbf{A}t}\mathbf{s}$$

$$e^{-\mathbf{A}t}\mathbf{x}(t)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{-\mathbf{A}\lambda}\mathbf{s}(\lambda)d\lambda$$

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\lambda)}\mathbf{s}(\lambda)d\lambda$$

零输入响应　　零状态响应

$$\mathbf{x}(t) = \mathbf{x}_\infty(t) + e^{\mathbf{A}(t-t_0)}\left(\mathbf{x}(t_0) - \mathbf{x}_\infty(t_0)\right)$$

稳态响应　　瞬态响应

LTI系统解的形态是确定的

$$\mathbf{x}(t)=\mathbf{x}_\infty(t)+e^{\mathbf{A}(t-t_0)}\left(\mathbf{x}(t_0)-\mathbf{x}_\infty(t_0)\right)$$

由激励决定的稳态响应，具有和激励相同的形态

通过状态转移矩阵的作用，当前状态是由之前状态转移过来的

状态转移矩阵 $e^{At}=I+At+\frac{1}{2!}A^2t^2+\frac{1}{3!}A^3t^3+\cdots$

$$=I+P\begin{bmatrix}\lambda_1 & & \\ & \cdot & \\ & & \lambda_n\end{bmatrix}P^{-1}t+\frac{1}{2!}P\begin{bmatrix}\lambda_1^2 & & \\ & \cdot & \\ & & \lambda_n^2\end{bmatrix}P^{-1}t^2+\cdots$$

$$=P\begin{bmatrix}1+\lambda_1 t+\frac{1}{2!}\lambda_1^2t^2+\cdots & & \\ & \cdot & \\ & & 1+\lambda_n t+\frac{1}{2!}\lambda_n^2t^2+\cdots\end{bmatrix}P^{-1}$$

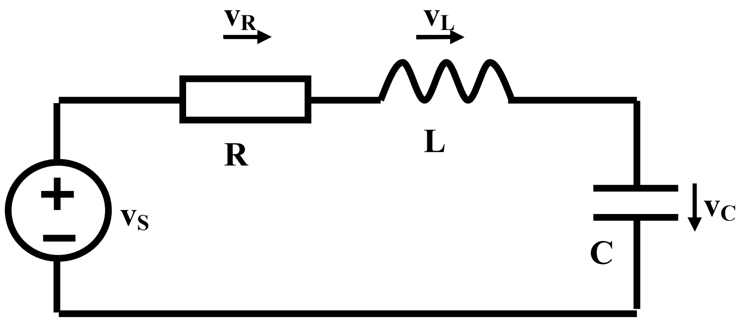
$$=P\begin{bmatrix}e^{\lambda_1 t} & & \\ & \cdot & \\ & & e^{\lambda_n t}\end{bmatrix}P^{-1}=\begin{bmatrix}\cdot & \cdot & \cdot \\ \cdot & \sum_{k=1}^{n}\alpha_k^{ij}e^{\lambda_k t} & \cdot \\ \cdot & \cdot & \cdot\end{bmatrix}_{n\times n}$$

状态变量的瞬态响应一定是$e^{\lambda_k t}$的叠加形式，其叠加加权系数待定(和初值、激励源/稳态响应均有关)；其中λ_k是状态矩阵**A**的特征根，也是**LTI**系统特征根

关键问题：特征根求取

二阶系统的特征参量是什么?

从微分方程的形态入手进行研究



$$v_S = v_R + v_L + v_C = i_C R + L\frac{d}{dt}i_C + v_C$$

$$= RC\frac{d}{dt}v_C + LC\frac{d^2}{dt^2}v_C + v_C$$

$$\frac{d^2}{dt^2}v_C + \frac{R}{L}\frac{d}{dt}v_C + \frac{1}{LC}v_C = \frac{1}{LC}v_S$$

$$v_S = v_R + v_L + v_C = v_R + L\frac{d}{dt}i_R + \frac{1}{C}\int i_R dt$$

$$= v_R + \frac{L}{R}\frac{d}{dt}v_R + \frac{1}{RC}\int v_R dt$$

$$\frac{d}{dt}v_S = \frac{d}{dt}v_R + \frac{L}{R}\frac{d^2}{dt^2}v_R + \frac{1}{RC}v_R$$

$$\frac{d^2}{dt^2}v_R + \frac{R}{L}\frac{d}{dt}v_R + \frac{1}{LC}v_R = \frac{R}{L}\frac{d}{dt}v_S$$

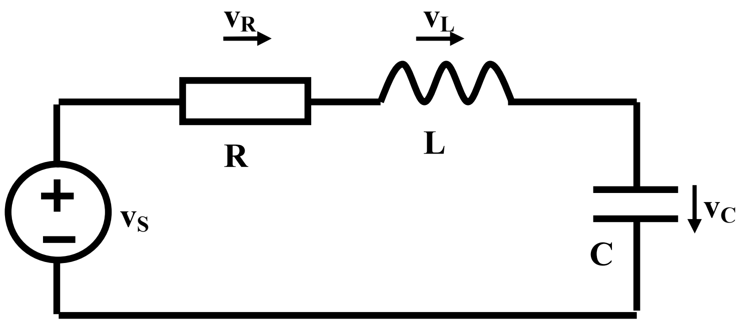
$$v_S = v_R + v_L + v_C = i_L R + v_L + \frac{1}{C}\int i_L dt$$

$$= \frac{R}{L}\int v_L dt + v_L + \frac{1}{LC}\iint v_L dt^2$$

$$\frac{d^2}{dt^2}v_S = \frac{R}{L}\frac{d}{dt}v_L + \frac{d^2}{dt^2}v_L + \frac{1}{LC}v_L$$

$$\frac{d^2}{dt^2}v_L + \frac{R}{L}\frac{d}{dt}v_L + \frac{1}{LC}v_L = \frac{d^2}{dt^2}v_S$$

二阶LTI系统微分方程的一般形式



用电路中的任意电量，均可得到形态完全一致的二阶微分电路方程，仅仅是激励的形态不同而已

$$\frac{d^2}{dt^2}v_C+\frac{R}{L}\frac{d}{dt}v_C+\frac{1}{LC}v_C=\frac{1}{LC}v_S$$

$$\frac{d^2}{dt^2}v_R+\frac{R}{L}\frac{d}{dt}v_R+\frac{1}{LC}v_R=\frac{R}{L}\frac{d}{dt}v_S$$

$$\frac{d^2}{dt^2}v_L+\frac{R}{L}\frac{d}{dt}v_L+\frac{1}{LC}v_L=\frac{d^2}{dt^2}v_S$$

$$\frac{d^2}{dt^2}x+2\xi\omega_0\frac{d}{dt}x+\omega_0^2x=s_x$$

获得这个微分方程形式是一定的，从频域看

$$(j\omega)^2\dot{X}+2\xi\omega_0(j\omega)\dot{X}+\omega_0^2\dot{X}=\dot{S}_X$$

$$\frac{\dot{X}}{\dot{S}}=\frac{\dot{S}_X/\dot{S}}{s^2+2\xi\omega_0 s+\omega_0^2}$$

从微分方程求特征根的一般过程

假设**n**阶**LTI**动态系统的**n**阶微分方程为 $\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} x(t) = f(s_1(t), s_2(t), \ldots)$ $a_n = 1$

其中， $x(t)$是电路中的某个电量，可以是状态变量，也可以不是状态变量，对于**LTI**系数，微分方程系数是由系统结构决定的常系数，前述分析表明，其齐次方程（零输入情况）的解一定是指数$X_0 e^{\lambda t}$形式，其中，λ为其特征根。代入齐次方程

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} x(t) = \sum_{k=0}^{n} a_k \frac{d^k}{dt^k}\left(X_0 e^{\lambda t}\right) = \sum_{k=0}^{n} a_k X_0 \lambda^k e^{\lambda t} = X_0 e^{\lambda t} \sum_{k=0}^{n} a_k \lambda^k = 0$$

获得的**n**次多项式方程，就是**LTI**系统的特征方程，其根就是特征根

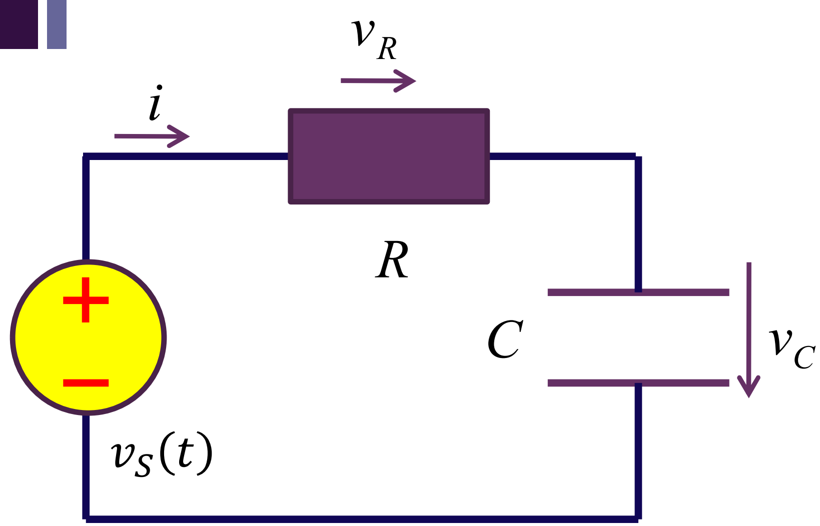
$$\sum_{k=0}^{n} a_k \lambda^k = 0$$

某种数学方法求解特征方程获得特征根后，电量$x(t)$的解可以表述为

$$x(t) = x_\infty(t) + A e^{\lambda_1 t} + B e^{\lambda_2 t} + \cdots$$

其中$x_\infty(t)$是激励源决定的稳态响应，**n**个待定系数**A**、**B**、**…**由**n**个初值$x(0^+)$、$\frac{d}{dt}x(0^+)$、 …确定

一阶系统特征根



$$v_S(t) = v_R(t) + v_C(t)$$

$$= RC\frac{d}{dt}v_C(t) + v_C(t)$$

电路方程为

$$RC\frac{d}{dt}v_C(t) + v_C(t) = v_S(t)$$

$$\frac{d}{dt}v_C(t) + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_S(t)$$

一阶系统微分方程的一般形式

$$\frac{d}{dt}x(t) + \frac{1}{\tau}x(t) = s(t)$$

其特征根方程为

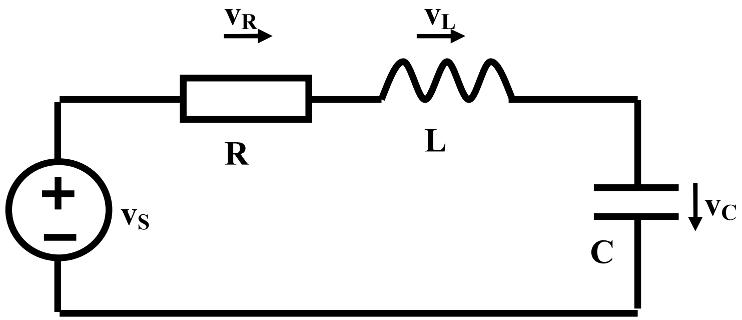
$$\lambda + \frac{1}{\tau} = 0$$

其特征根为 $\lambda = -\frac{1}{\tau}$

特征根位于左半平面

特征根量纲：1/s

二阶系统特征根：自微分方程求取



$$v_S = v_R + v_L + v_C = i_C R + L\frac{d}{dt}i_C + v_C$$

$$= RC\frac{d}{dt}v_C + LC\frac{d^2}{dt^2}v_C + v_C$$

$$\frac{d^2}{dt^2}v_C + \frac{R}{L}\frac{d}{dt}v_C + \frac{1}{LC}v_C = \frac{1}{LC}v_S$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\frac{d^2}{dt^2}v_C + 2\xi\omega_0\frac{d}{dt}v_C + \omega_0^2 v_C = \omega_0^2 v_S$$

$$\xi = \frac{1}{2}R\sqrt{\frac{C}{L}} = \frac{R}{2Z_0}$$

$$\lambda^2 + 2\xi\omega_0\lambda + \omega_0^2 = 0$$

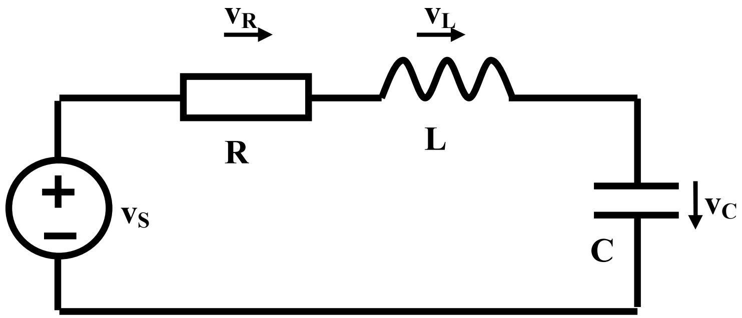
$$Z_0 = \sqrt{\frac{L}{C}}$$

$$\lambda_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_0$$

特征根位于左半平面

特征根量纲：**1/s**

二阶系统特征根：自状态矩阵求取



$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} v_S \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 Z_0 \\ -\omega_0 Y_0 & -2\xi\omega_0 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\det\begin{bmatrix} -\lambda & \omega_0 Z_0 \\ -\omega_0 Y_0 & -2\xi\omega_0 - \lambda \end{bmatrix} = 0$$

特征方程 $\lambda^2 + 2\xi\omega_0\lambda + \omega_0^2 = 0$

特征根 $\lambda_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_0$

$\xi > 1$ 过阻尼：两个负实根

$\xi = 1$ 临界阻尼：两个负实重根

$0 < \xi < 1$ 欠阻尼：两个共轭复根

$\xi = 0$ 无阻尼：两个共轭纯虚根

阻尼系数：对电路中能量损耗的描述

待定系数法：过阻尼

$$\lambda_{1,2}=\left(-\xi\pm\sqrt{\xi^2-1}\right)\omega_0$$

$$\xi>1$$

$$x(t)=x_\infty(t)+Ae^{\lambda_1 t}+Be^{\lambda_2 t}=x_\infty(t)+Ae^{-\frac{t}{\tau_1}}+Be^{-\frac{t}{\tau_2}}$$

指数衰减
长寿命项
短寿命项

$$\frac{d}{dt}x(t)=\frac{d}{dt}x_\infty(t)+A\lambda_1 e^{\lambda_1 t}+B\lambda_2 e^{\lambda_2 t}$$

$$x(0^+)=x_\infty(0^+)+A+B$$

$$\dot{x}(0^+)=\dot{x}_\infty(0^+)+A\lambda_1+B\lambda_2$$

短期行为看短寿命项（高频）；
长期行为看长寿命项（低频）

$$A=\frac{\lambda_2}{\lambda_2-\lambda_1}(X_0-X_{\infty 0})-\frac{1}{\lambda_2-\lambda_1}(\dot{X}_0-\dot{X}_{\infty 0})$$

$$B=\frac{\lambda_1}{\lambda_1-\lambda_2}(X_0-X_{\infty 0})-\frac{1}{\lambda_1-\lambda_2}(\dot{X}_0-\dot{X}_{\infty 0})$$

待定系数法：欠阻尼

$$0<\xi<1$$

$$\lambda_{1,2}=\left(-\xi\pm\sqrt{\xi^2-1}\right)\omega_0=\left(-\xi\pm j\sqrt{1-\xi^2}\right)\omega_0$$

左半平面共轭复根

$$x(t)=x_\infty(t)+Ae^{\lambda_1 t}+Be^{\lambda_2 t}$$

$$=x_\infty(t)+Ae^{-\xi\omega_0 t+j\sqrt{1-\xi^2}\omega_0 t}+Be^{-\xi\omega_0 t-j\sqrt{1-\xi^2}\omega_0 t}$$

$$=x_\infty(t)+e^{-\xi\omega_0 t}\left(Ae^{+j\sqrt{1-\xi^2}\omega_0 t}+Be^{-j\sqrt{1-\xi^2}\omega_0 t}\right)$$

$$=x_\infty(t)+e^{-\xi\omega_0 t}\begin{pmatrix}A\cos\sqrt{1-\xi^2}\omega_0 t+jA\sin\sqrt{1-\xi^2}\omega_0 t\\+B\cos\sqrt{1-\xi^2}\omega_0 t-jB\sin\sqrt{1-\xi^2}\omega_0 t\end{pmatrix}$$

$$=x_\infty(t)+e^{-\xi\omega_0 t}\begin{pmatrix}(A+B)\cos\sqrt{1-\xi^2}\omega_0 t\\+j(A-B)\sin\sqrt{1-\xi^2}\omega_0 t\end{pmatrix}$$

重定义待定系数：欠阻尼

$$0 < \xi < 1$$

$$\lambda_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_0 = \left(-\xi \pm j\sqrt{1 - \xi^2}\right)\omega_0$$

$$x(t) = x_\infty(t) + e^{-\xi\omega_0 t}\left(A\cos\sqrt{1-\xi^2}\omega_0 t + B\sin\sqrt{1-\xi^2}\omega_0 t\right)$$

$$\frac{d}{dt}x(t) = \frac{d}{dt}x_\infty(t) - \xi\omega_0 e^{-\xi\omega_0 t}\left(A\cos\sqrt{1-\xi^2}\omega_0 t + B\sin\sqrt{1-\xi^2}\omega_0 t\right) + \sqrt{1-\xi^2}\omega_0 e^{-\xi\omega_0 t}\left(-A\sin\sqrt{1-\xi^2}\omega_0 t + B\cos\sqrt{1-\xi^2}\omega_0 t\right)$$

$$x(0^+) = x_\infty(0^+) + A$$

$$\frac{d}{dt}x(0^+) = \frac{d}{dt}x_\infty(0^+) - \xi\omega_0 A + \sqrt{1-\xi^2}\omega_0 B$$

$$A = X_0 - X_{\infty 0}$$

$$B = \left(X_0 - X_{\infty 0} + \frac{\dot{X}_0 - \dot{X}_{\infty 0}}{\xi\omega_0}\right)\frac{\xi}{\sqrt{1-\xi^2}}$$

五要素法

稳态响应　初值　阻尼系数　自由振荡频率

$$x(t)=x_\infty(t)+(X_0-X_{\infty 0})e^{-\xi\omega_0 t}\cos\sqrt{1-\xi^2}\omega_0 t$$
$$+\left(X_0-X_{\infty 0}+\frac{\dot{X}_0-\dot{X}_{\infty 0}}{\xi\omega_0}\right)\frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_0 t}\sin\sqrt{1-\xi^2}\omega_0 t$$

微分初值

$$0<\xi<1$$

$$x(t)=x_\infty(t)+(X_0-X_{\infty 0})e^{-\omega_0 t}+\left(X_0-X_{\infty 0}+\frac{\dot{X}_0-\dot{X}_{\infty 0}}{\omega_0}\right)\omega_0 t e^{-\omega_0 t}$$

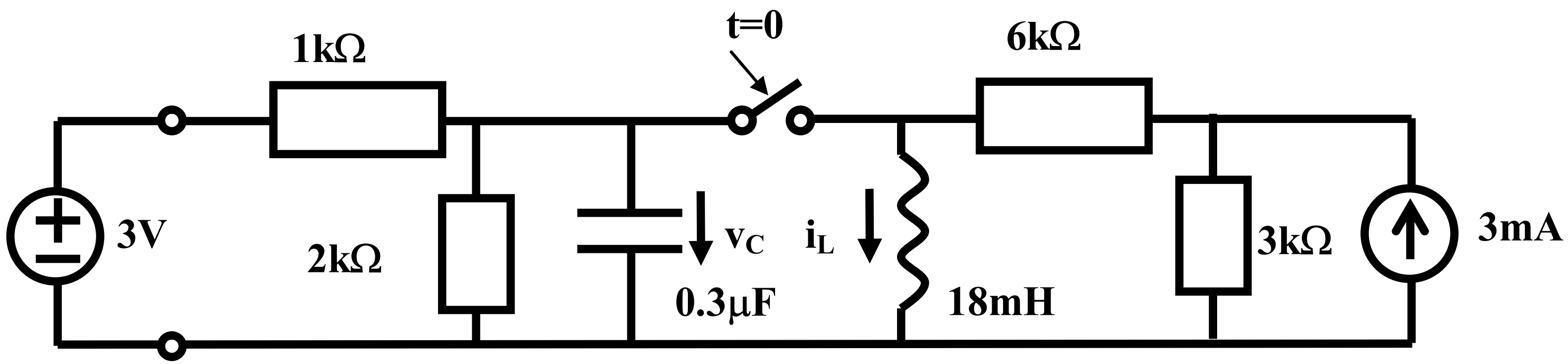
$$\xi=1$$

$$x(t)=x_\infty(t)+(X_0-X_{\infty 0})e^{-\xi\omega_0 t}\cosh\sqrt{\xi^2-1}\omega_0 t$$
$$+\left(X_0-X_{\infty 0}+\frac{\dot{X}_0-\dot{X}_{\infty 0}}{\xi\omega_0}\right)\frac{\xi}{\sqrt{\xi^2-1}}e^{-\xi\omega_0 t}\sinh\sqrt{\xi^2-1}\omega_0 t$$

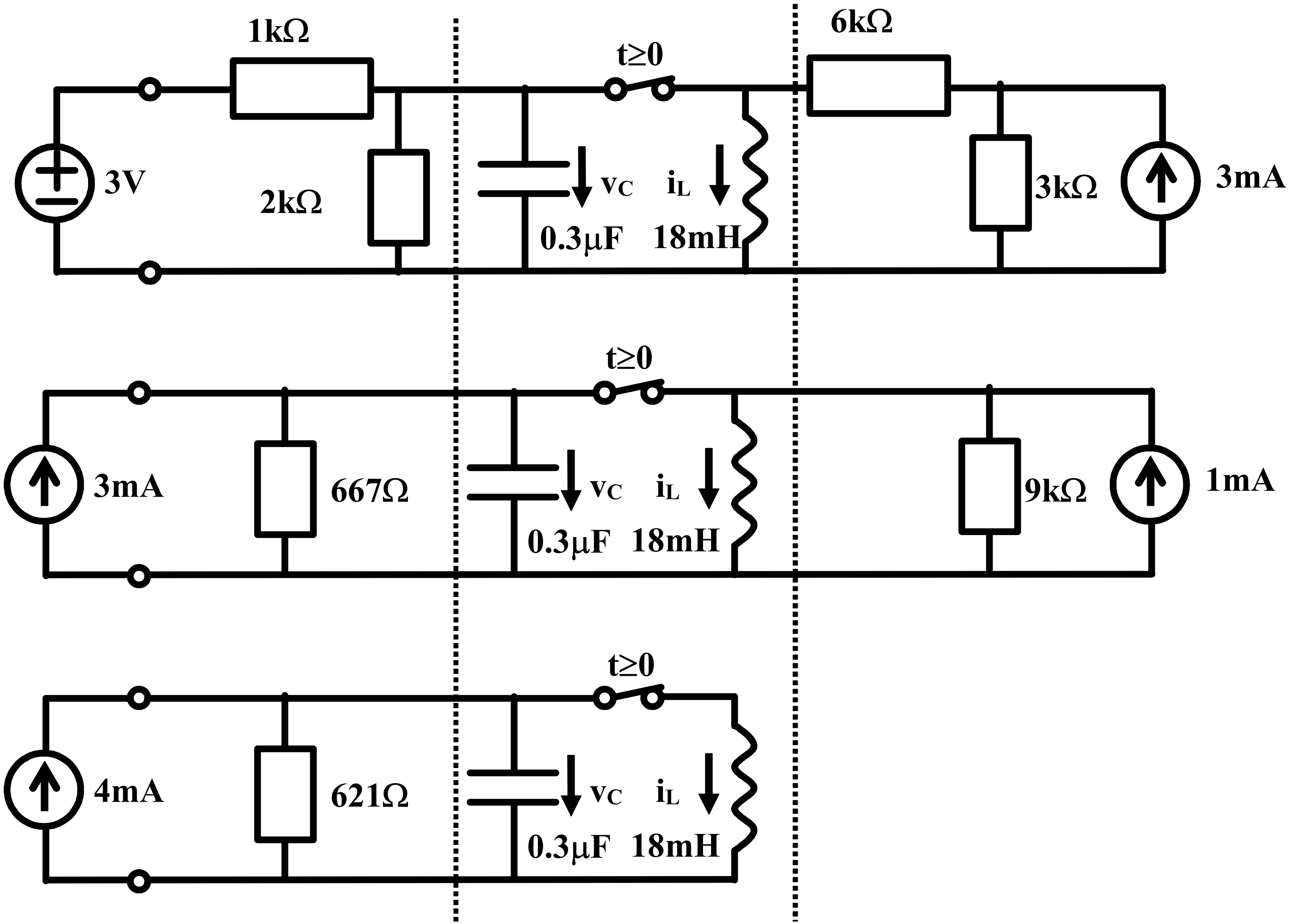
$$\xi>1$$

例1

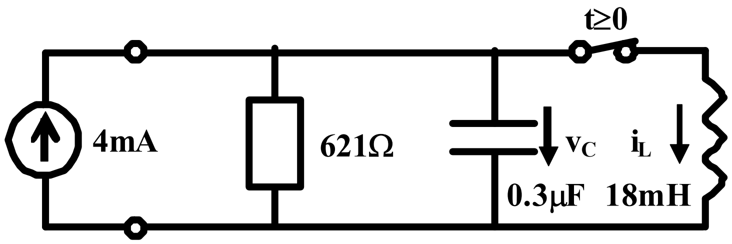
- 开关在t=0时刻闭合。开关闭合前电路已经稳定。求开关闭合后，电容电压$v_C(t)$和电感电流$i_L(t)$的变化规律



RLC并联


1kΩ
6kΩ
t≥0
3V
2kΩ
vC
iL
0.3μF
18mH
3kΩ
3mA
t≥0
3mA
667Ω
vC
iL
0.3μF
18mH
9kΩ
1mA
t≥0
4mA
621Ω
vC
iL
0.3μF
18mH

五要素：阻尼系数和自由振荡频率



串联RLC

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Z_0 = \sqrt{\frac{L}{C}} = \omega_0 L = \frac{1}{\omega_0 C}$$

$$Q = \frac{Z_0}{R} = \frac{\text{虚功}}{\text{实功}}$$

$$\xi = \frac{1}{2Q} = \frac{R}{2Z_0}$$

并联RLC

$$\omega_0 = \frac{1}{\sqrt{CL}} = \frac{1}{\sqrt{LC}}$$

$$Y_0 = \sqrt{\frac{C}{L}} = \omega_0 C = \frac{1}{\omega_0 L}$$

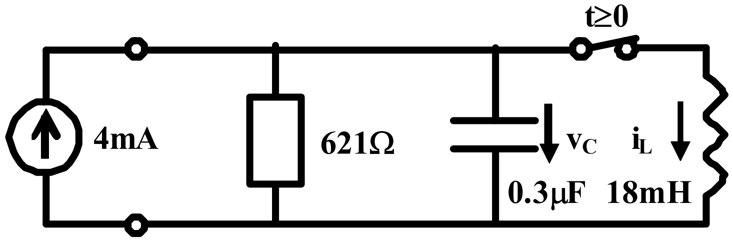
$$Q = \frac{Y_0}{G} = \frac{\text{虚功}}{\text{实功}}$$

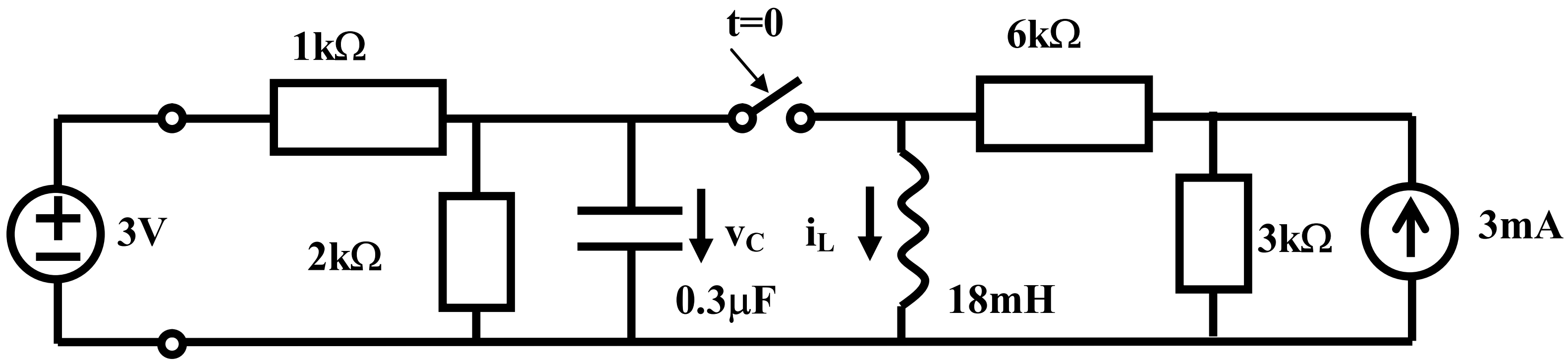
$$\xi = \frac{1}{2Q} = \frac{G}{2Y_0}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{18m \times 0.3\mu}} = 13.6krad/s$$

$$\xi = \frac{G}{2Y_0} = \frac{1}{2R}\sqrt{\frac{L}{C}} = \frac{1}{2 \times 621}\sqrt{\frac{18m}{0.3\mu}} = 0.1973$$

五要素：两个初值





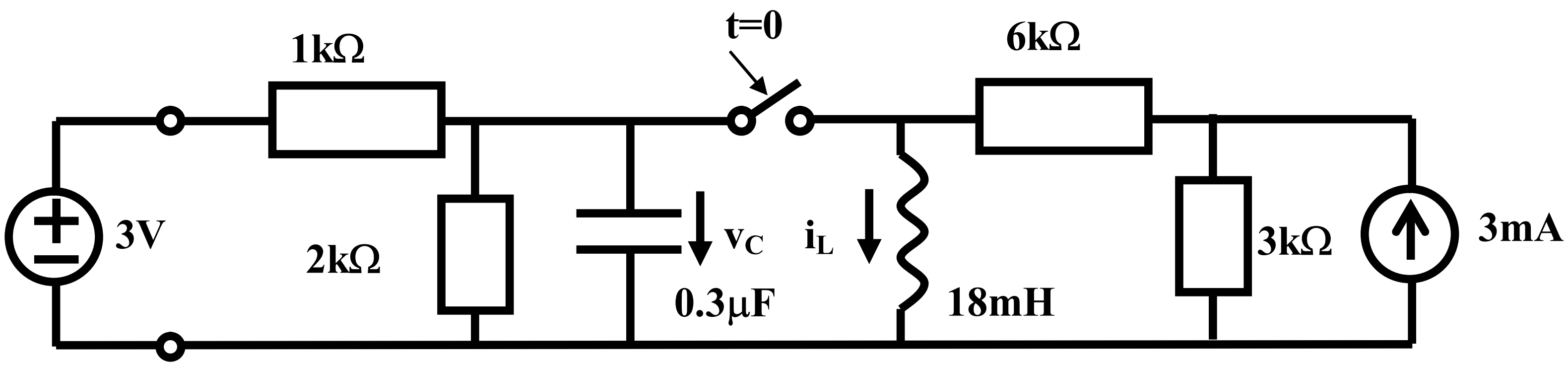
$$v_C\left(0^-\right)=\frac{2k\Omega}{1k\Omega+2k\Omega}\times 3V=2V \qquad i_L\left(0^-\right)=\frac{3k\Omega}{6k\Omega+3k\Omega}\times 3mA=1mA$$

$$v_C\left(0^+\right)=v_C\left(0^-\right)=2V \qquad i_L\left(0^+\right)=i_L\left(0^-\right)=1mA$$

$$\frac{dv_C\left(0^+\right)}{dt}=\frac{1}{C}i_C\left(0^+\right)=\frac{1}{C}\left(i_S\left(0^+\right)-i_L\left(0^+\right)-i_R\left(0^+\right)\right)=\frac{1}{C}\left(4mA-1mA-\frac{v_C\left(0^+\right)}{R}\right)$$

$$=\frac{1}{0.3\mu F}\left(4mA-1mA-\frac{2V}{621\Omega}\right)=-\frac{0.2222mA}{0.3\mu F}=-0.7407\,V/ms$$

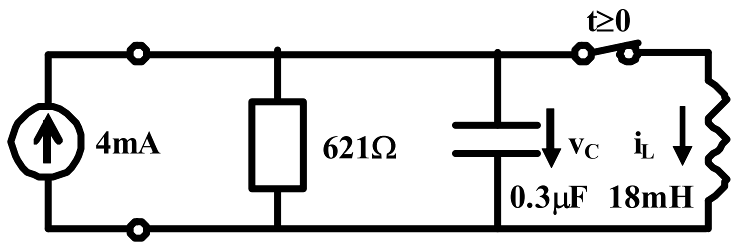
五要素：稳态响应



$$v_{C,\infty}(t)=0$$

$$v_{C,\infty}(0^+)=0$$

$$\frac{dv_{C,\infty}(0^+)}{dt}=0$$



五要素解

$$v_C(t)=v_{C,\infty}(t)+\left(V_0-V_{\infty,0}\right)e^{-\xi\omega_0 t}\cos\left(\sqrt{1-\xi^2}\,\omega_0 t\right)$$

$$+\left(V_0-V_{\infty,0}+\frac{\dot{V}_0-\dot{V}_{\infty,0}}{\xi\omega_0}\right)\frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_0 t}\sin\left(\sqrt{1-\xi^2}\,\omega_0 t\right)$$

$$=0+(2-0)e^{-\xi\omega_0 t}\cos\left(\sqrt{1-\xi^2}\,\omega_0 t\right)$$

$$+\left(2-0+\frac{-0.7407\times10^3-0}{0.1973\times13.6\times10^3}\right)\frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_0 t}\sin\left(\sqrt{1-\xi^2}\,\omega_0 t\right)$$

$$=2e^{-\xi\omega_0 t}\cos\left(\sqrt{1-\xi^2}\,\omega_0 t\right)+1.7241\frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_0 t}\sin\left(\sqrt{1-\xi^2}\,\omega_0 t\right)$$

$$=2e^{-\frac{t}{0.3724\times10^{-3}}}\cos\left(13.34\times10^3 t\right)+0.347e^{-\frac{t}{0.3724\times10^{-3}}}\sin\left(13.34\times10^3 t\right)$$

$$=2.03e^{-\frac{t}{0.3724\times10^{-3}}}\sin\left(13.34\times10^3 t+1.4\right)$$

幅度指数衰减的正弦振荡波形

欠阻尼减幅振荡

$0 < \xi < 1$

$Q = \frac{1}{2\xi} > 0.5$

$$x(t) = x_\infty(t) + e^{-\xi\omega_0 t}\left(A\cos\sqrt{1-\xi^2}\omega_0 t + B\sin\sqrt{1-\xi^2}\omega_0 t\right)$$

$$= x_\infty(t) + \sqrt{A^2+B^2}e^{-\xi\omega_0 t}\cos\left(\sqrt{1-\xi^2}\omega_0 t - \arctan\frac{B}{A}\right)$$

$$= x_\infty(t) + \sqrt{A^2+B^2}e^{-\frac{t}{\tau}}\cos(\omega_d t - \varphi_0)$$

$$\tau = \frac{1}{\xi\omega_0}$$

幅度指数衰减的正弦振荡波形：振铃

$$\omega_d = \sqrt{1-\xi^2}\omega_0$$

$$e^{-\xi\omega_0 t}\Big|_{t=QT} = e^{-\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{1-\xi^2}\omega_0 QT} = e^{-\frac{\pi}{\sqrt{1-\xi^2}}} < e^{-\pi} = 0.043 = 4.3\%$$

$$e^{-\xi\omega_0 t}\Big|_{t=1.5QT} = e^{-1.5\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{1-\xi^2}\omega_0 QT} = e^{-\frac{1.5\pi}{\sqrt{1-\xi^2}}} < e^{-1.5\pi} = 0.009 = 0.9\%$$

经过**Q**个周期，振铃幅度衰减为**4.3%**以下
经过**1.5Q**个周期，振铃幅度衰减为**1%**以下
经过**2.2Q**个周期，振铃幅度衰减为**0.1%**以下

无阻尼等幅振荡

$0 < \xi < 1$

$$x(t) = x_\infty(t) + (X_0 - X_{\infty 0})e^{-\xi\omega_0 t}\cos\sqrt{1-\xi^2}\omega_0 t + \left(X_0 - X_{\infty 0} + \frac{\dot{X}_0 - \dot{X}_{\infty 0}}{\xi\omega_0}\right)\frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_0 t}\sin\sqrt{1-\xi^2}\omega_0 t$$

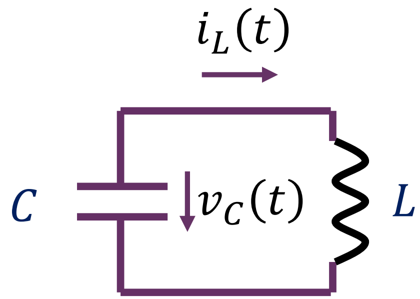
$\xi = 0$

$$x(t) = x_\infty(t) + (X_0 - X_{\infty 0})\cos\omega_0 t + \frac{\dot{X}_0 - \dot{X}_{\infty 0}}{\omega_0}\sin\omega_0 t$$

$$= x_\infty(t) + \sqrt{(X_0 - X_{\infty 0})^2 + \left(\frac{\dot{X}_0 - \dot{X}_{\infty 0}}{\omega_0}\right)^2}\cos\left(\omega_0 t - \arctan\frac{\dot{X}_0 - \dot{X}_{\infty 0}}{\omega_0(X_0 - X_{\infty 0})}\right)$$

无阻尼：振荡幅度不变　　LC自由振荡频率：无阻尼振荡频率

无阻尼LC谐振腔的自由振荡



稳态响应由源决定，本例无外加激励源（自由振荡）

$$v_{C\infty}(t)=0 \qquad i_{L\infty}(t)=0 \qquad \xi=0 \qquad \omega_0=\frac{1}{\sqrt{LC}}$$

$$v_C(0^+)=v_C(0^-)=V_0 \qquad i_L(0^+)=i_L(0^-)=I_0$$

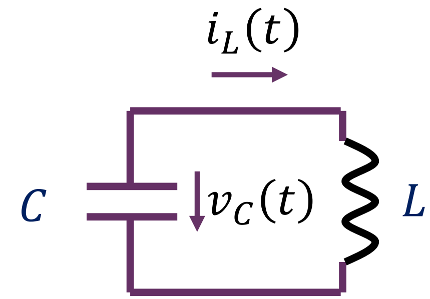
$$\frac{d}{dt}v_C(0^+)=\frac{i_C(0^+)}{C}=-\frac{i_L(0^+)}{C}=-\frac{i_L(0^-)}{C}=-\frac{I_0}{C}$$

$$\frac{d}{dt}i_L(0^+)=\frac{v_L(0^+)}{L}=\frac{v_C(0^+)}{L}=\frac{v_C(0^-)}{L}=\frac{V_0}{L}$$

$$v_C(t)=v_{C\infty}(t)+(V_{C0}-V_{C\infty0})\cos\omega_0 t+\frac{\dot{V}_{C0}-\dot{V}_{C\infty0}}{\omega_0}\sin\omega_0 t$$

$$=0+V_0\cos\omega_0 t-\frac{I_0}{\omega_0 C}\sin\omega_0 t=V_0\sqrt{1+\frac{LI_0^2}{CV_0^2}}\cos\left(\omega_0 t+\arctan\frac{Z_0 I_0}{V_0}\right)$$

无阻尼LC谐振腔的自由振荡



$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Z_0 = \sqrt{\frac{L}{C}}$$

$$v_C(t) = V_0\cos\omega_0 t - \frac{I_0}{\omega_0 C}\sin\omega_0 t = V_0\sqrt{1+\frac{LI_0^2}{CV_0^2}}\cos\left(\omega_0 t + \arctan\frac{Z_0 I_0}{V_0}\right)$$

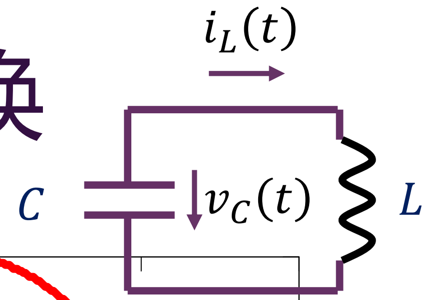
$$i_L(t) = I_0\cos\omega_0 t + \frac{V_0}{\omega_0 L}\sin\omega_0 t = I_0\sqrt{1+\frac{CV_0^2}{LI_0^2}}\sin\left(\omega_0 t + \arctan\frac{Z_0 I_0}{V_0}\right)$$

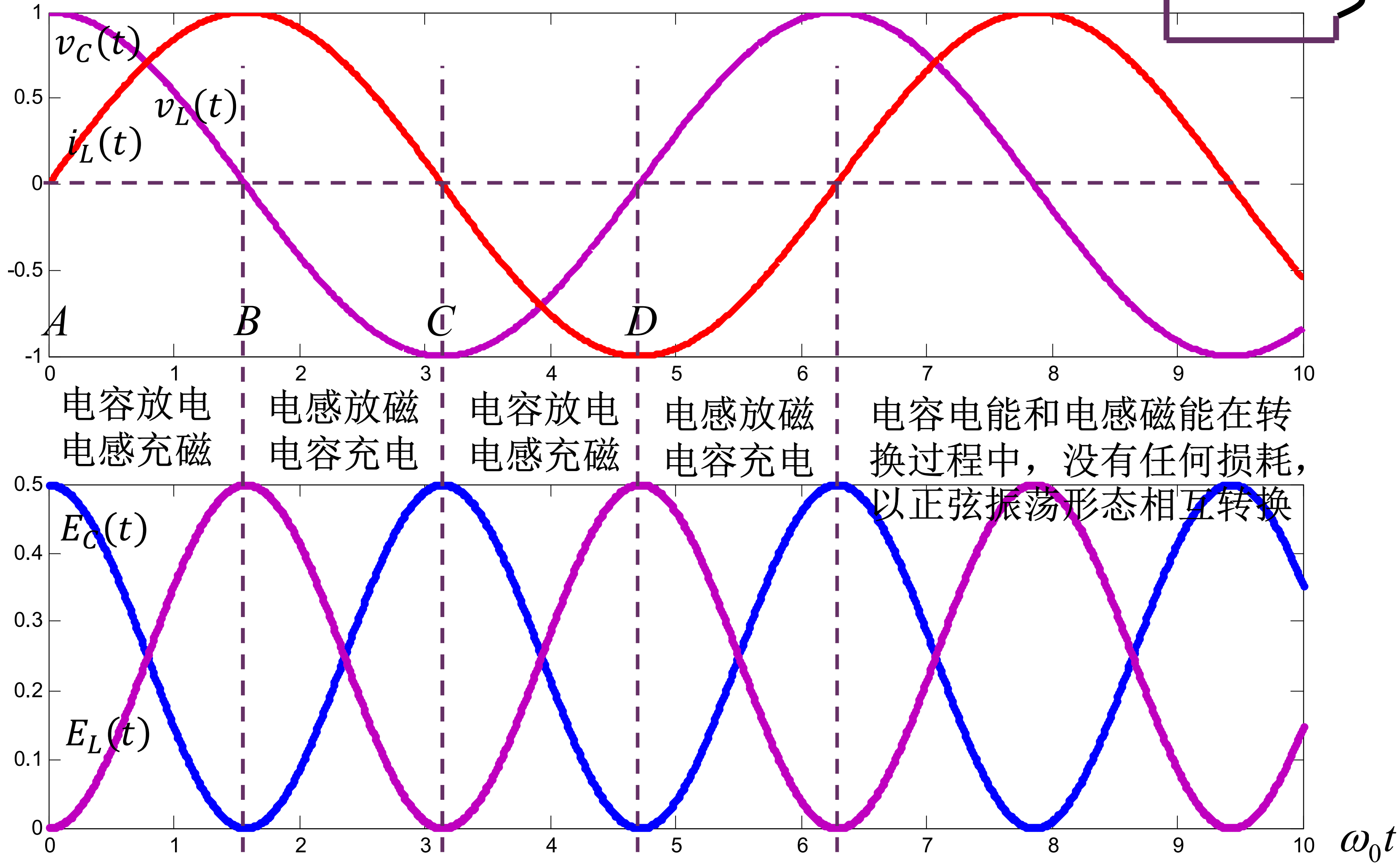
可以验证 $v_C(t) = v_L(t) = L\frac{d}{dt}i_L(t)$ 满足元件约束方程

$$E(t) = E_C(t) + E_L(t) = \frac{1}{2}Cv_C^2(t) + \frac{1}{2}Li_L^2(t) = \frac{1}{2}CV_0^2 + \frac{1}{2}LI_0^2 = E(0)$$

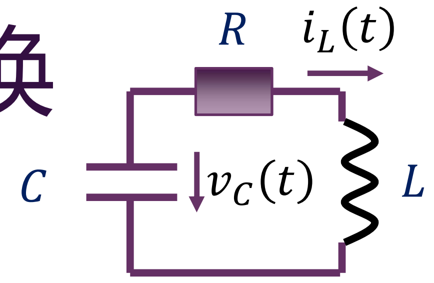
无阻尼自由谐振时，电容电能和电感磁能以正弦规律无损失相互转换，
谐振腔内总储能始终不变（振荡幅度不变，振荡幅度由初始储能决定）

无阻尼自由振荡波形与能量转换

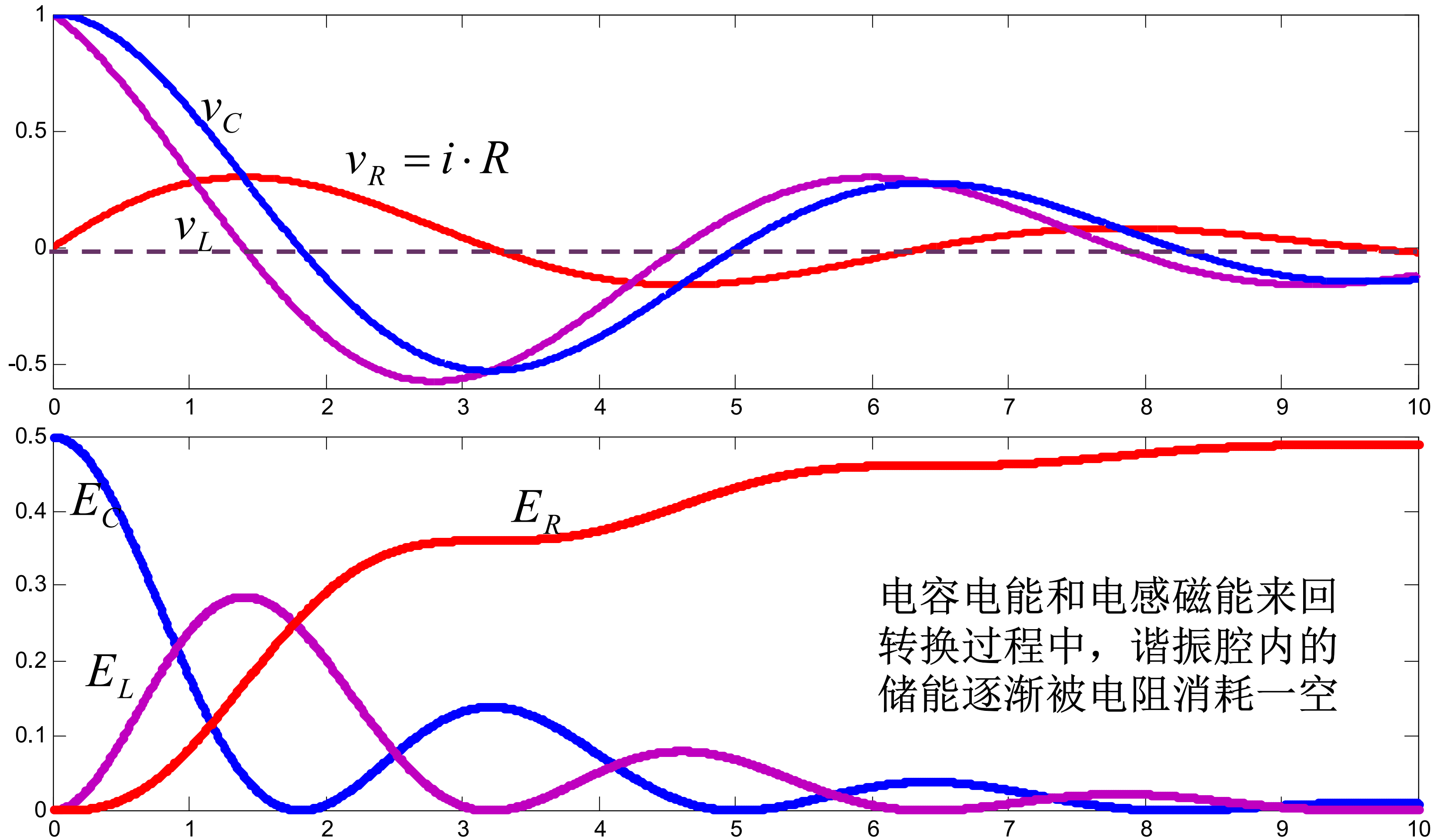




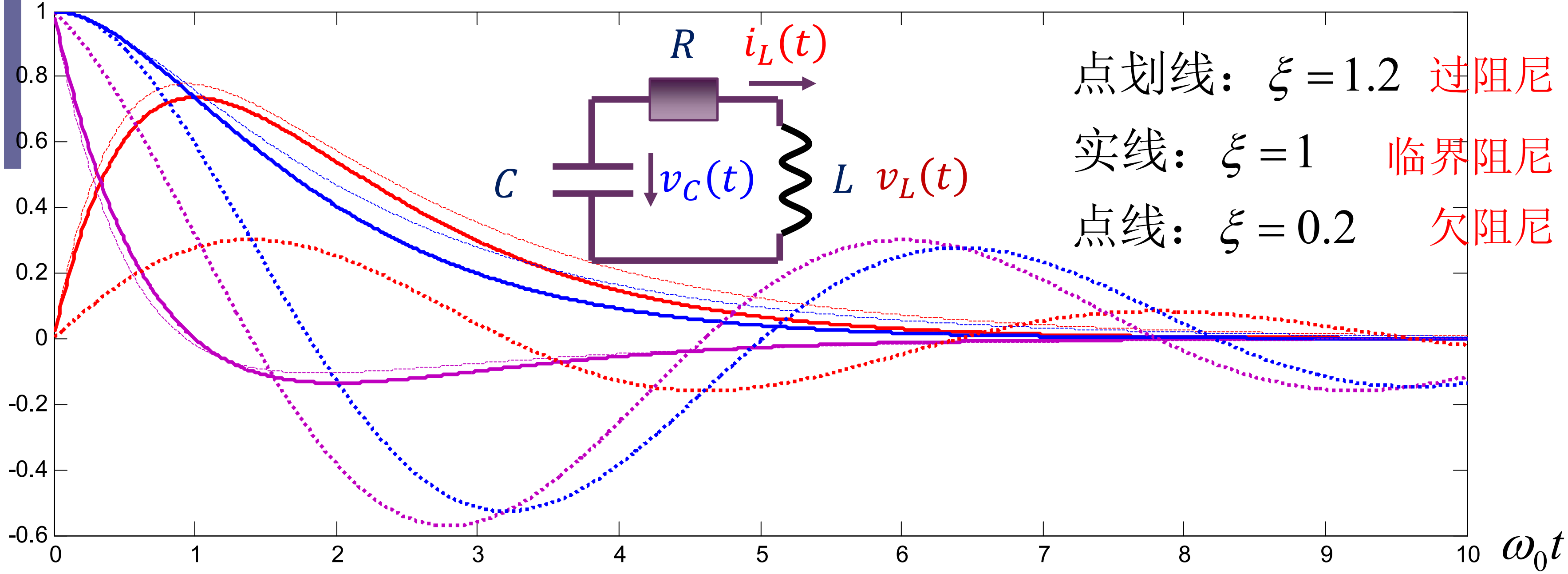
欠阻尼自由振荡波形与能量转换

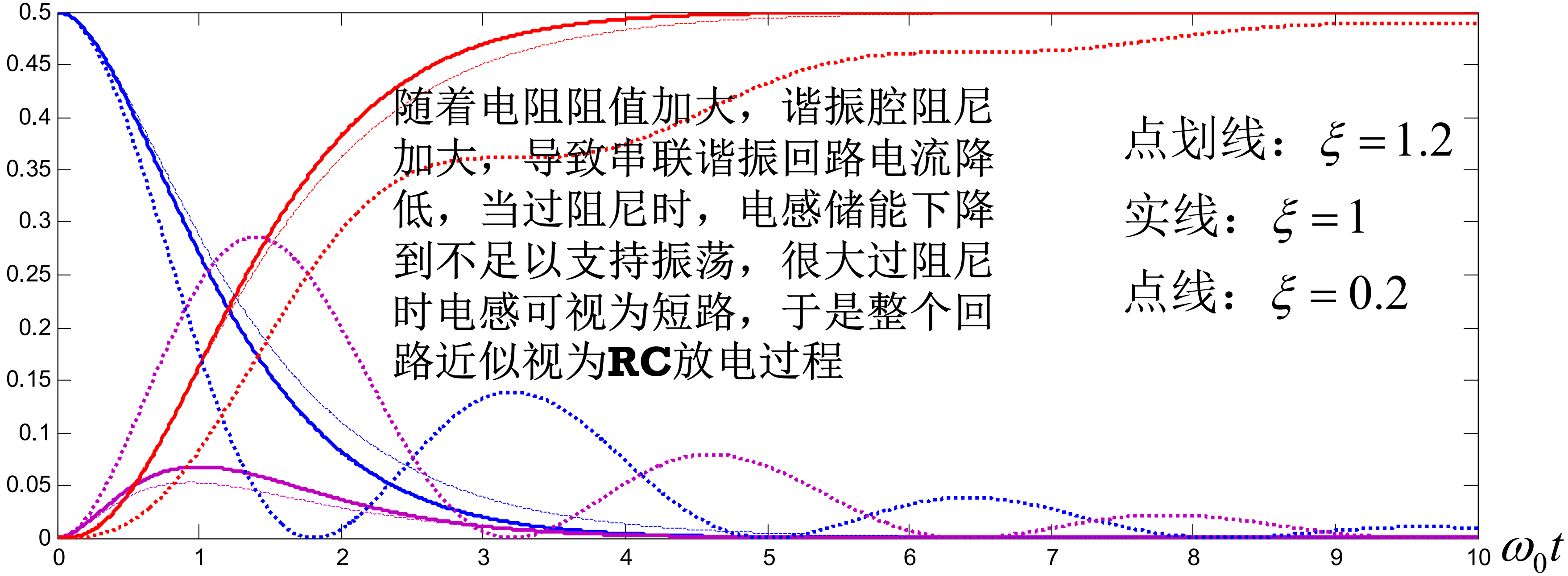


$$\xi = \frac{R}{2Z_0} = 0.2$$



电容电能和电感磁能来回转换过程中，谐振腔内的储能逐渐被电阻消耗一空





随着电阻阻值加大，谐振腔阻尼加大，导致串联谐振回路电流降低，当过阻尼时，电感储能下降到不足以支持振荡，很大过阻尼时电感可视为短路，于是整个回路近似视为**RC**放电过程

本节内容小结

- 描述一阶LTI系统的关键参量是时间常数$\boldsymbol{\tau}$
  - 一阶系统特征根$\boldsymbol{\lambda} = -\frac{1}{\tau}$：指数衰减特性
  - 一阶RC：$\boldsymbol{\tau = RC}$
  - 一阶RL：$\boldsymbol{\tau = GL}$

- 描述二阶LTI系统的关键参量是阻尼系数$\boldsymbol{\xi}$和自由振荡频率$\boldsymbol{\omega_0}$
  - 二阶系统特征根$\boldsymbol{\lambda_{1,2}} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_0$
    - $\xi > 1$：过阻尼，两个不等负实根：指数衰减特性
    - $\xi = 1$：临界阻尼，两个负实等根：指数衰减特性（偏）
    - $0 < \xi < 1$：欠阻尼，两个共轭复根：幅度指数衰减的正弦振荡特性
    - $\xi = 0$：无阻尼，两个共轭纯虚根：等幅正弦振荡特性
  - 串联RLC：$\omega_0 = \frac{1}{\sqrt{LC}}, \xi = \frac{R}{2Z_0}, Z_0 = \omega_0 L = \frac{1}{\omega_0 C} = \sqrt{\frac{L}{C}}$
  - 并联RLC：$\omega_0 = \frac{1}{\sqrt{LC}}, \xi = \frac{G}{2Y_0}, Y_0 = \omega_0 C = \frac{1}{\omega_0 L} = \sqrt{\frac{C}{L}}$

- n阶LTI系统的时域分析需要2n+1个要素
  - 1个稳态响应
  - n个初值
  - n个特征根

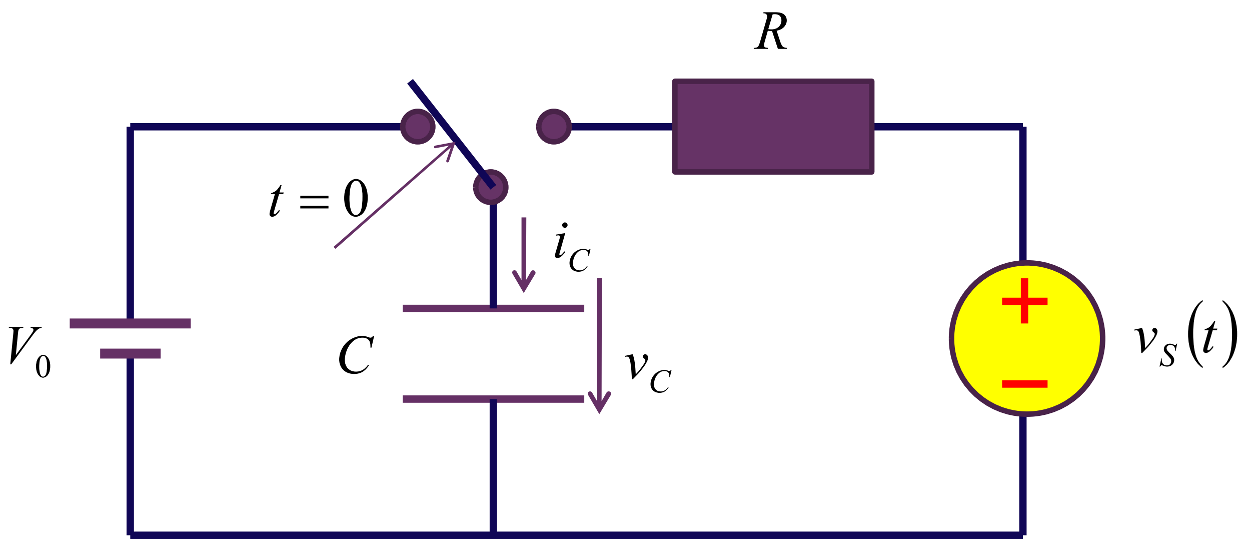
作业讲解：有关稳态响应

- 三要素法和五要素法都有求稳态响应的需求
- 稳态响应指的是激励源在$t=-\infty$时就将激励源加载到系统输入端导致的系统响应
  - （1）冲激激励，阶跃激励，直流激励：等待足够长时间，均认为是直流电路：电容开路，电感短路可获得直流激励下的稳态解
  - （2）正弦波激励：相量法（电容C用jωC导纳替代，电感L用jωL阻抗替代）可获得正弦激励下的稳态解
  - （3）方波激励：分时段阶跃激励
  - （4）其他激励：稳态响应形式应当和激励形态相类似，猜测稳态响应解的形态，代入到原始方程中验证，确定猜测形态中的系数

一阶系统的稳态响应不妨用理论表达式验证一下

$$x_{\infty}(t)=\int_{-\infty}^{t} e^{\frac{\lambda-t}{\tau}} s(\lambda) d\frac{\lambda}{\tau}$$

作业6.7 稳态响应



$$v_S(t) = \frac{V_{S0}}{\tau_S} t$$

假设激励源是一个斜升信号，请用三要素法给出电容电压表达式。

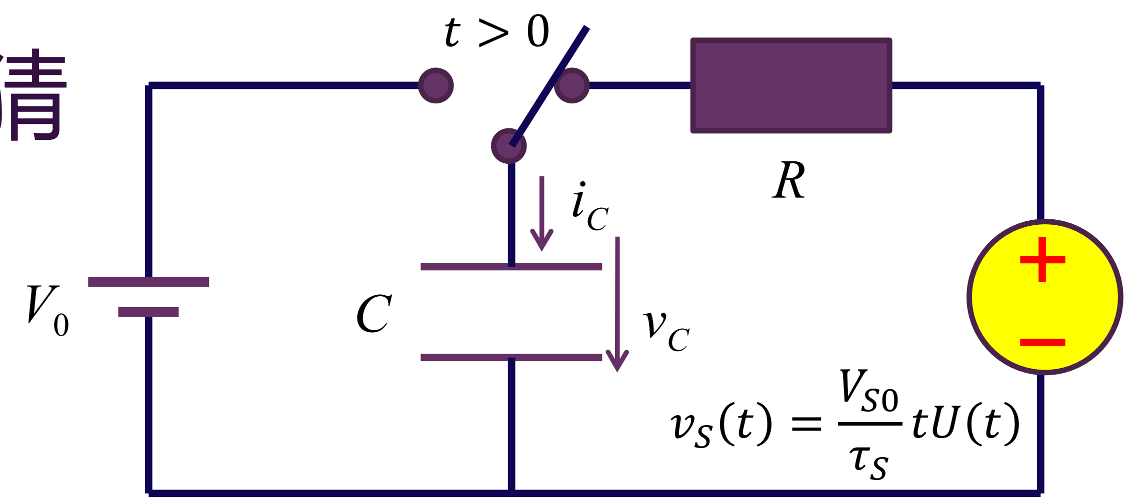
稳态响应可以猜

三要素

$$\tau = RC$$

$$v_C(0^+) = V_0$$

$$v_{C\infty}(t) = ?$$



稳态响应可以猜：稳态响应和激励具有相同的形态，同时电容充放电有滞后效应，猜测为

$$v_{C\infty}(t) = \alpha(t - \tau_0)$$

稳态响应必然满足电路方程，因此可通过代入电路方程确认两个待定参量

$$v_S(t) = v_R(t) + v_C(t) = RC\frac{d}{dt}v_c(t) + v_c(t)$$

$$\frac{V_{S0}}{\tau_S}t = v_{S\infty}(t) = RC\frac{d}{dt}v_{c\infty}(t) + v_{c\infty}(t) = RC\alpha + \alpha t - \alpha\tau_0$$

$$\alpha = \frac{V_{S0}}{\tau_S} \qquad \tau_0 = RC = \tau \qquad v_{C\infty}(t) = \frac{V_{S0}}{\tau_S}(t - \tau)$$

故而电容电压为

$$v_C(t) = \frac{V_{S0}}{\tau_S}(t - \tau) + \left(V_0 + \frac{\tau}{\tau_S}V_{S0}\right)e^{-\frac{t}{\tau}} \qquad (t > 0)$$

稳态响应也可以推衍出

零输入响应都是一样的，不一样的是零状态响应，完全由源决定

$$v_S(t)=V_{S0}U(t) \longrightarrow v_{C,ZSR}(t)=V_{S0}g(t)=V_{S0}\left(1-e^{-\frac{t}{\tau}}\right)U(t)$$

由线性时不变系统特性：激励积分，响应也积分

$$v_S(t)=\frac{V_{S0}}{\tau_S}tU(t)=\frac{1}{\tau_S}\int_{-\infty}^{t}V_{S0}U(\lambda)d\lambda \longrightarrow v_{C,ZSR}(t)=\frac{1}{\tau_S}\int_{-\infty}^{t}V_{S0}g(\lambda)d\lambda$$

$$=\frac{1}{\tau_S}\int_{-\infty}^{t}V_{S0}\left(1-e^{-\frac{\lambda}{\tau}}\right)U(\lambda)d\lambda$$

$$=\frac{V_{S0}}{\tau_S}\int_{0}^{t}\left(1-e^{-\frac{\lambda}{\tau}}\right)d\lambda=\frac{V_{S0}}{\tau_S}\left(\lambda+\tau e^{-\frac{\lambda}{\tau}}\right)\Big|_0^t$$

$$=\frac{V_{S0}}{\tau_S}\left(t+\tau e^{-\frac{t}{\tau}}-\tau\right)$$

$$v_C(t)=v_{C,ZSR}(t)+v_{C,ZIR}(t)=\frac{V_{S0}}{\tau_S}\left(t+\tau e^{-\frac{t}{\tau}}-\tau\right)+V_0e^{-\frac{t}{\tau}}$$

$$=\frac{V_{S0}}{\tau_S}(t-\tau)+\left(\frac{V_{S0}}{\tau_S}\tau+V_0\right)e^{-\frac{t}{\tau}} \qquad (t>0)$$

两个零状态响应的对比

阶跃响应与斜升响应

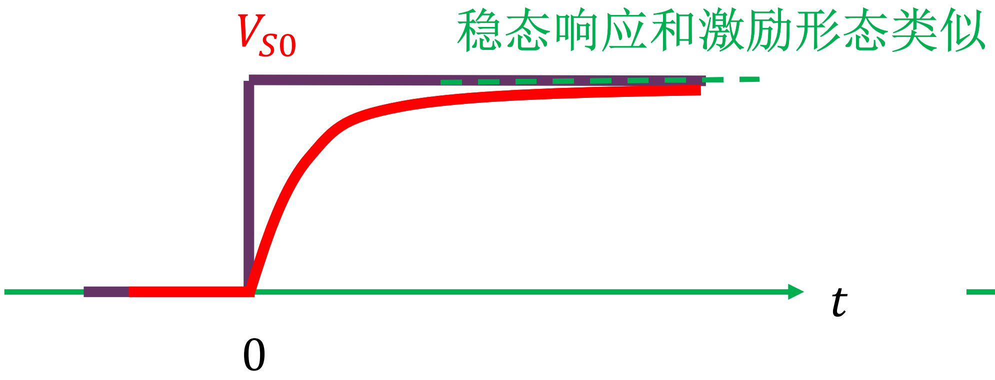
$$v_S(t) = V_{S0}U(t)$$

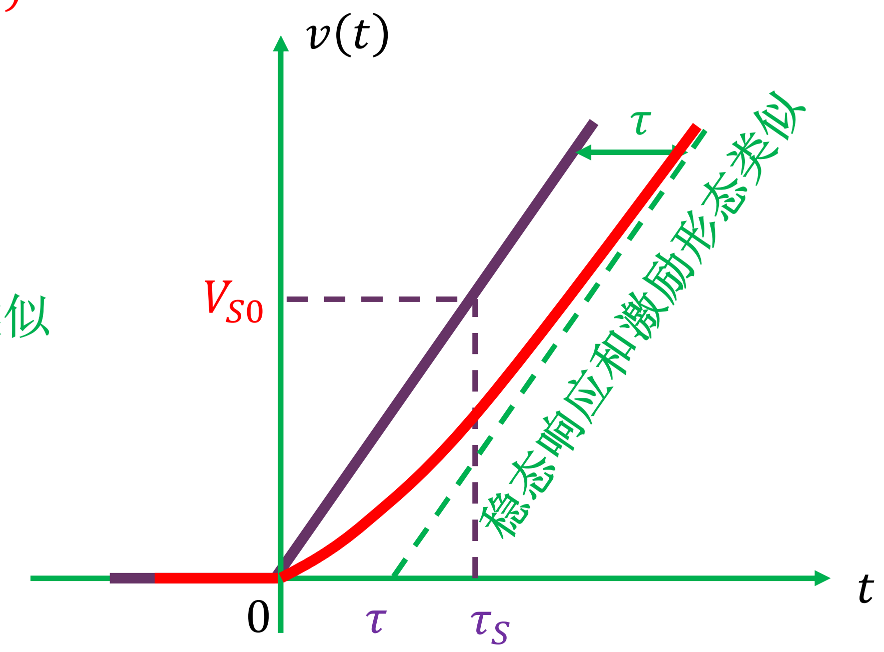
$$v_S(t) = \frac{V_{S0}}{\tau_S}tU(t)$$

$$v_{C,ZSR}(t) = V_{S0}\left(1 - e^{-\frac{t}{\tau}}\right)U(t)$$

$$v_{C,ZSR}(t) = \frac{V_{S0}}{\tau_S}(t-\tau) + \frac{\tau}{\tau_S}V_{S0}e^{-\frac{t}{\tau}}$$

$$(t > 0)$$





不妨代入理论表达式验证一番

$$v_C(t) = V_0 \cdot e^{-\frac{t}{\tau}} + \int_0^t v_S(\lambda) \cdot e^{\frac{\lambda - t}{\tau}} d\frac{\lambda}{\tau} = v_{C\infty}(t) + \left(V_0 - v_{C\infty}(0)\right) \cdot e^{-\frac{t}{\tau}} \qquad (t \geq 0)$$

何谓稳态？瞬态结束即为稳态！
如何结束瞬态？开关启动时间退至-∞!

稳态响应 $v_{C\infty}(t) = \int_{-\infty}^t v_S(\lambda) \cdot e^{\frac{\lambda - t}{\tau}} d\frac{\lambda}{\tau}$

$$v_{C\infty}(t) = \int_{-\infty}^{t} v_S(\lambda) e^{\frac{\lambda-t}{\tau}} d\frac{\lambda}{\tau} = \int_{-\infty}^{t} \frac{V_{S0}}{\tau_S} \lambda e^{\frac{\lambda-t}{\tau}} d\frac{\lambda}{\tau} = \frac{V_{S0}}{\tau_S} e^{-\frac{t}{\tau}} \int_{-\infty}^{t} \lambda e^{\frac{\lambda}{\tau}} d\frac{\lambda}{\tau}$$

$$= \frac{V_{S0}}{\tau_S} e^{-\frac{t}{\tau}} \int_{-\infty}^{t} \lambda d e^{\frac{\lambda}{\tau}} = \frac{V_{S0}}{\tau_S} e^{-\frac{t}{\tau}} \left(\lambda e^{\frac{\lambda}{\tau}} - \int_{-\infty}^{t} e^{\frac{\lambda}{\tau}} d\lambda \right) = \frac{V_{S0}}{\tau_S} e^{-\frac{t}{\tau}} \left(\lambda e^{\frac{\lambda}{\tau}} - \tau e^{\frac{\lambda}{\tau}} \right) \Big|_{-\infty}^{t}$$

$$= \frac{V_{S0}}{\tau_S} e^{-\frac{t}{\tau}} \left(t e^{\frac{t}{\tau}} - \tau e^{\frac{t}{\tau}} \right) = \frac{V_{S0}}{\tau_S} (t - \tau)$$

$$v_C(t) = \frac{V_{S0}}{\tau_S}(t-\tau) + \left(V_0 + \frac{\tau}{\tau_S} V_{S0} \right) e^{-\frac{t}{\tau}} \qquad (t > 0)$$

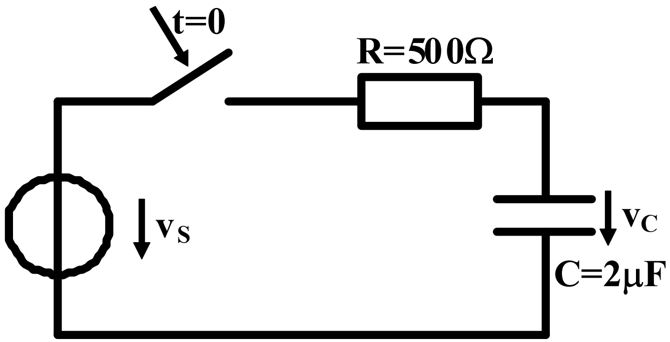
作业7.3 三要素法求解正弦激励

- 如图所示，t=0时刻开关闭合，正弦波电压激励源加载到一阶RC串联电路端口
  - 其中，

$$v_S(t) = 2\cos\omega_0 t$$

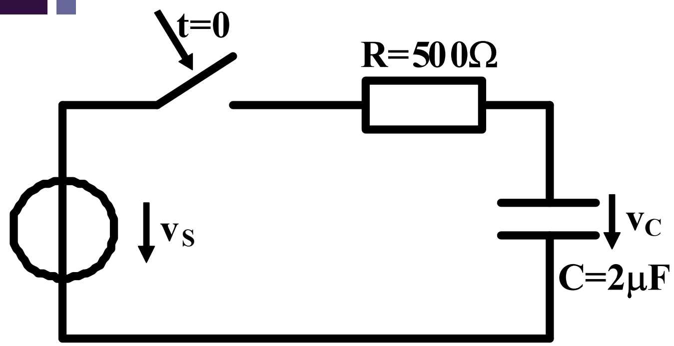
$$\omega_0 = 2\pi f_0 \qquad f_0 = 500Hz$$

- 假设电容初始电压为0，$v_C(0)=0$，请给出电容电压时域表达式



三要素法之稳态响应

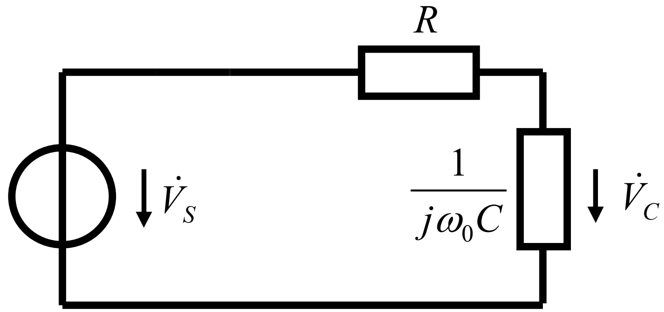
相量法求正弦稳态响应



$$\tau = RC = 1ms$$

$$v_C(0^+) = 0$$

$$v_{c\infty}(t) = ?$$



$v_S(t) = 2\cos\omega_0 t \qquad \omega_0 = 2\pi f_0 \qquad f_0 = 500Hz$

$$\dot{V}_C = \frac{\frac{1}{j\omega_0 C}}{R + \frac{1}{j\omega_0 C}}\dot{V}_S = \frac{1}{1 + j\omega_0 RC}\dot{V}_S$$

$$\dot{V}_C = \frac{1}{1 + j\omega_0 RC}\dot{V}_S = \frac{1}{1 + j \times 2\pi \times f_0 \times R \times C}\dot{V}_S$$

$$= \frac{1}{1 + j \times 2\pi \times 500 \times 500 \times 0.000002} \times 2 = \frac{2}{1 + j3.1416}$$

$$= 0.6066e^{-j72.34°}$$

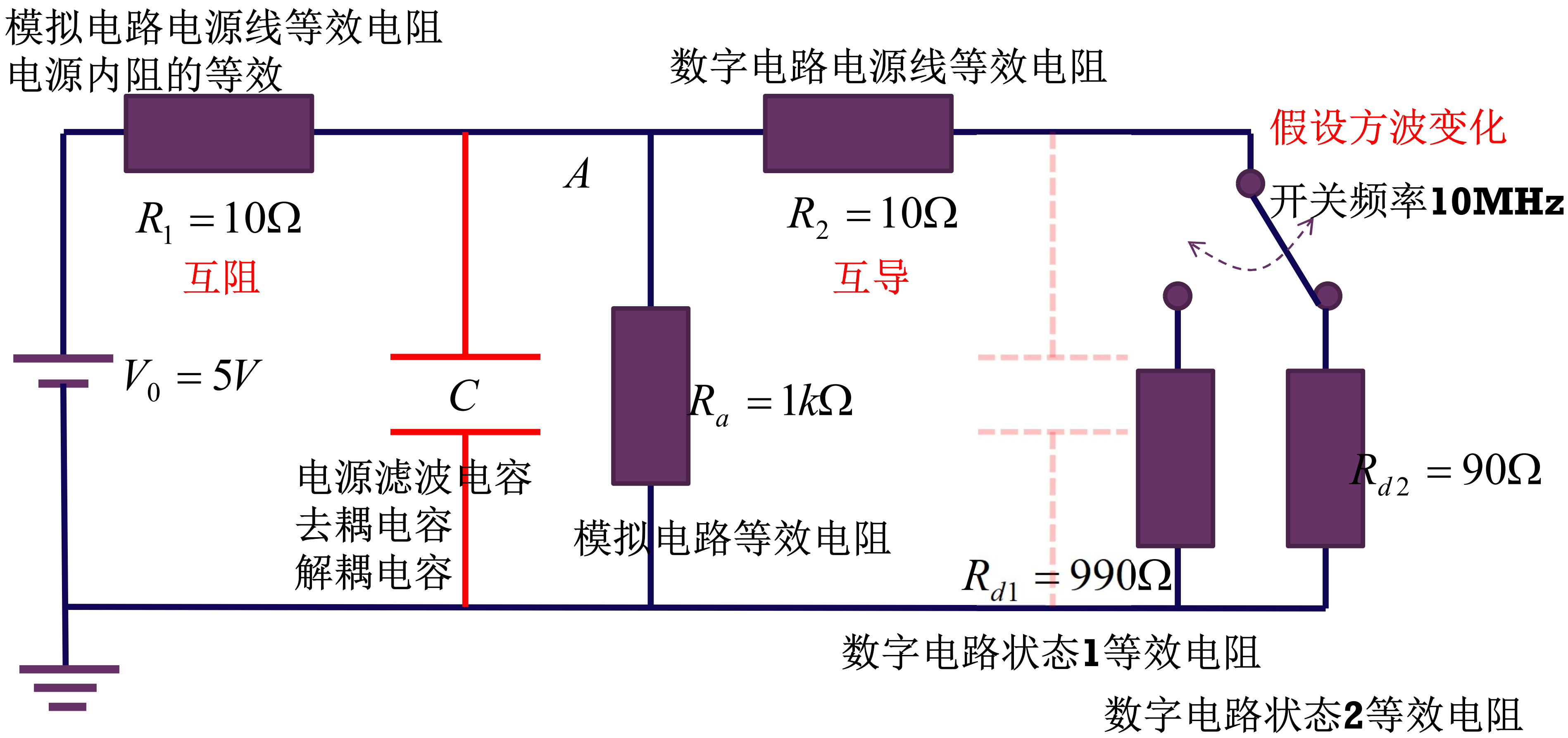
$$v_{c\infty}(t) = 0.6066\cos(\omega_0 t - 72.34°)$$

$$v_C(t) = v_{C\infty}(t) + \left(v_C(0^+) - v_{C\infty}(0^+)\right)e^{-\frac{t}{\tau}} = 0.6066\cos(\omega_0 t - 72.34°) - 0.1840e^{-\frac{t}{\tau}}$$

$$(t > 0)$$

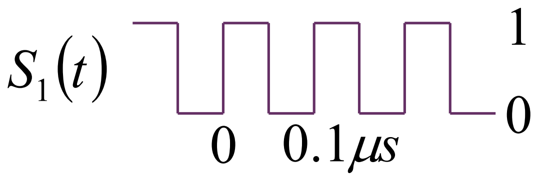
作业7.6 用电容做电源滤波（选作）

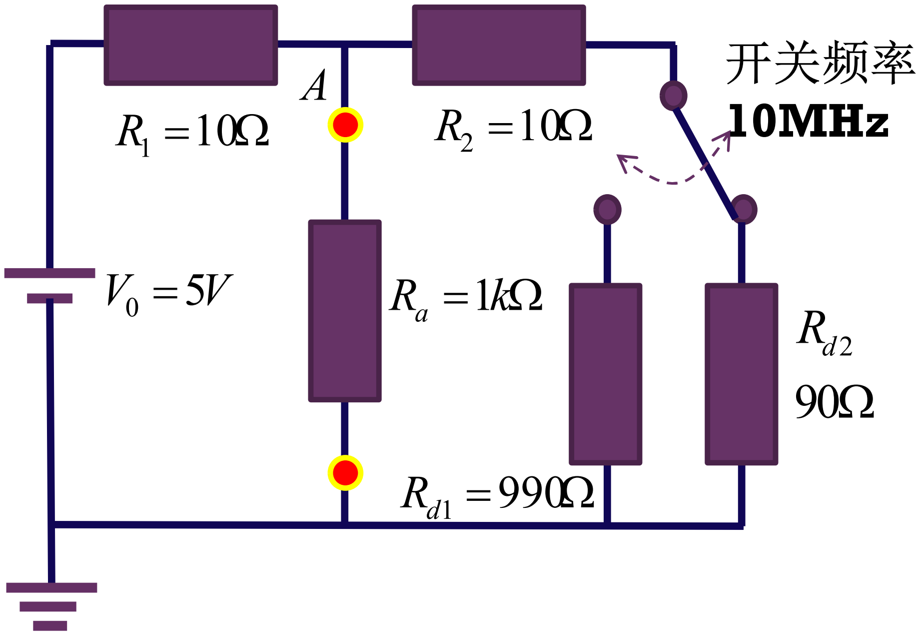
- 1）假设没有滤波电容，求模拟电路电源端A点的电压波形
- 2）多大的电容，可以使得A点电压波形起伏是没有电容时的1/10

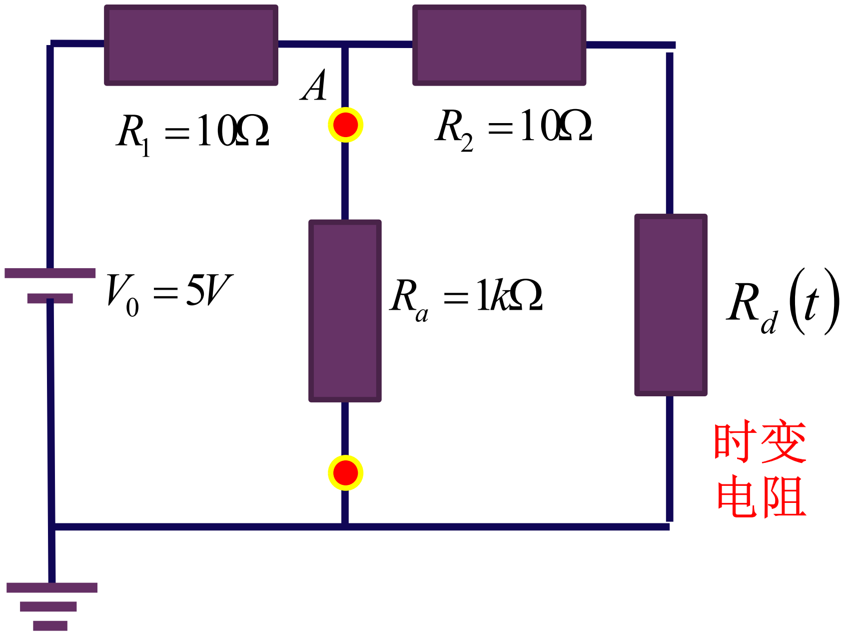


将数字芯片抽象为时变电阻

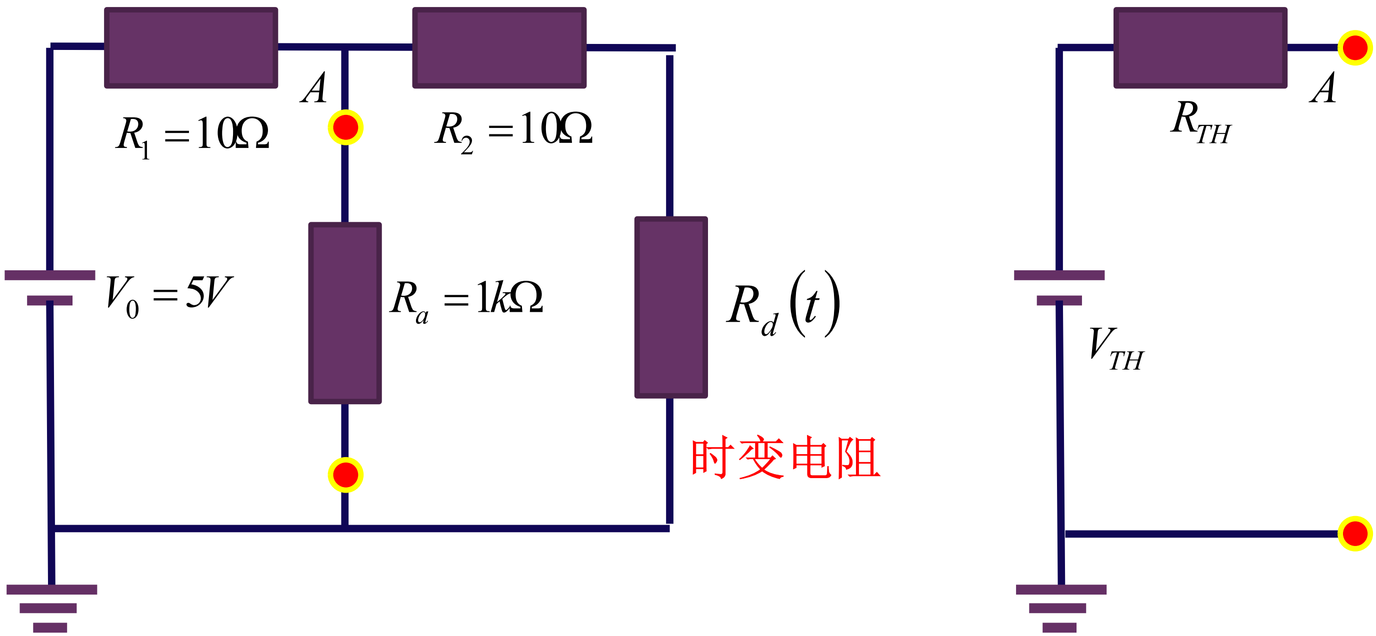
$$R_d(t) = R_{d1}S_1(t) + R_{d2}\left(1 - S_1(t)\right)$$







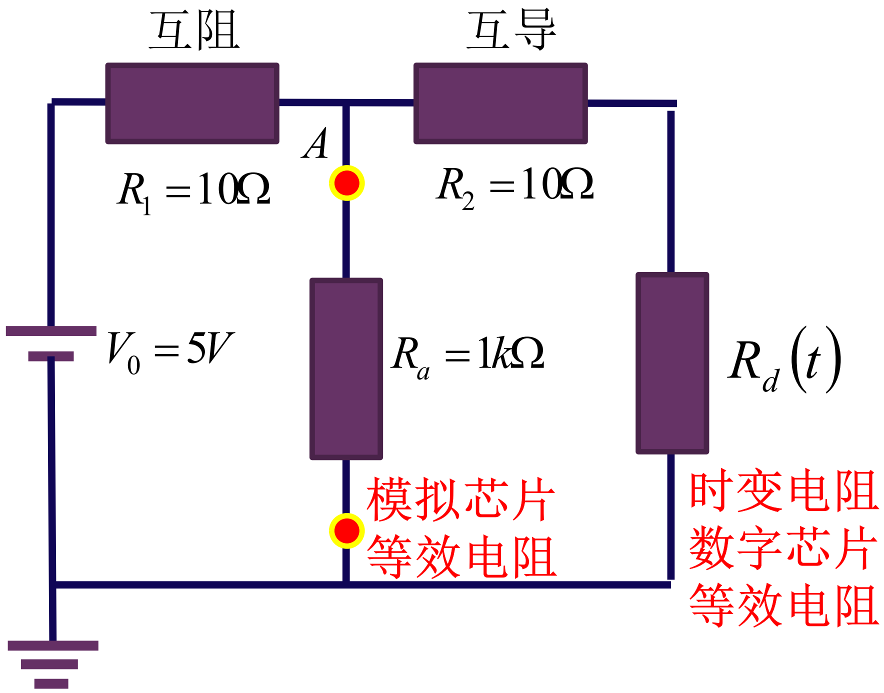
对模拟芯片的电源-地端口做戴维南等效



$$R_{TH} = R_1 \parallel R_a \parallel (R_2 + R_d) = \frac{R_a R_1}{R_1 + R_a + \dfrac{R_a R_1}{R_2 + R_d}}$$

$$V_{TH} = \frac{(R_2 + R_d) \parallel R_a}{R_1 + (R_2 + R_d) \parallel R_a} V_0 = \frac{R_a}{R_1 + R_a + \dfrac{R_1 R_a}{R_2 + R_d}} V_0$$

互阻和互导形成芯片间的串扰



$$R_{TH}=R_1\parallel R_a\parallel(R_2+R_d)=\frac{R_aR_1}{R_1+R_a+\frac{R_aR_1}{R_2+R_d}}$$

$$V_{TH}=\frac{(R_2+R_d)\parallel R_a}{R_1+(R_2+R_d)\parallel R_a}V_0=\frac{R_a}{R_1+R_a+\frac{R_1R_a}{R_2+R_d}}V_0$$

$$R_{TH}\overset{R_1=0}{=}0$$

$$V_{TH}\overset{R_1=0}{=}V_0$$

如果没有互阻（互阻**R_1=0**），**AG**端口等效为恒压源，数字芯片的变化（时变电阻**R_d**）无法影响模拟芯片电压

互阻**R_1**和互导**G_2**导致数字芯片和模拟芯片相互耦合，数字芯片的电流变化导致模拟芯片电源电压波动：不开避免的串扰

（1）电源内阻不为0

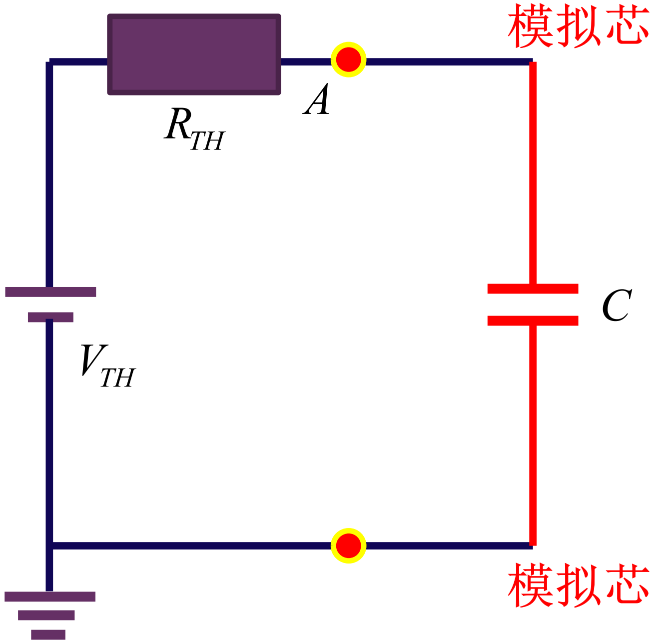
（2）无法为每个芯片单独供电

$$R_{TH}\overset{R_2=\infty}{=}\frac{R_aR_1}{R_1+R_a}$$

$$V_{TH}\overset{R_2=\infty}{=}\frac{R_a}{R_1+R_a}V_0$$

如果没有互导（**G_2=0**，分别单独提供供电电源），**AG**端口等效源电压和源内阻均不受数字芯片（时变电阻**R_d**）的影响，即数字芯片的电流变化无法影响模拟芯片的电源电压

滤波电容　降低波动
去耦电容　解除耦合



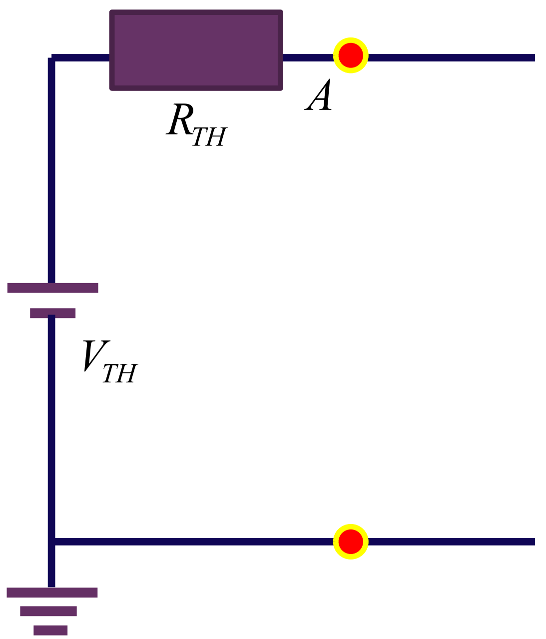
电容**C**具有电压保持功能，具有求平均功能，只要电容足够大，$\mathbf{V_{TH}}$的变化就会被电容抹平：电源滤波，芯片解耦

$$R_{TH}(t)=\frac{R_aR_1}{R_1+R_a+\dfrac{R_aR_1}{R_2+R_d(t)}}$$

$$V_{TH}(t)=\frac{R_a}{R_1+R_a+\dfrac{R_1R_a}{R_2+R_d(t)}}V_0$$

数字芯片的电流变化用时变电阻$\mathbf{R_d(t)}$抽象，$\mathbf{R_d(t)}$对模拟芯片的影响通过互阻、互导实现

末加去耦电容时的串扰情况



$$R_{TH}(t)=\frac{R_aR_1}{R_1+R_a+\frac{R_aR_1}{R_2+R_d(t)}}\qquad V_{TH}(t)=\frac{R_a}{R_1+R_a+\frac{R_1R_a}{R_2+R_d(t)}}V_0$$

$$R_d(t)=R_{d1}S_1(t)+R_{d2}(1-S_1(t))$$

$$V_{TH,1}(t)=\frac{R_a}{R_1+R_a+\frac{R_1R_a}{R_2+R_{d1}}}V_0=\frac{1000}{10+1000+\frac{10\cdot 1000}{10+990}}\times 5=\frac{1000}{10+1000+\underline{\underline{10}}}\times 5=4.902V$$

$$V_{TH,2}(t)=\frac{R_a}{R_1+R_a+\frac{R_1R_a}{R_2+R_{d2}}}V_0=\frac{1000}{10+1000+\frac{10\cdot 1000}{10+90}}\times 5=\frac{1000}{10+1000+\underline{\underline{100}}}\times 5=4.505V$$

$R_{TH1}=9.804\Omega$

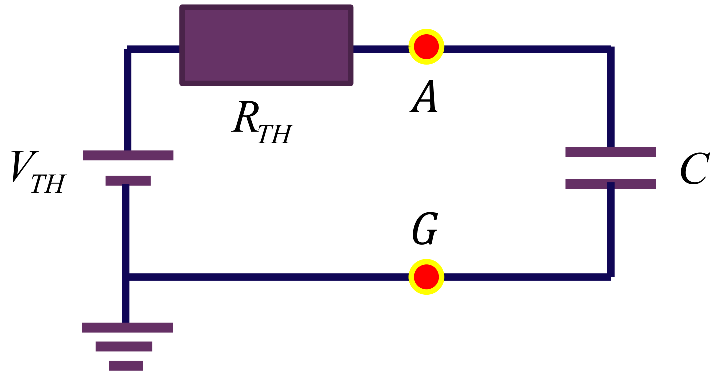
$R_{TH2}=9.010\Omega$

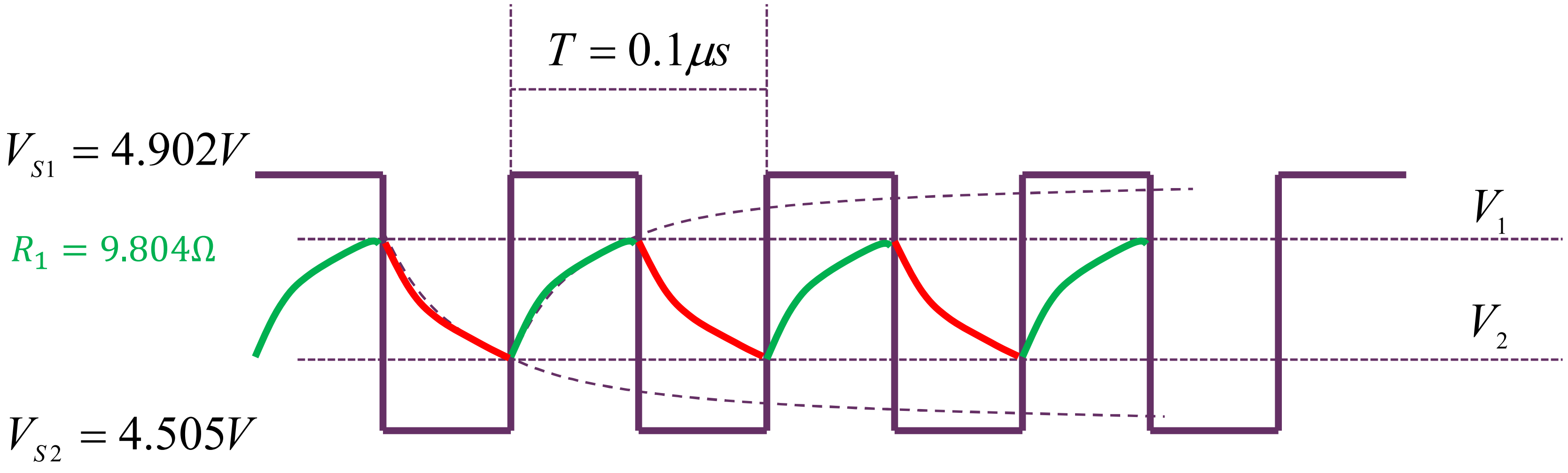
等效内阻同时变化

$$\Delta V_{A0}=4.902-4.505=0.397V$$

电压波动**0.4V**

去耦电容使得串扰幅度降低



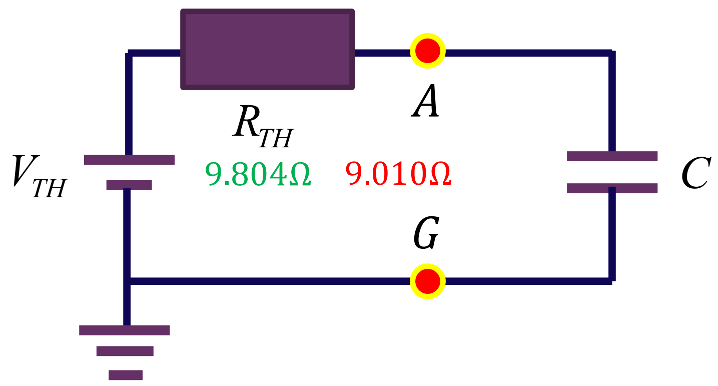


$R_2 = 9.010\Omega$

$\Delta V_{A0} = 0.397V$

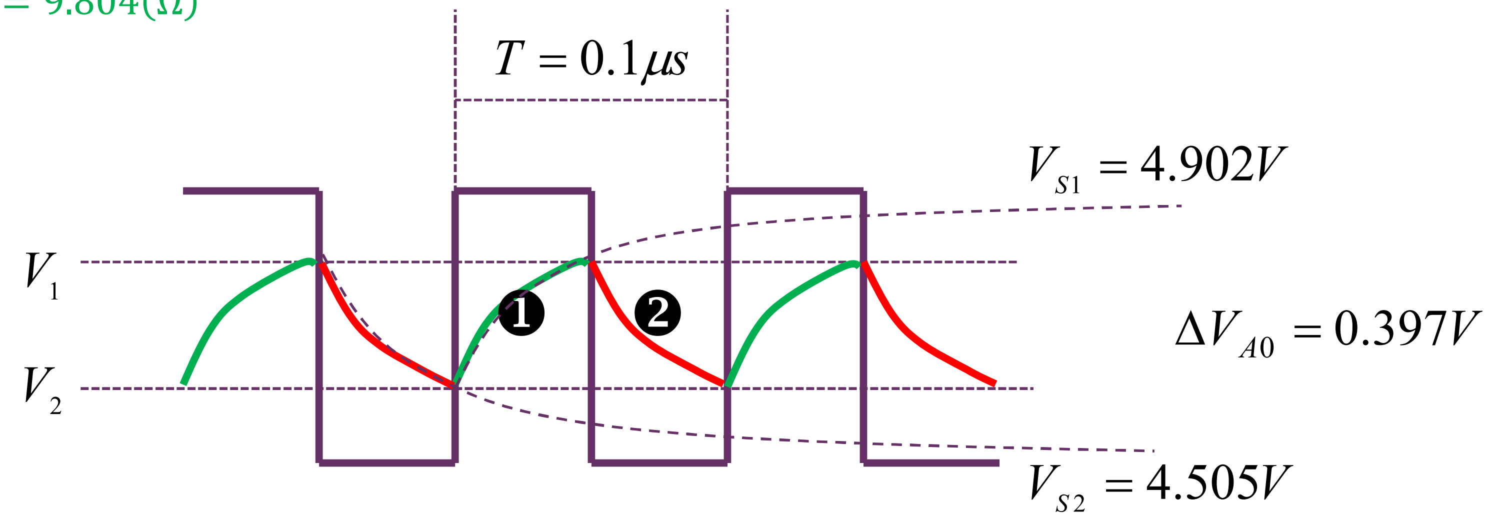
$\Delta V_A = V_1 - V_2 \le 0.1\Delta V_{A0} = 0.0397V$

充放电抑制突变



❶ $v_r(t) = V_{S1} + (V_2 - V_{S1}) e^{-\frac{\Delta t_1}{\tau_1}}$ $\tau_1 = R_1 C$

$R_1 = 9.804(\Omega)$



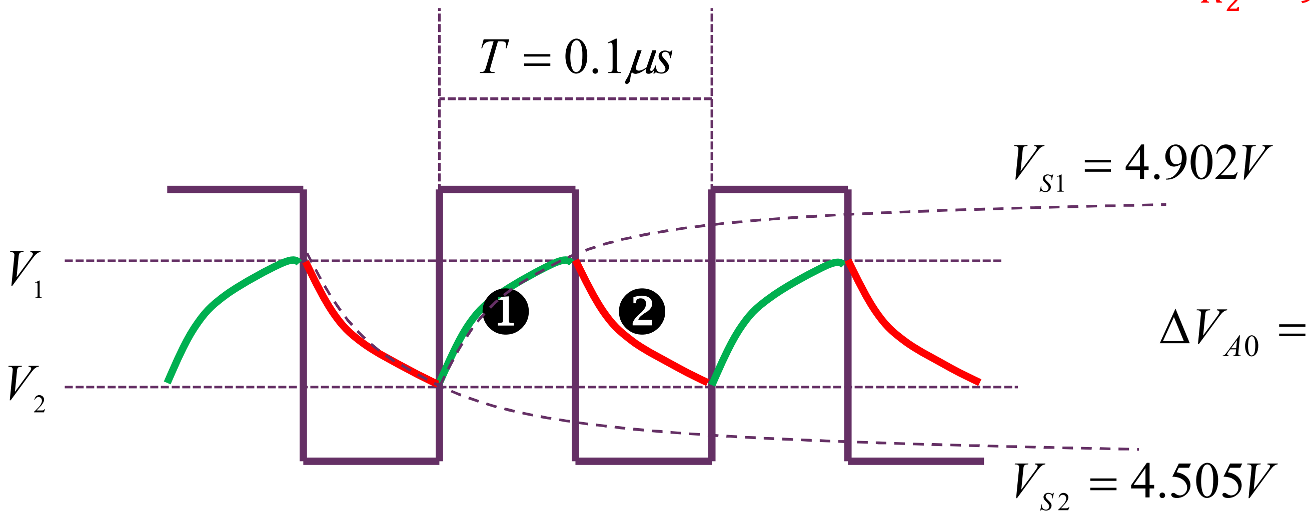
$\Delta V_A = V_1 - V_2 \le 0.1 \Delta V_{A0} = 0.0397V$

❷ $v_f(t) = V_{S2} + (V_1 - V_{S2}) e^{-\frac{\Delta t_2}{\tau_2}}$ $\tau_2 = R_2 C$ $R_2 = 9.010(\Omega)$

电容使得波动变小了

$R_1 = 9.804(\Omega)$

$R_2 = 9.010(\Omega)$



❶ $$v_r(t) = V_{S1} + (V_2 - V_{S1})e^{-\frac{\Delta t_1}{\tau_1}}$$

$$V_1 = V_{S1} + (V_2 - V_{S1})e^{-\frac{0.5T}{\tau_1}}$$

❷ $$v_f(t) = V_{S2} + (V_1 - V_{S2})e^{-\frac{\Delta t_2}{\tau_2}}$$

$$V_2 = V_{S2} + (V_1 - V_{S2})e^{-\frac{0.5T}{\tau_2}}$$

$$\tau_1 = R_1 C$$

$$\tau_2 = R_2 C$$

$$\Delta V = V_1 - V_2 \le 0.1\Delta V_S$$

$$T = 0.1\mu s$$

$$C = ?$$

电容如何取值达到要求?

$$V_1 = V_{S1} + (V_2 - V_{S1})e^{-\frac{0.5T}{\tau_1}} = V_2 a_1 + V_{S1}(1 - a_1) \qquad a_1 = e^{-\frac{0.5T}{\tau_1}}$$

$$V_2 = V_{S2} + (V_1 - V_{S2})e^{-\frac{0.5T}{\tau_2}} = V_1 a_2 + V_{S2}(1 - a_2) \qquad a_2 = e^{-\frac{0.5T}{\tau_2}}$$

$$V_1 - V_2 a_1 = V_{S1}(1 - a_1) \qquad V_1 = \frac{V_{S1}(1 - a_1) + V_{S2}(1 - a_2)a_1}{1 - a_1 a_2}$$

$$V_2 - V_1 a_2 = V_{S2}(1 - a_2) \qquad V_2 = \frac{V_{S2}(1 - a_2) + V_{S1}(1 - a_1)a_2}{1 - a_1 a_2}$$

$$\Delta V_A = V_1 - V_2 = (V_{S1} - V_{S2})\frac{(1 - a_1)(1 - a_2)}{1 - a_1 a_2} \le 0.1(V_{S1} - V_{S2})$$

$$\frac{(1 - a_1)(1 - a_2)}{1 - a_1 a_2} \le 0.1$$

需要求解非线性方程获得电容**C**大小

$$a_1 = e^{-\frac{0.5T}{\tau_1}} \qquad a_2 = e^{-\frac{0.5T}{\tau_2}} \qquad \tau_1 = R_1 C \qquad \tau_2 = R_2 C \qquad T = 0.1\mu s$$

保留主项获得足够精确的近似解

$$\frac{(1-a_1)(1-a_2)}{1-a_1a_2} \le 0.1 \qquad a_1 = e^{-\frac{0.5T}{R_1C}} \qquad a_2 = e^{-\frac{0.5T}{R_2C}}$$

假设**C**足够大，时间常数足够大，充放电时间足够长，可以做如下估算：

$$a_1 = e^{-\frac{0.5T}{R_1C}} \approx 1-\frac{0.5T}{R_1C} \qquad a_2 = e^{-\frac{0.5T}{R_2C}} \approx 1-\frac{0.5T}{R_2C}$$

$$0.1 \ge \frac{(1-a_1)(1-a_2)}{1-a_1a_2} \approx \frac{\frac{0.5T}{\tau_1}\frac{0.5T}{\tau_2}}{\frac{0.5T}{\tau_1}+\frac{0.5T}{\tau_2}-\frac{0.5T}{\tau_1}\frac{0.5T}{\tau_2}} \approx \frac{\frac{0.5T}{\tau_1}\frac{0.5T}{\tau_2}}{\frac{0.5T}{\tau_1}+\frac{0.5T}{\tau_2}} = \frac{0.5T}{\tau_1+\tau_2} = \frac{0.5T}{C(R_1+R_2)}$$

$$C \ge \frac{0.5T}{0.1(R_1+R_2)} = \frac{5T}{R_1+R_2} = \frac{5\times 0.1\mu s}{9.8+9.0} = 0.0266\mu F$$

验算设计正确性

$$C \ge \frac{0.5T}{0.1(R_1+R_2)} = \frac{5T}{R_1+R_2} = \frac{5\times 0.1\mu s}{9.8+9.0} = 0.0266\mu F \quad \text{取} \quad C = 0.03\mu F$$

验证正确性 $\tau_1 = R_1 C = 9.8\times 0.03\mu F = 0.294\mu s$ $\tau_2 = R_2 C = 9.0\times 0.03\mu F = 0.270\mu s$

$$a_1 = e^{-\frac{0.5T}{\tau_1}} = e^{-\frac{0.05}{0.294}} = e^{-0.1701} = 0.847 \qquad a_2 = e^{-\frac{0.5T}{\tau_2}} = e^{-\frac{0.05}{0.270}} = e^{-0.1852} = 0.831$$

$$V_1 = \frac{V_{S1}(1-a_1)+V_{S2}(1-a_2)a_1}{1-a_1a_2} = \frac{4.902\times 0.153+4.505\times 0.169\times 0.847}{0.296} = 4.712V$$

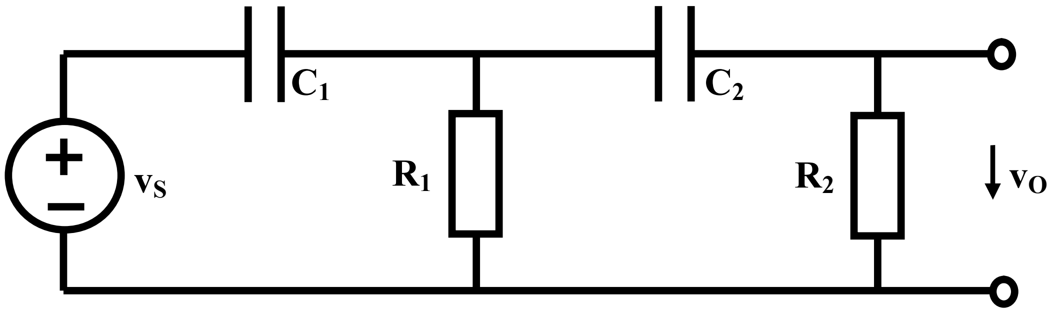
$$V_2 = \frac{V_{S2}(1-a_2)+V_{S1}(1-a_1)a_2}{1-a_1a_2} = \frac{4.505\times 0.169+4.902\times 0.153\times 0.831}{0.296} = 4.678V$$

$$\Delta V_A = V_1 - V_2 = 4.712-4.678 = 0.034V \qquad \Delta V_{A0} = 4.902-4.505 = 0.397V$$

确实满足 $\Delta V_A \le 0.1 V_{A0}$

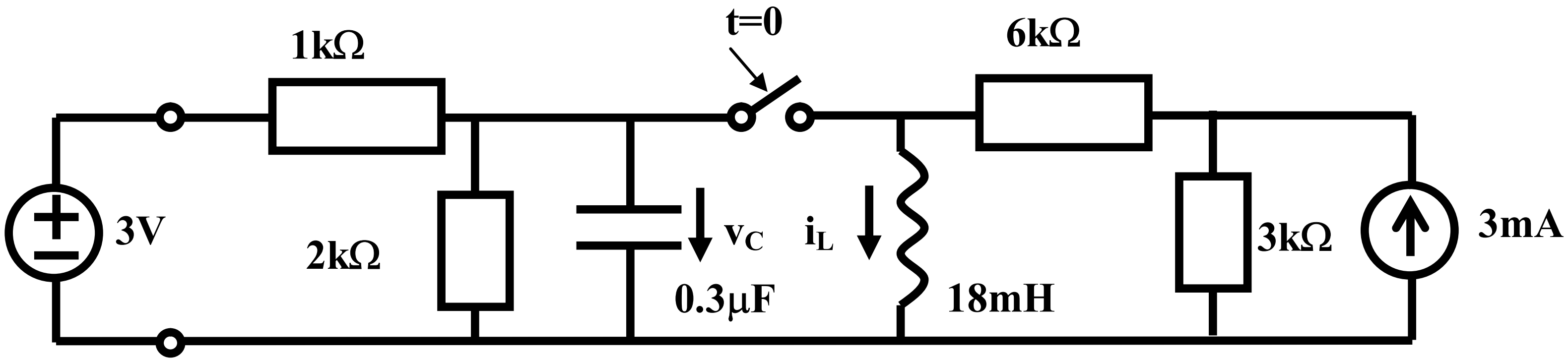
C=0.3μF，ΔV<0.01ΔV$_S$： 电容越大，滤波效果（去耦效果）越好

作业1 二阶RC高通滤波器



- 1、列写电路状态方程
- 2、列写以v_O为未知量的二阶微分方程
- 3、列写频域传递函数
- 4、从微分方程（或频域传递函数）说明关键参量：ξ，ω_0
- 5、假设两个电容初始电压均为0，激励源为阶跃信号源$v_S(t)=V_0U(t)$，用五要素法获得输出电压表达式（考察$R_1=R_2=R$，$C_1=C_2=C$的特殊情况）

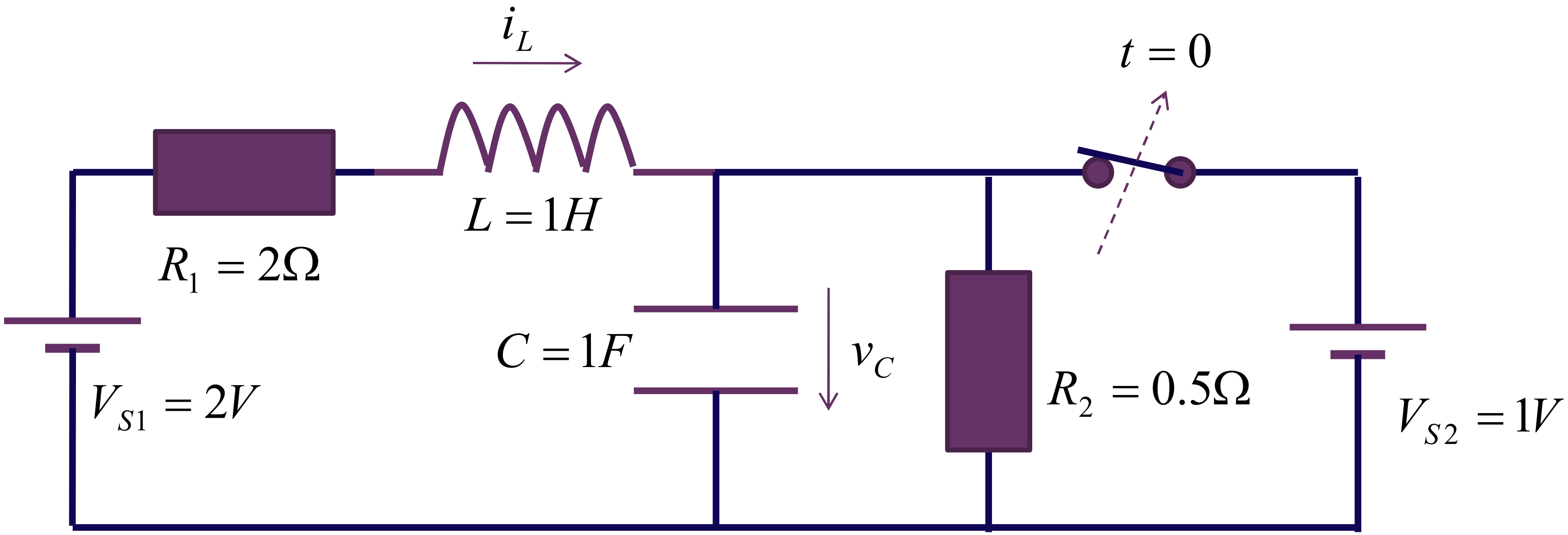
作业2 分析与验证



- 开关在t=0时刻闭合。开关闭合前电路已经稳定。求开关闭合后，电容电压$v_C(t)$和电感电流$i_L(t)$的变化规律
  - 课件已给电容电压$v_C(t)$的变化规律，求$i_L(t)$的变化规律
  - 验证

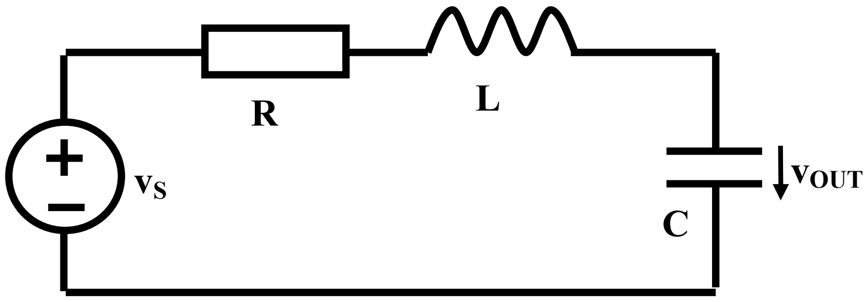
$$v_C(t)=v_L(t)=L\frac{di_L(t)}{dt} \qquad (t\geq 0)$$

作业3 非简单串并联电路
需要首先确定系统参量



- 1/用五要素法或待定系数法获得电容电压和电感电流
- 2/选作：利用状态方程求解，求A矩阵特征根，用待定系数法求解

CAD作业



- 研究该电路
  - 频域传递函数幅频特性和相频特性
  - 时域阶跃响应时域波形
- 参量
  - 阻尼系数=0.01,0.1,0.5,0.707,0.866,1,2,10,50,100

本节课内容在教材中的章节对应

- P756-780: 二阶动态系统时域分析，五要素法