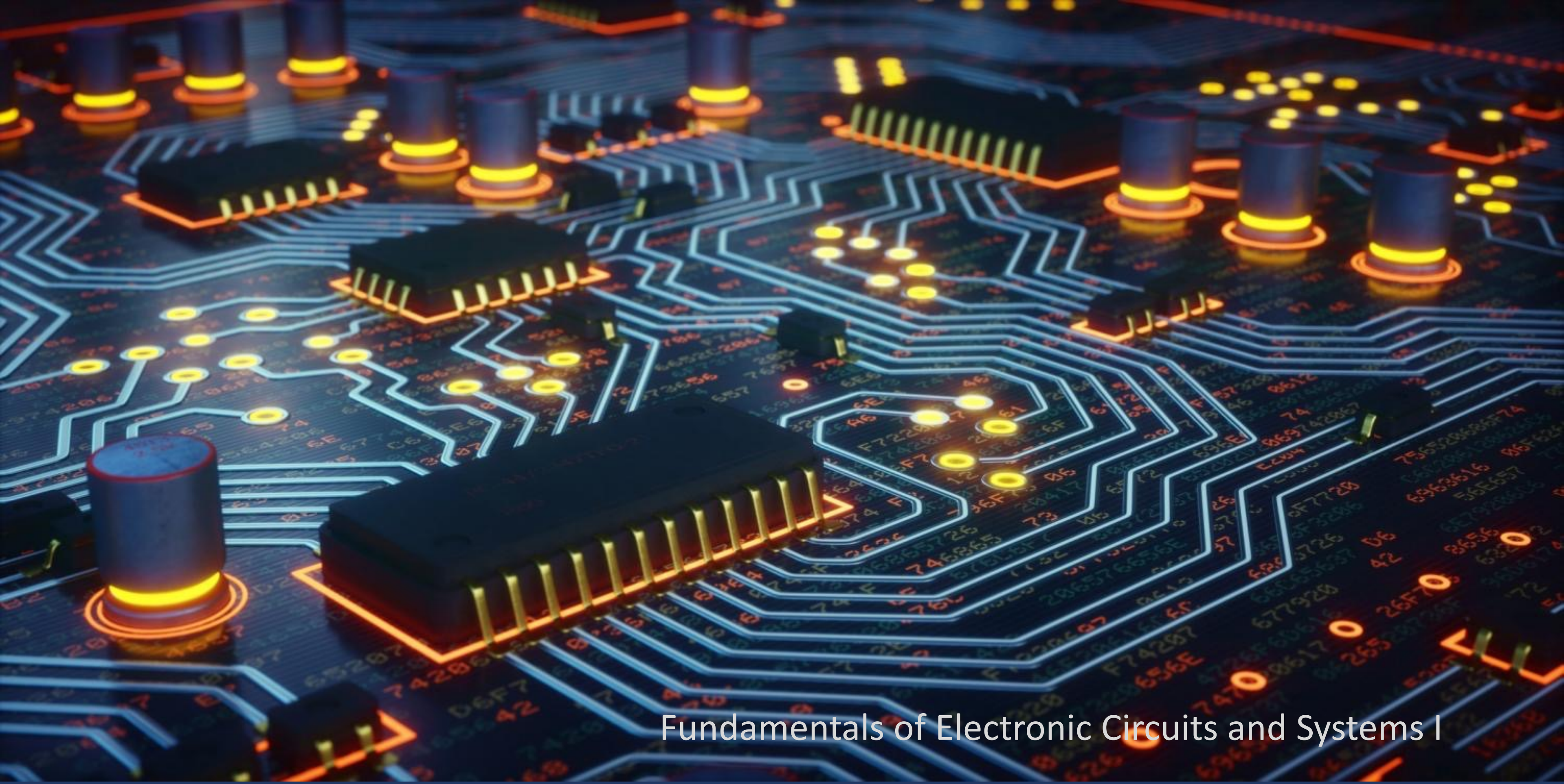

Fundamentals of Electronic Circuits and Systems I

Circuit Analysis in Frequency Domain

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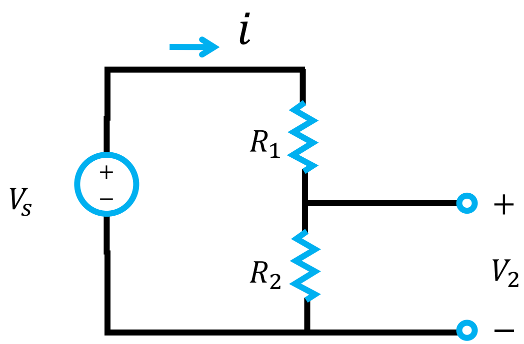
Dept of EE, Tsinghua University

Outlines

- Transfer function
- Filters
- Bode plot
- Circuit element models in s-domain

Recall: voltage divider

VOLTAGE DIVIDER



According to KVL $V_s = iR_1 + iR_2$

➔ $i = \frac{V_s}{R_1+R_2}$

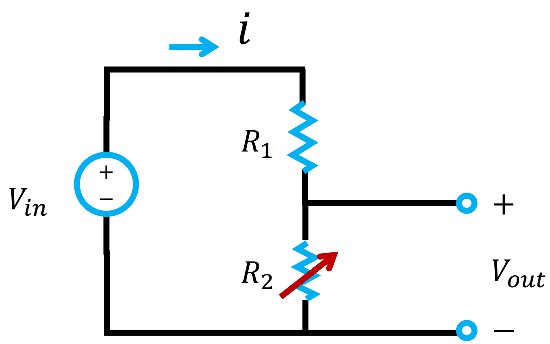
$$V_2 = iR_2 = \frac{R_2}{R_1 + R_2} V_s$$

Voltage divided over resistors

Transfer function

QUESTION: calculate the voltage transfers from input to output based on varying R_2

VOLTAGE DIVIDER



- According to KVL

$$V_{out} = iR_2 = \frac{R_2}{R_1 + R_2} V_{in}$$

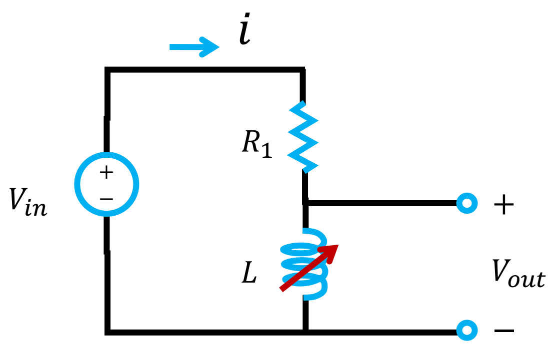
- Ratio between input & output voltages

$$G(R_2) = \frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2}$$

Transfer function

QUESTION: calculate the voltage transfers from input to output based on varying L

VOLTAGE DIVIDER



- According to KVL

$$\mathbb{V}_{out} = iZ_L = \frac{j\omega L}{R_1 + j\omega L}\mathbb{V}_{in}$$

- The transfer function

$$\mathbb{G}(\omega) = \frac{\mathbb{V}_{out}}{\mathbb{V}_{in}} = \frac{j\omega L}{R_1 + j\omega L} = \frac{1}{\frac{R_1}{j\omega L} + 1}$$

This is a "frequency-dependent" variable voltage divider.

Transfer is dependent on the value of ω

Transfer function

TRANSFER FUNCTION of a circuit or system describes the output response to an input excitation as a function of the angular frequency ω

$$\mathbb{G}(\omega) = \frac{\mathbb{V}_{out}(\omega)}{\mathbb{V}_{in}(\omega)}$$

← **Voltage gain**

$$= M(\omega)e^{j\phi(\omega)}$$

where

$$M(\omega) = |\mathbb{G}(\omega)|$$

Magnitude

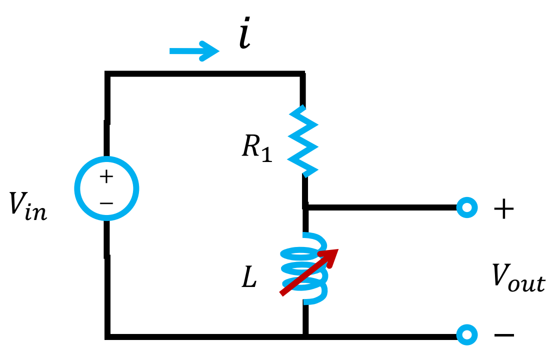
$$\phi(\omega) = \tan^{-1}\left[\frac{\Im\mathfrak{m}[\mathbb{G}(\omega)]}{\Re\mathfrak{e}[\mathbb{G}(\omega)]}\right]$$

Phase

Transfer function

QUESTION: calculate the voltage transfers from input to output based on varying L

VOLTAGE DIVIDER



$$\mathbb{G}(\omega) = \frac{1}{1 - j\frac{R_1}{\omega L}}$$

- The magnitude of $\mathbb{G}(\omega)$

$$|\mathbb{G}(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{R_1}{\omega L}\right)^2}}$$

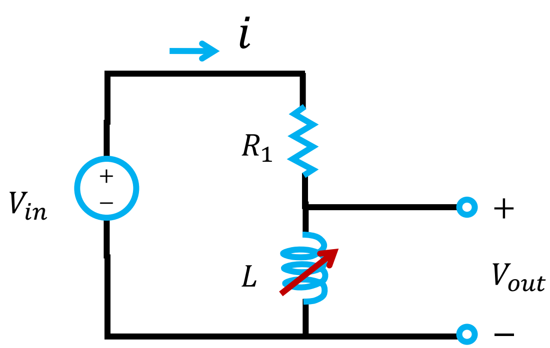
- The phase of $\mathbb{H}(\omega)$

$$\phi = \tan^{-1}\left(-\frac{R_1}{\omega L}\right)$$

Transfer function

QUESTION: calculate the voltage transfers from input to output based on varying L

VOLTAGE DIVIDER



$$\mathbb{G}(R_2) = \frac{\mathbb{V}_{out}}{\mathbb{V}_{in}} = \frac{j\omega L}{R_1 + j\omega L} = \frac{1}{\frac{R_1}{j\omega L} + 1}$$

$$\omega = 2\pi f$$

- If frequency is very high

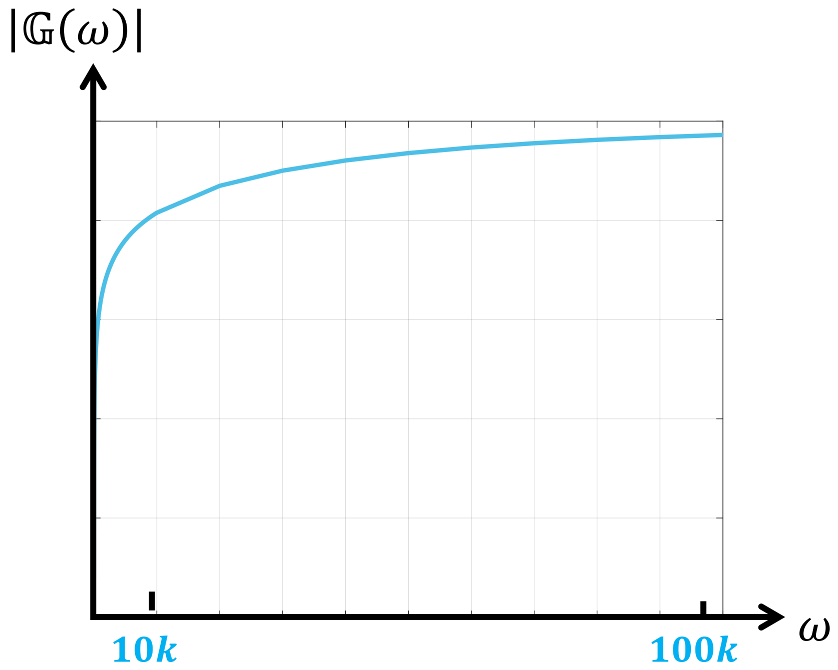
$$\mathbb{G}(R_2) \to 1$$

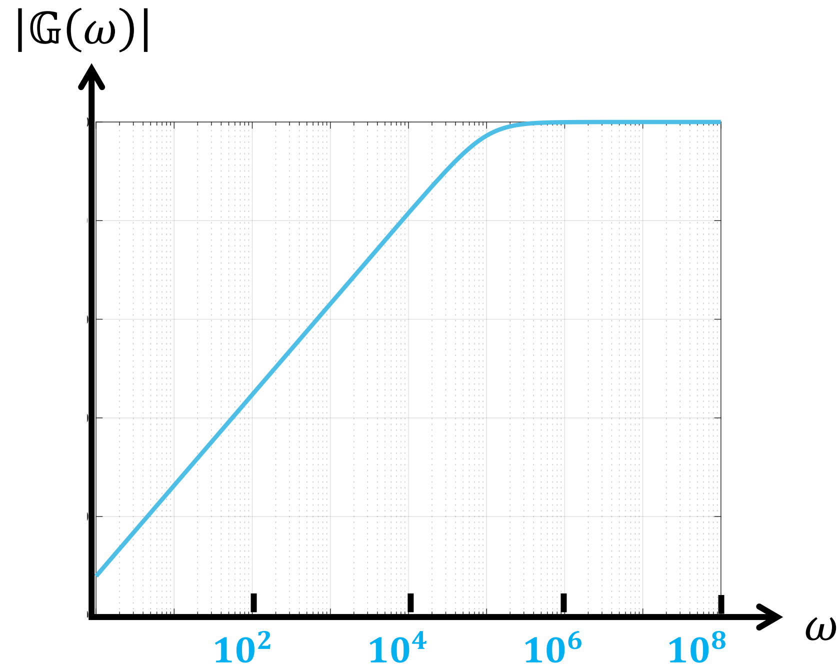
- If frequency is very low

$$\mathbb{G}(R_2) \approx \frac{j\omega L}{R_1} \to 0$$

Transfer function

Let's try to plot the transfer function $|\mathbb{G}(\omega)| = \dfrac{1}{\sqrt{1+\left(\frac{R_1}{\omega L}\right)^2}}$



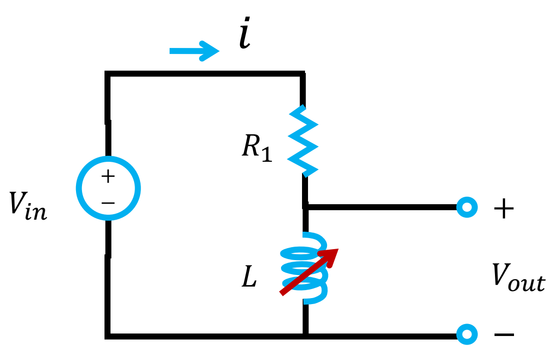


While plot the transfer function, the x-axis is usually plot in logarithmic scale

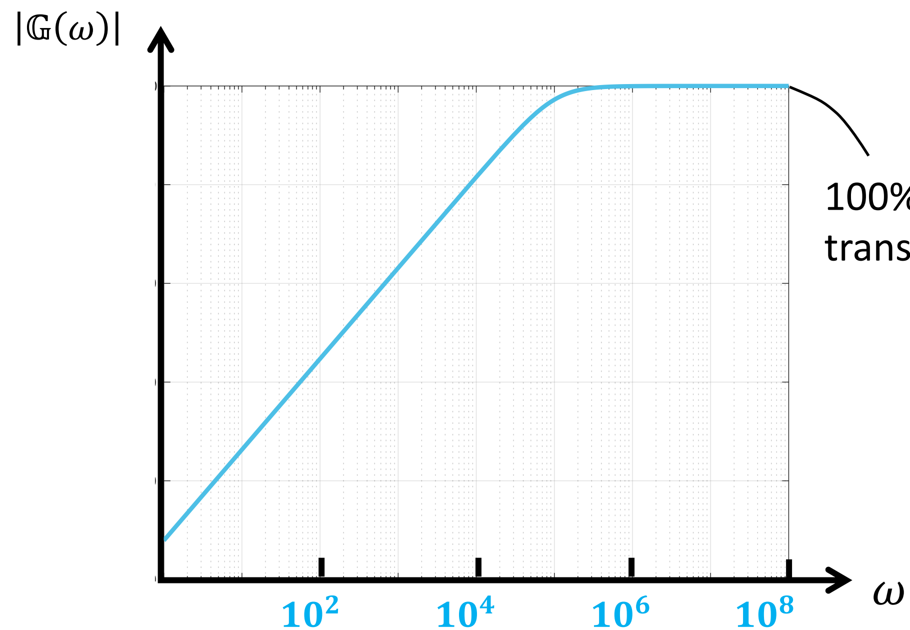
Transfer function

QUESTION: calculate the voltage transfers from input to output based on varying L

VOLTAGE DIVIDER



$$|\mathbb{G}(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{R_1}{\omega L}\right)^2}}$$

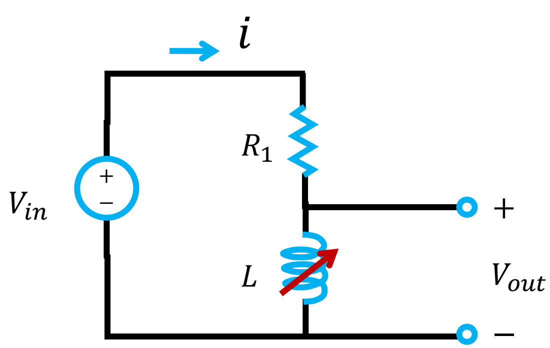


- Transfer is dependent on ω
- If frequency is very high $\mathbb{G}(\omega) \rightarrow 1$
- If frequency is very low $\mathbb{G}(\omega) \rightarrow 0$

Transfer function

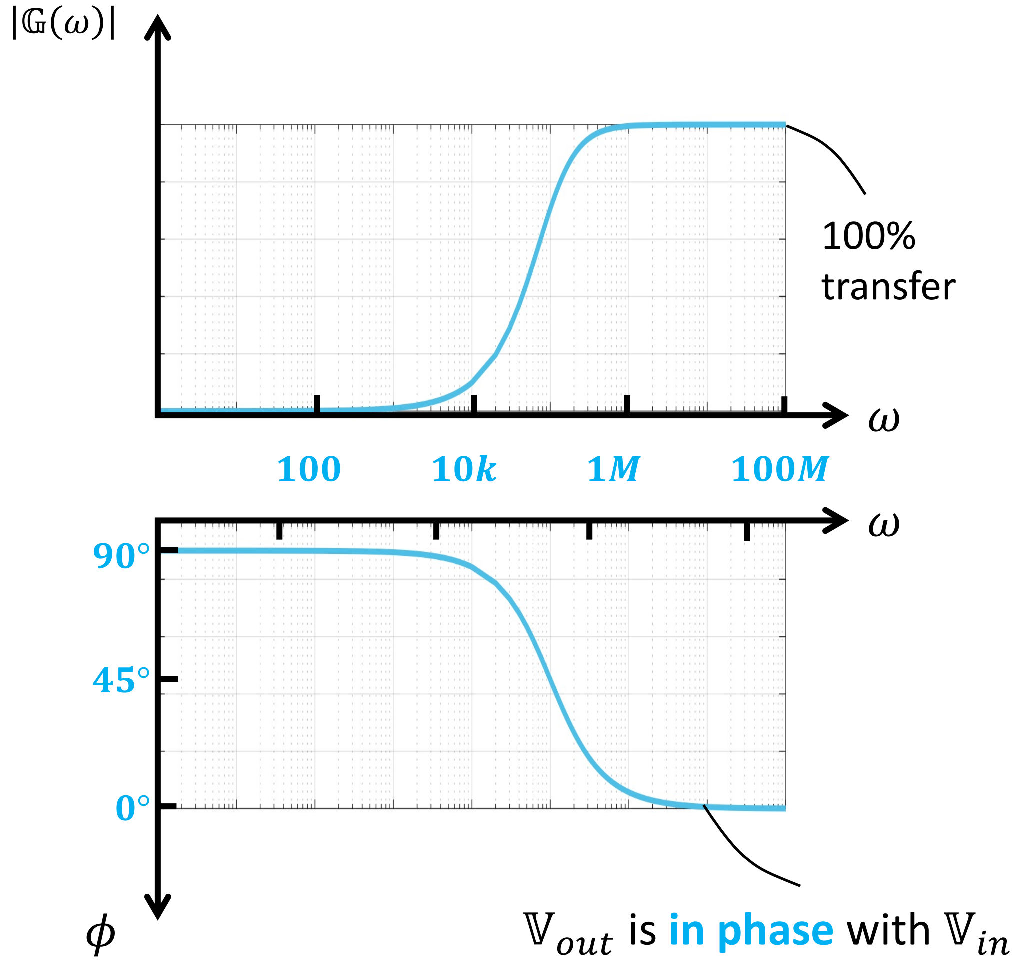
QUESTION: calculate the voltage transfers from input to output based on varying L_2

VOLTAGE DIVIDER



$$|\mathbb{G}(\omega)| = \frac{1}{\sqrt{1+\left(\frac{R_1}{\omega L}\right)^2}}$$

$$\phi = \tan^{-1}\left(-\frac{R_1}{\omega L}\right)$$



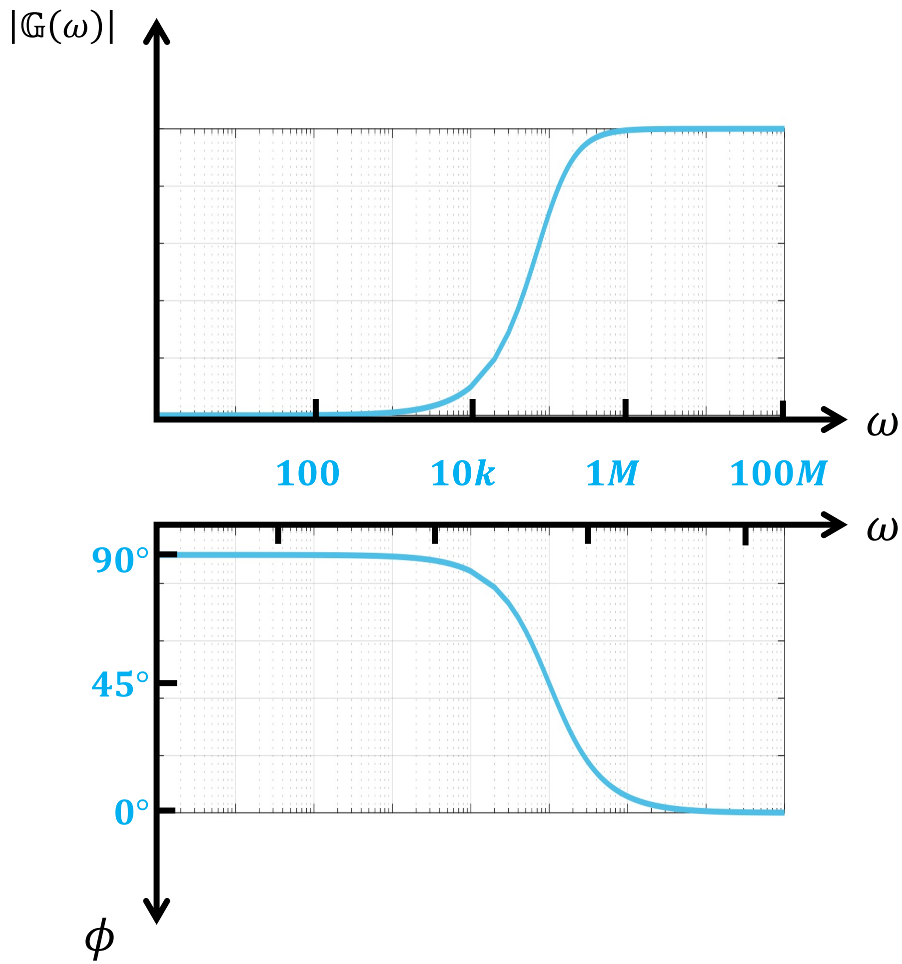
High pass filter

This is a HIGH PASS FILTER

Which passes high frequencies

and block low frequencies

- If $\mathbb{V}_{in}$ has **high** frequency, **nearly all** of $\mathbb{V}_{in}$ will transfer to output
- If $\mathbb{V}_{in}$ has **low** frequency, **little** of $\mathbb{V}_{in}$ will transfer to output



Different forms of the transfer function

TRANSFER FUNCTION of a circuit or system describes the output response to an input excitation as a function of the angular frequency ω

$$\mathbb{G}_v(\omega) = \frac{\mathbb{V}_{out}(\omega)}{\mathbb{V}_{in}(\omega)}$$ ← Voltage gain

$$\mathbb{G}_i(\omega) = \frac{\mathbb{I}_{out}(\omega)}{\mathbb{I}_{in}(\omega)}$$ ← Current gain

$$\mathbb{Z}(\omega) = \frac{\mathbb{V}_{out}(\omega)}{\mathbb{I}_{in}(\omega)}$$ ← Transimpedance

$$\mathbb{Y}(\omega) = \frac{\mathbb{I}_{out}(\omega)}{\mathbb{V}_{in}(\omega)}$$ ← Transadmittance

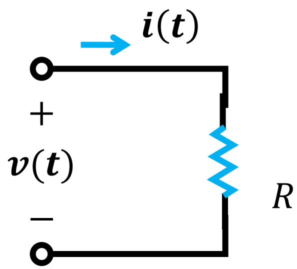
Recall: Impedance

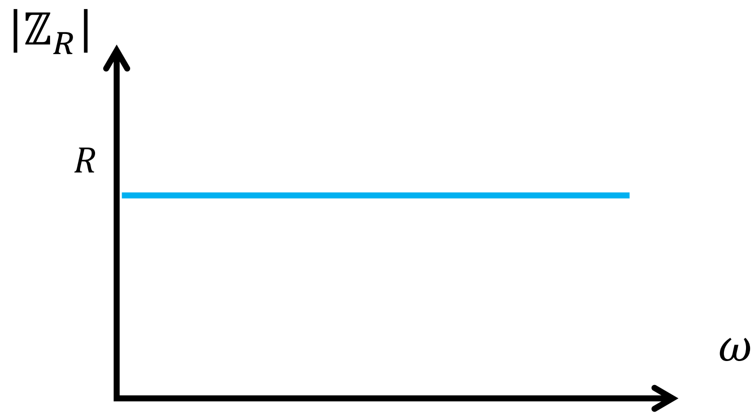
Impedance, Z,

is defined as the ratio of the phasor voltage to the phasor current

			
i-v relation	$i = \frac{v}{R}$	$i(t) = C\frac{dv(t)}{dt}$	$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(x)dx$
v-i relation	$v = iR$	$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)dx$	$v(t) = L\frac{di(t)}{dt}$
Impedance	$\boldsymbol{R}$	$\boldsymbol{\frac{1}{j\omega C}}$	$\boldsymbol{j\omega L}$

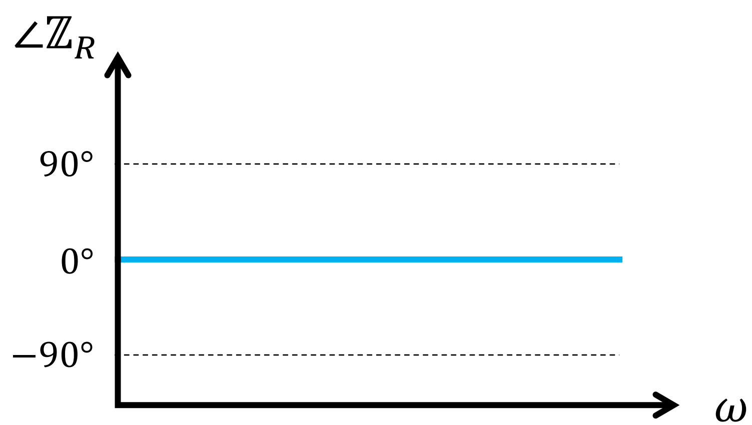
Transfer function of a resistor



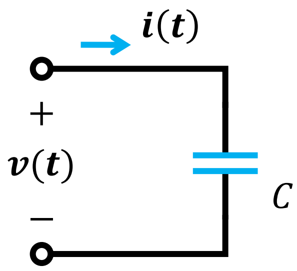


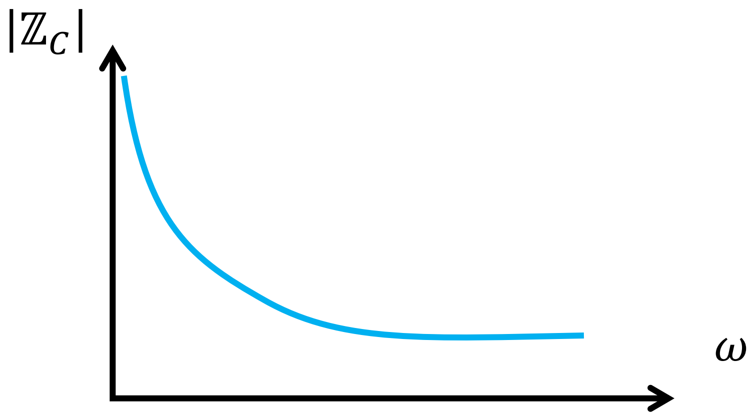
$$\mathbb{Z}_R(\omega) = R$$

$$\begin{cases} |\mathbb{Z}_R| = R \\ \angle \mathbb{Z}_R = 0° \end{cases}$$



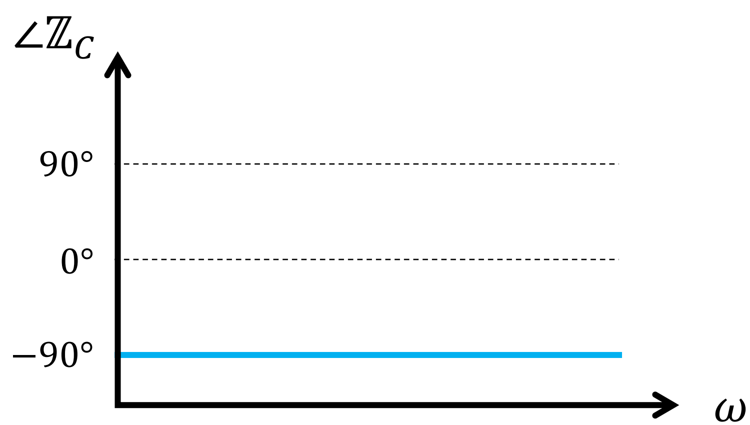
Transfer function of a capacitor



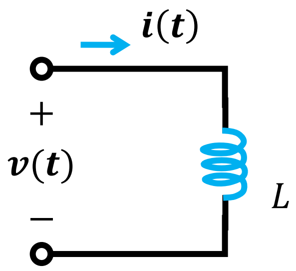


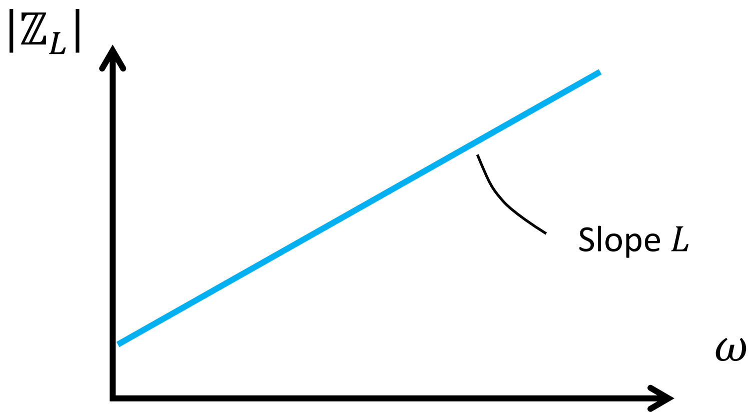
$$\mathbb{Z}_C(\omega) = \frac{1}{j\omega C}$$

$$\begin{cases} |\mathbb{Z}_C| = \dfrac{1}{\omega C} \\ \angle\mathbb{Z}_C = -90° \end{cases}$$



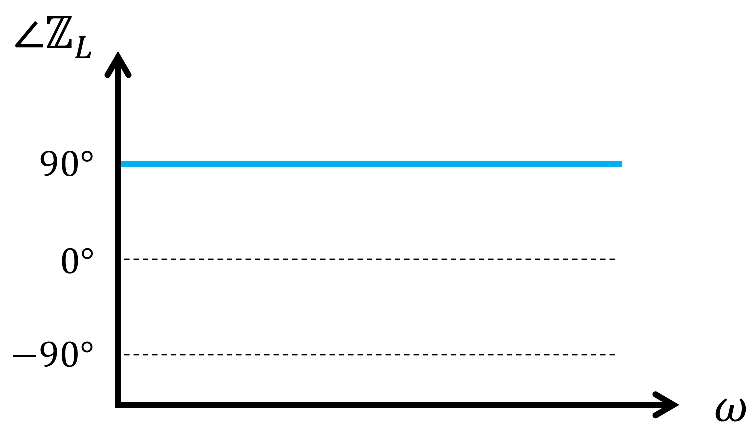
Transfer function of an inductor





$$\mathbb{Z}_L(\omega) = j\omega L$$

$$\begin{cases} |\mathbb{Z}_C| = \omega L \\ \angle\mathbb{Z}_C = 90° \end{cases}$$

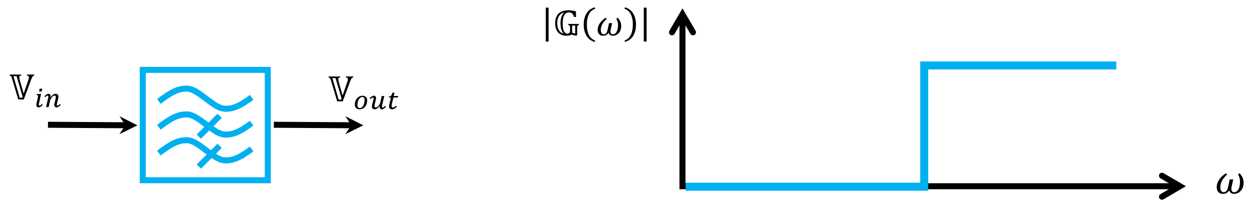


Outlines

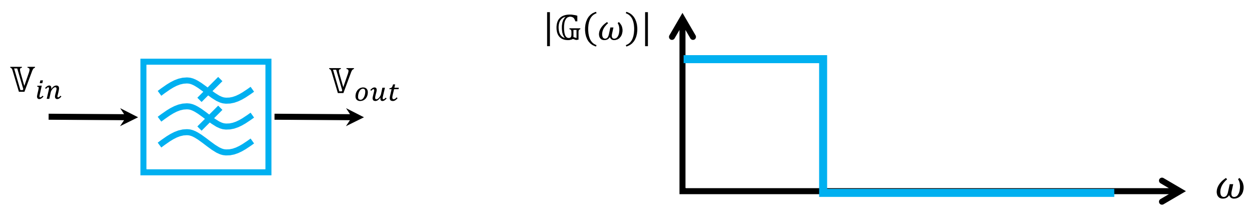
- Transfer function
  - Voltage gain
  - Transimpedance
- **Filters**

Common ideal filters

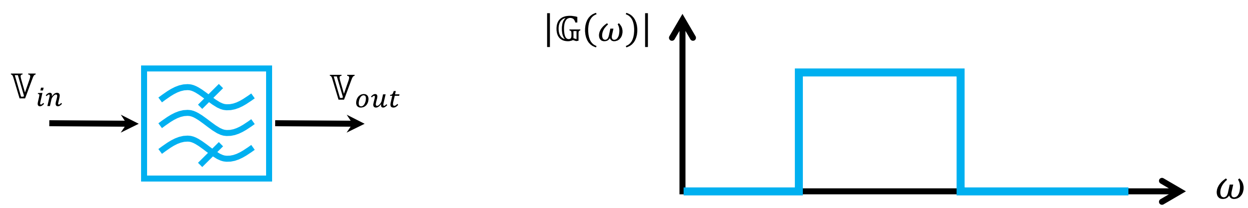
High pass filter



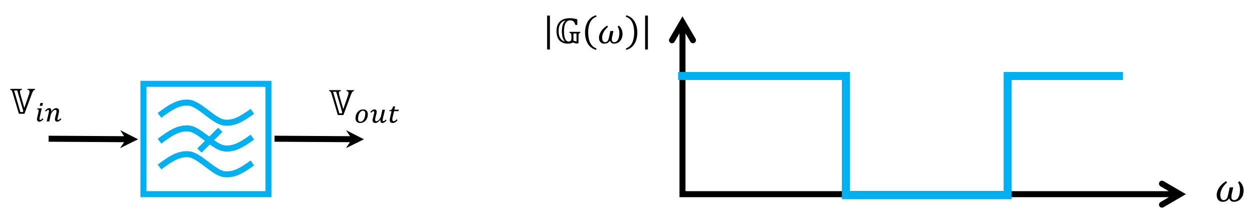
Low pass filter



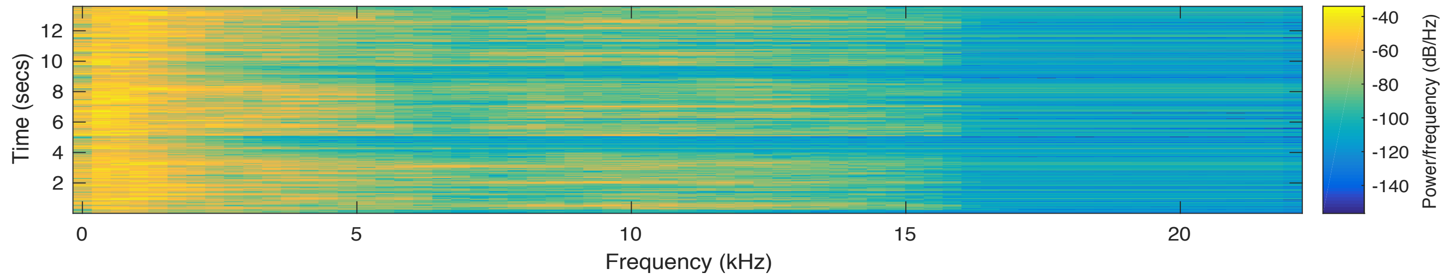
Band pass filter



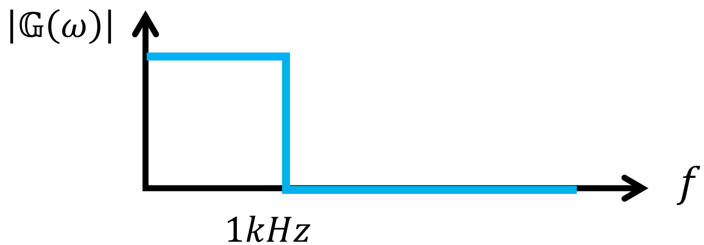
Band stop filter



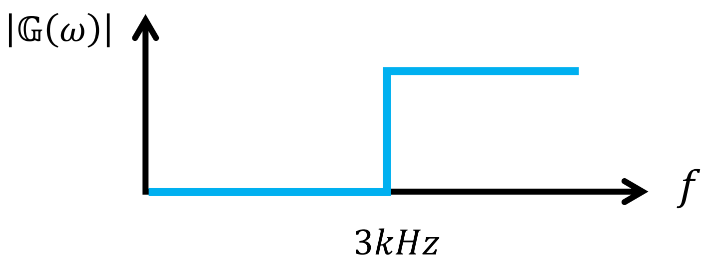
Common ideal filters



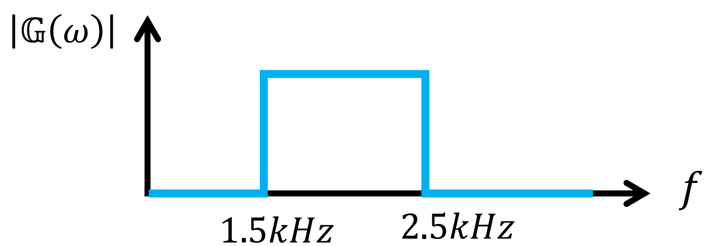
Low pass filter



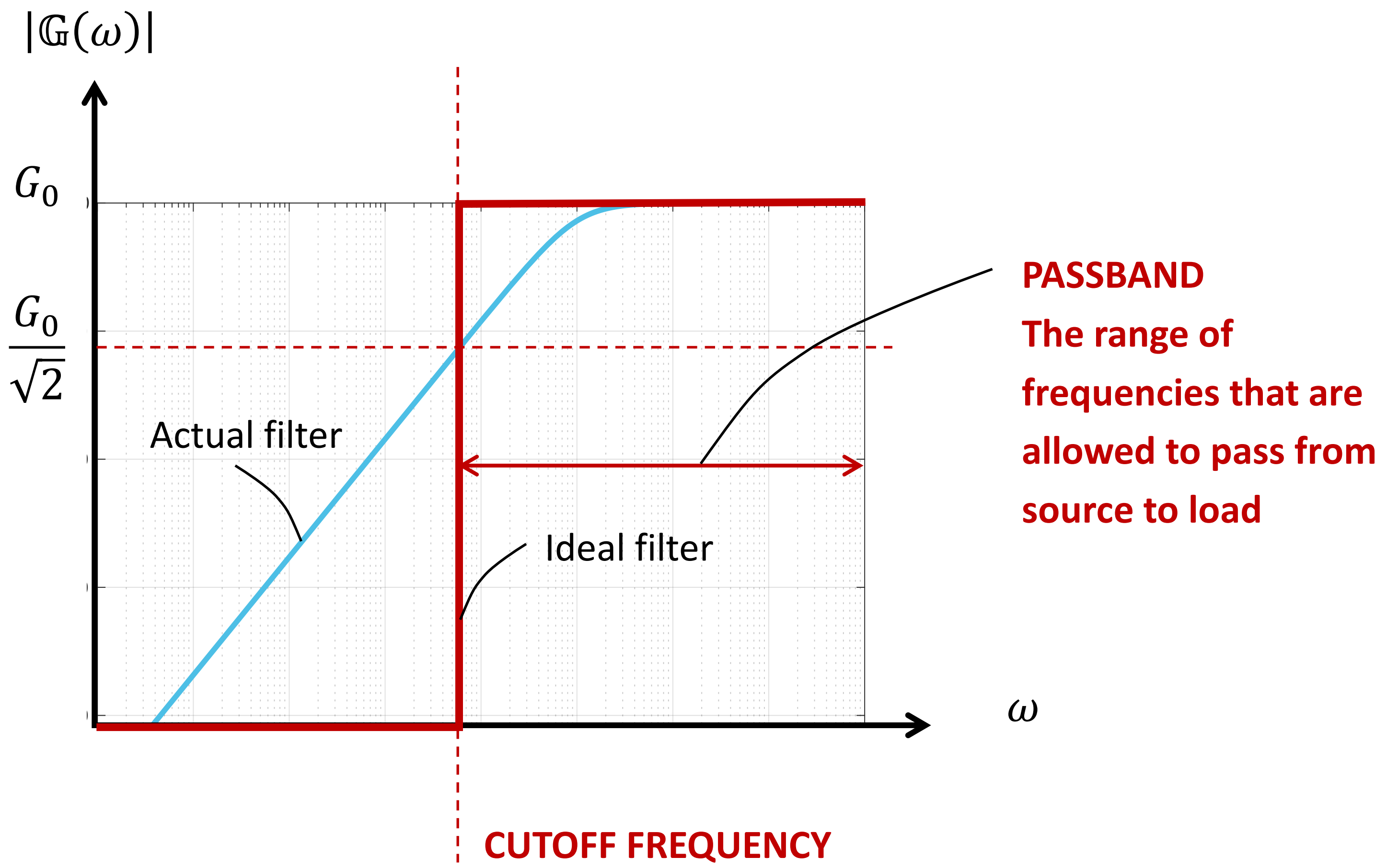
High pass filter



Band pass filter

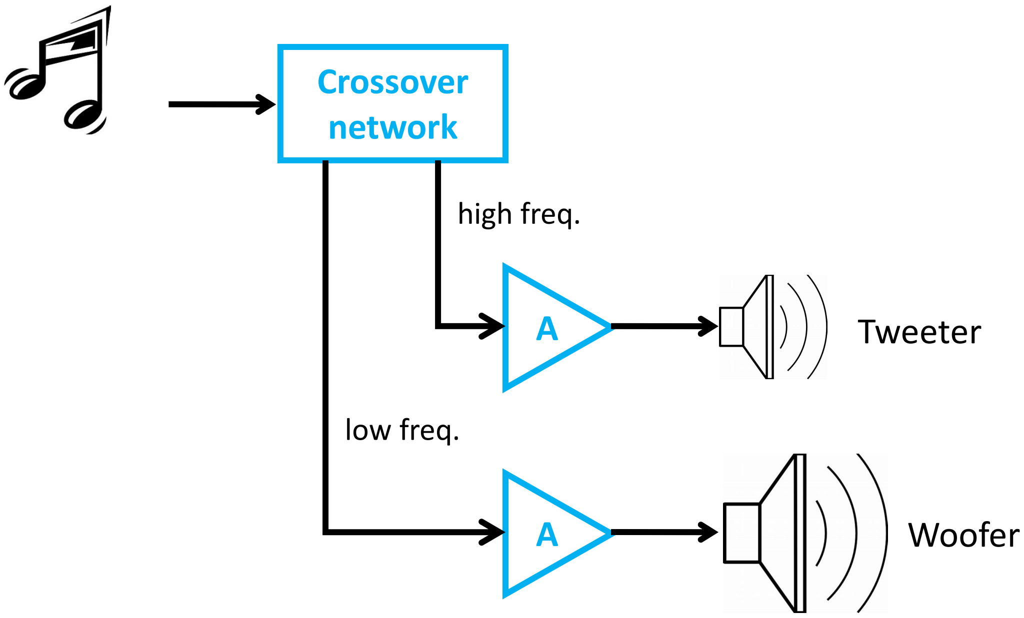


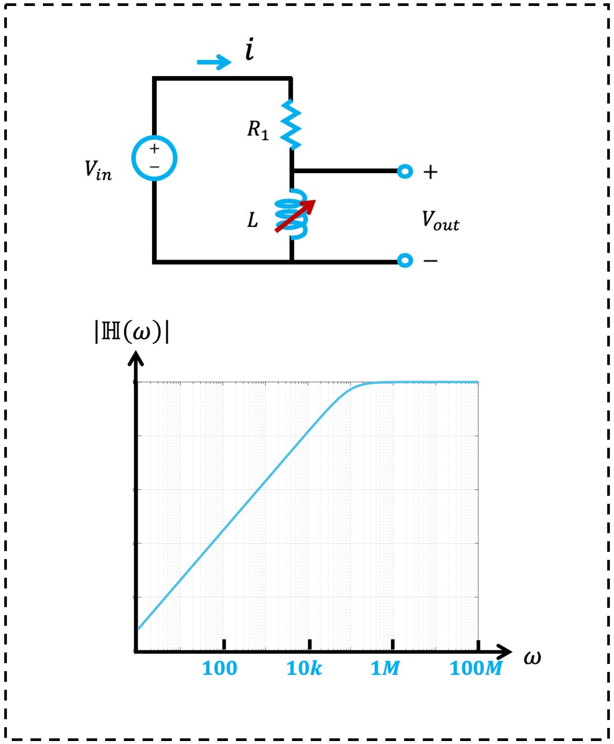
Ideal v.s. actual filters



Example 1

QUESTION: design a stereo amplifier with two output channels to split the high and low frequencies.





Example 1

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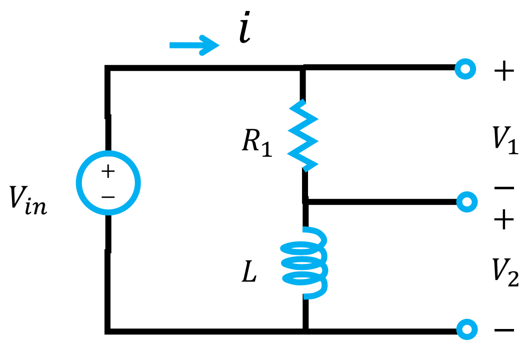


- According to KVL

$$\mathbb{V}_1 = iR_1 = \frac{R_1}{R_1 + j\omega L}\mathbb{V}_{in}$$

$$\mathbb{V}_2 = iZ_L = \frac{j\omega L}{R_1 + j\omega L}\mathbb{V}_{in}$$

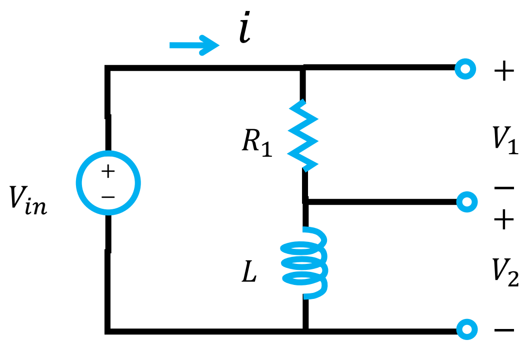
- The transfer function

$$\mathbb{G}_1(\omega) = \frac{\mathbb{V}_1}{\mathbb{V}_{in}} = \frac{R_1}{R_1 + j\omega L}$$

$$\mathbb{G}_2(\omega) = \frac{\mathbb{V}_2}{\mathbb{V}_{in}} = \frac{j\omega L}{R_1 + j\omega L}$$

Example 1

QUESTION: design a stereo amplifier with two output channels to split the high and low frequencies.



$$\mathbb{G}_1(\omega) = \frac{\mathbb{V}_1}{\mathbb{V}_{in}} = \frac{R_1}{R_1 + j\omega L}$$

$$\mathbb{G}_2(\omega) = \frac{\mathbb{V}_2}{\mathbb{V}_{in}} = \frac{j\omega L}{R_1 + j\omega L}$$

$$|\mathbb{G}_1(\omega)| = \frac{1}{\sqrt{1 + \frac{\omega^2 L^2}{R_1^2}}}$$

When $|\mathbb{G}_1(\omega_c)| = \frac{1}{\sqrt{2}}$

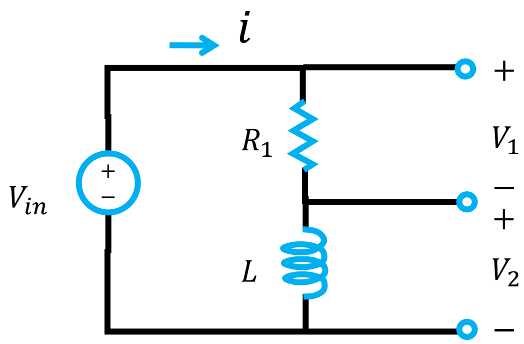
$\rightarrow \quad \omega_{c1} = \frac{R_1}{L}$

Similar for the cutoff freq. of $\mathbb{G}_2(\omega)$

$$\omega_{c2} = \frac{R_1}{L}$$

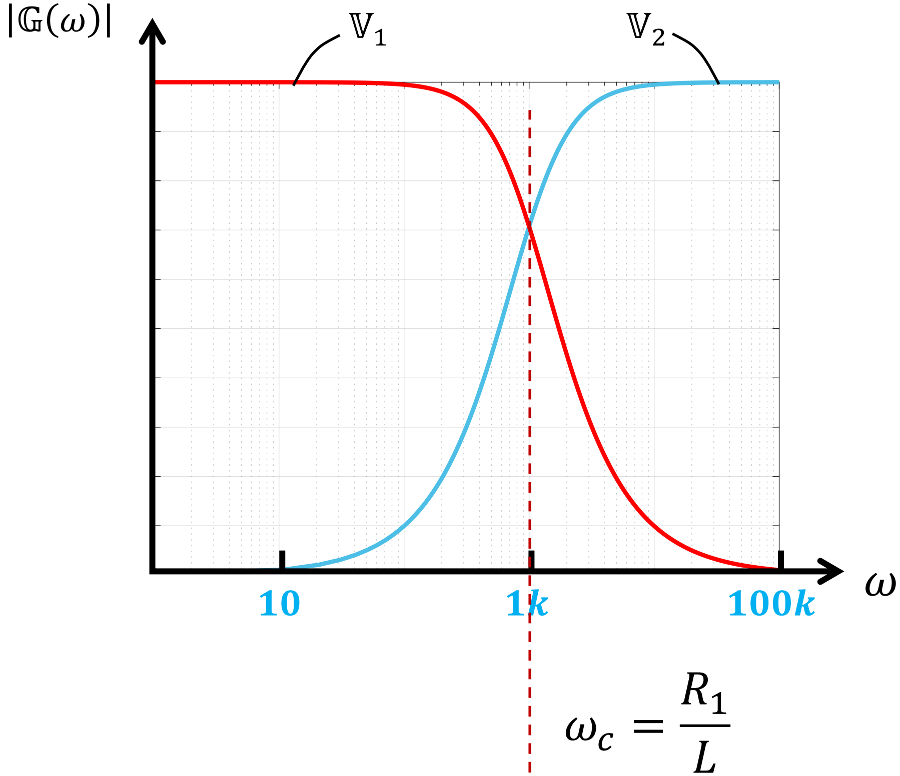
Example 1

QUESTION: design a stereo amplifier with two output channels to split the high and low frequencies.



$$\mathbb{G}_1(\omega) = \frac{\mathbb{V}_1}{\mathbb{V}_{in}} = \frac{R_1}{R_1 + j\omega L}$$

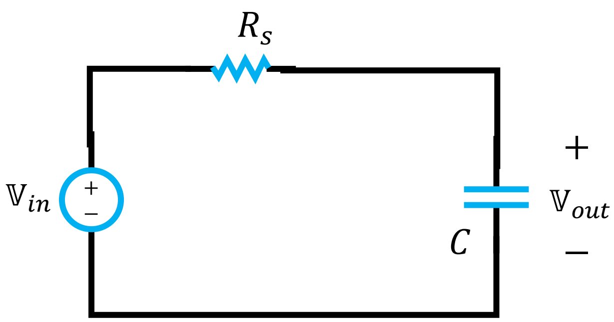
$$\mathbb{G}_2(\omega) = \frac{\mathbb{V}_2}{\mathbb{V}_{in}} = \frac{j\omega L}{R_1 + j\omega L}$$



$$\omega_c = \frac{R_1}{L}$$

Example 2

QUESTION: find the transfer function of the circuit below



How to vary Z_C ?

- Not practical to vary C
- Much easier to vary ω

- According to KVL

$$\mathbb{V}_{out} = iZ_C = \frac{Z_C}{R_s + Z_C}\mathbb{V}_{in}$$

- If Z_C is very low

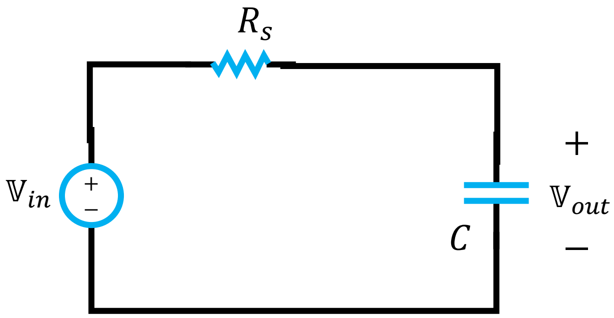
$$\mathbb{G}(\omega) = \frac{Z_C}{R_s + Z_C} \xrightarrow{Z_C \to 0} 0$$

- If Z_C is very high

$$\mathbb{G}(\omega) = \frac{Z_C}{R_s + Z_C} \xrightarrow{Z_C \to \infty} 1$$

Example 2

QUESTION: find the transfer function of the circuit below



- The transfer function

$$\mathbb{G}(\omega) = \frac{Z_C}{R_s + Z_C}$$

$$= \frac{\frac{1}{j\omega C}}{R_s + \frac{1}{j\omega C}}$$

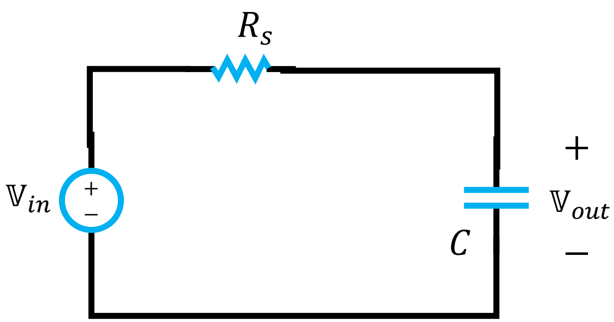
$$= \frac{1}{j\omega C R_s + 1}$$

This is also a "frequency-dependent" variable voltage divider.

Transfer is dependent on the value of ω

Example 2

QUESTION: find the transfer function of the circuit below



$$\left[\ \mathbb{G}(\omega) = \frac{1}{j\omega C R_s + 1}\ \right]$$

- If frequency is very high, Z_C is very low

$$Z_C = \frac{1}{j\omega C} \to 0$$ **short circuit**

$$\mathbb{G}(\omega) \to 0$$

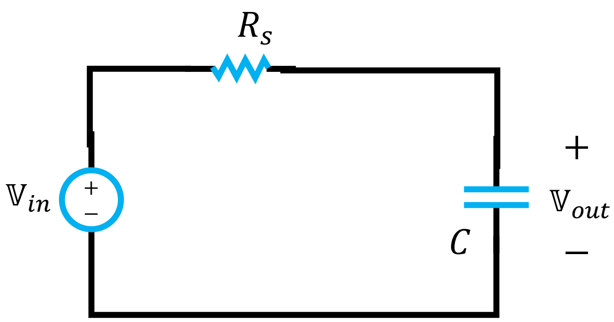
- If frequency is very low, Z_C is very high

$$Z_C = \frac{1}{j\omega C} \to \infty$$ **open circuit**

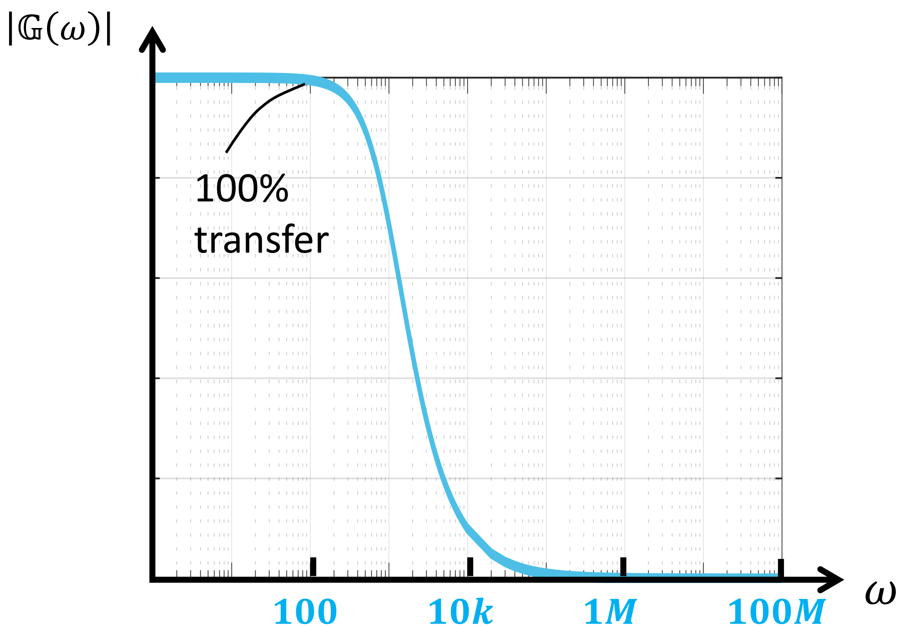
$$\mathbb{G}(\omega) \to 1$$

Example 2

QUESTION: find the transfer function of the circuit below



$$\mathbb{G}(\omega) = \frac{1}{j\omega C R_s + 1}$$

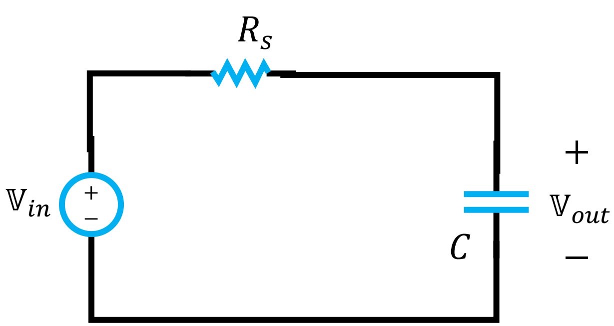


This is a LOW PASS FILTER
Which passes low frequencies
and block high frequencies

- Transfer is dependent on ω
- If frequency is very low $\mathbb{G}(\omega) \to 1$
- If frequency is very high $\mathbb{G}(\omega) \to 0$

Example 2

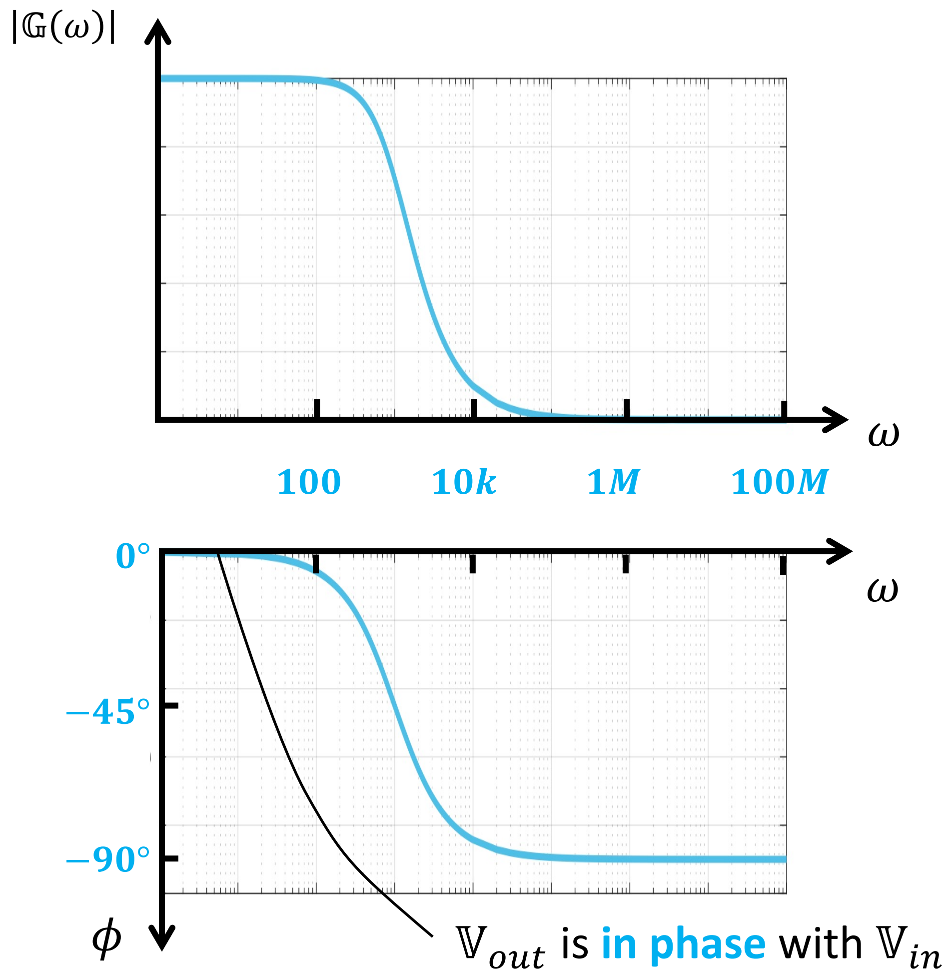
QUESTION: find the transfer function of the circuit below



$$\mathbb{G}(\omega) = \frac{1}{j\omega C R_s + 1}$$

$$|\mathbb{G}(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R_s^2 C^2}}$$

$$\phi = -\tan^{-1}(\omega C R_s)$$



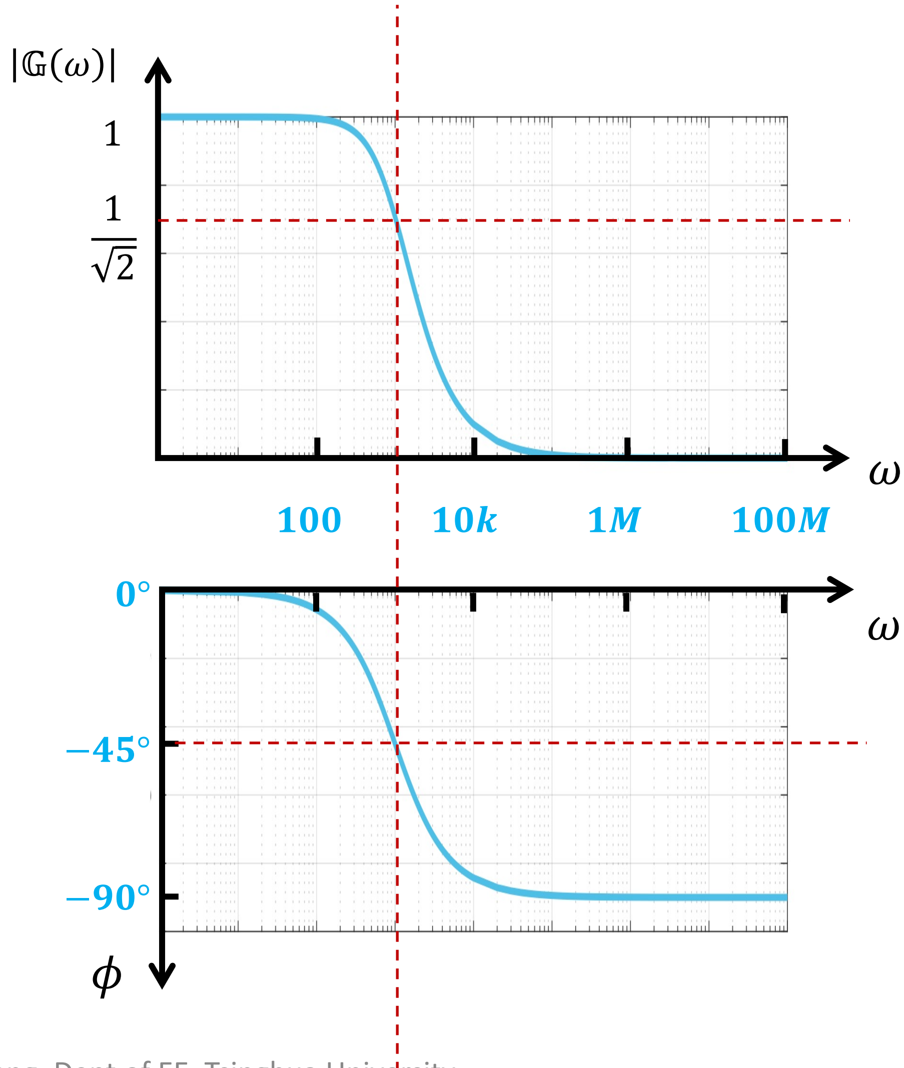
Example 2

$$\begin{cases} |\mathbb{G}(\omega)| = \dfrac{1}{\sqrt{1+\omega^2 R_s^2 C^2}} \\ \phi = -\tan^{-1}(\omega C R_s) \end{cases}$$

When $|\mathbb{G}(\omega_c)| = \dfrac{1}{\sqrt{2}}$

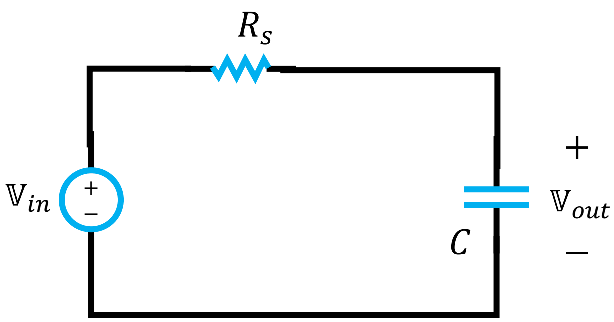
→ $\omega_c = \dfrac{1}{R_s C}$

$\phi_c = -\tan^{-1}(\omega_c C R_s) = -45°$



Example 2

QUESTION: find the transfer function of the circuit below



$$\left[\begin{aligned} \mathbb{G}(\omega) &= \frac{1}{j\omega C R_s + 1} \\ \phi &= -\tan^{-1}(\omega C R_s) \end{aligned}\right]$$

- If $v_{in}(t) = V_S \sin(\omega t)$
- According to KVL

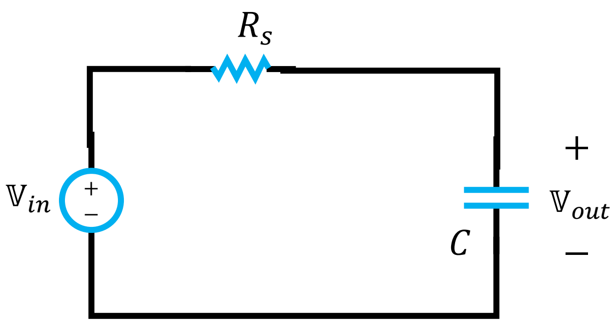
$$\frac{d}{dt}v(t) + \frac{1}{R_s C}v(t) = \frac{1}{R_s C}v_{in}(t)$$

$$v(t) = K_1 \cos(\omega t) + K_2 \sin(\omega t) + K_3 e^{-\frac{1}{R_S C}t}$$

where $\begin{cases} K_1 = \dfrac{-\omega R_s C}{1+\omega^2 R_S^2 C^2} V_S \\ K_2 = \dfrac{1}{1+\omega^2 R_S^2 C^2} V_S \\ K_3 = \dfrac{\omega R_s C}{1+\omega^2 R_S^2 C^2} V_S \end{cases}$

Example 2

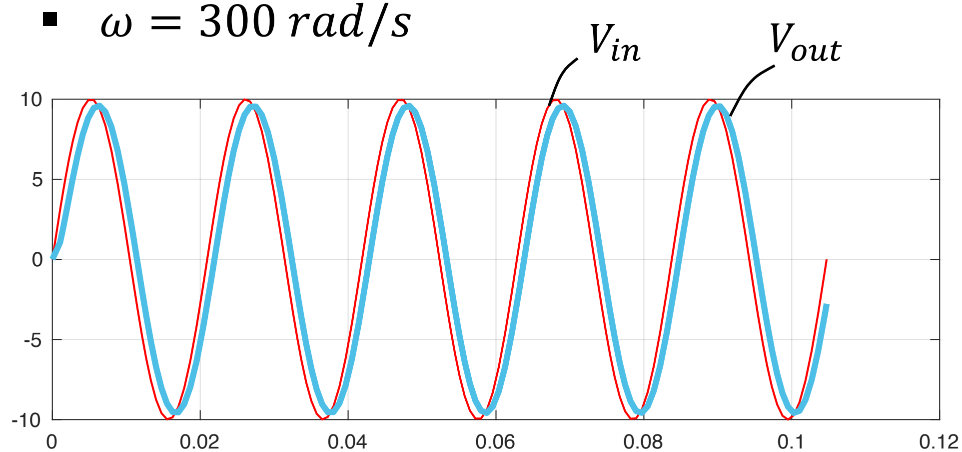
QUESTION: find the transfer function of the circuit below



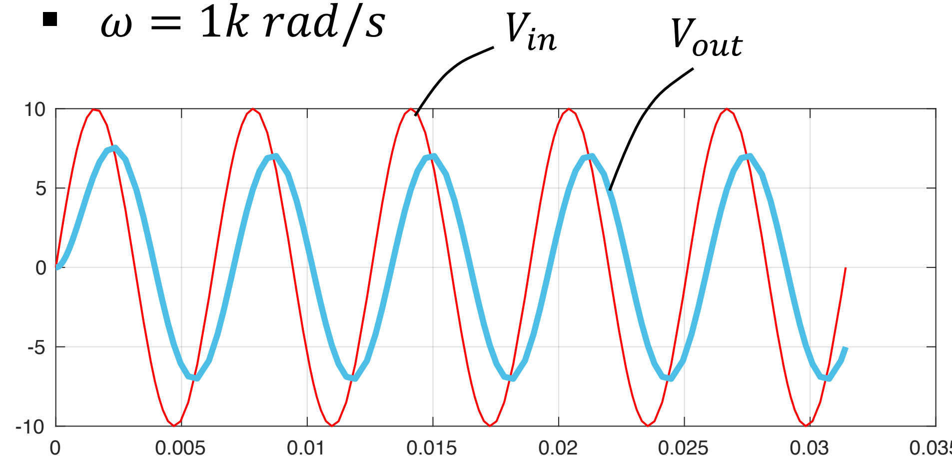
$$\mathbb{G}(\omega) = \frac{1}{j\omega C R_s + 1}$$

$$\phi = -\tan^{-1}(\omega C R_s)$$

- $\omega = 300\ rad/s$

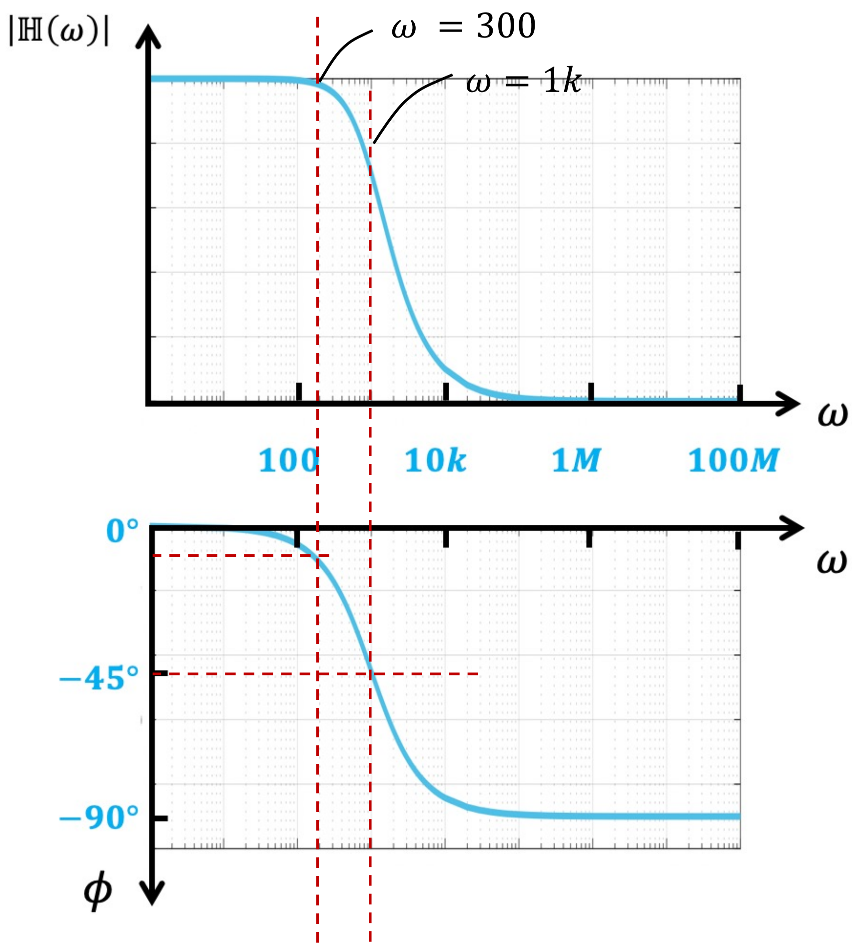


- $\omega = 1k\ rad/s$

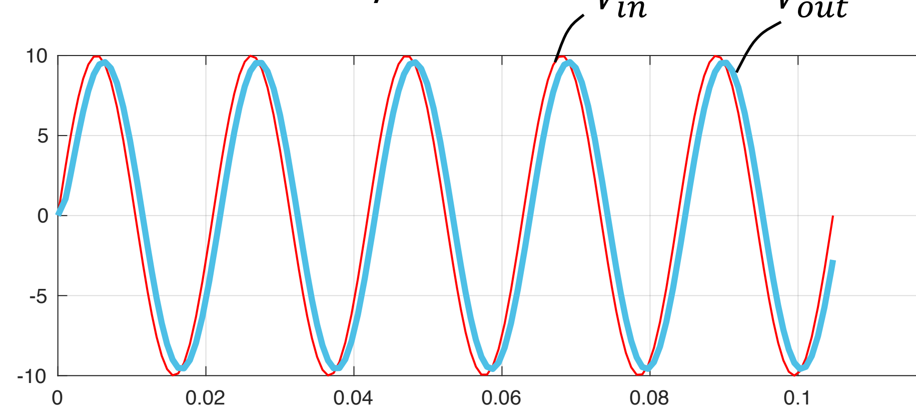


Example 3

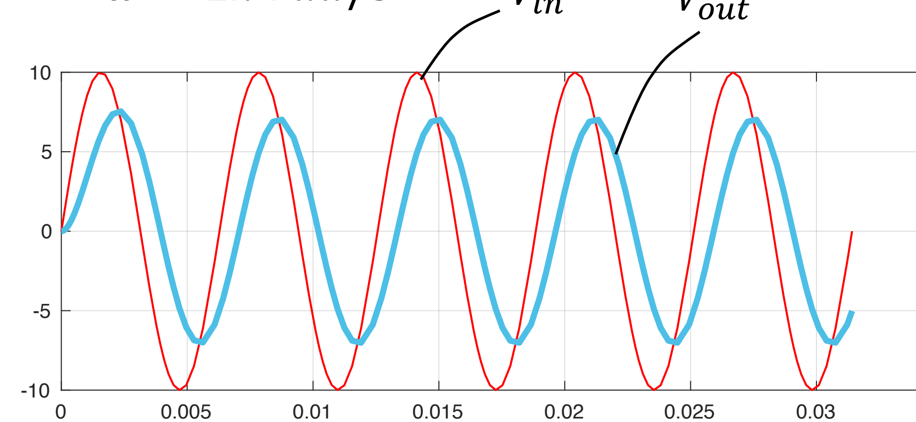
QUESTION: find the transfer function of the circuit below



- $\omega = 300\ rad/s$

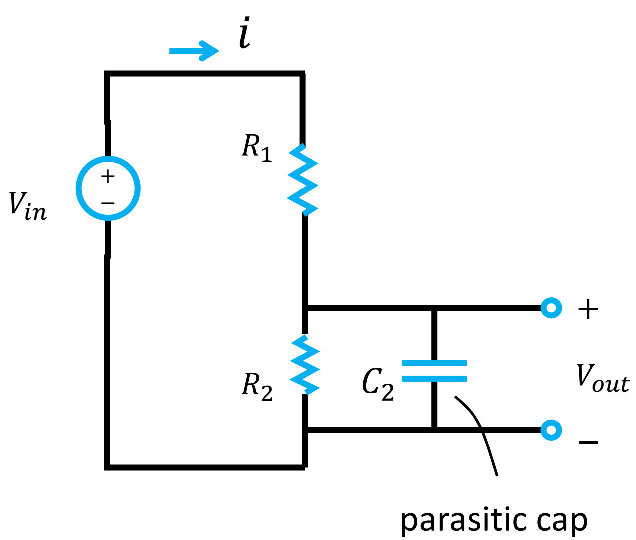


- $\omega = 1k\ rad/s$



Example 3

QUESTION: a voltage divider usually does work well at high frequency due to the parasitic capacitance. In order to remedy this capacitance issue, a small capacitor in parallel with the series resistor is applied. This capacitor is called compensate capacitor. Find the proper value of the compensate capacitor.



- If there is no parasitic cap, according to KVL

$$\mathbb{V}_{out} = iR_2 = \frac{R_2}{R_1 + R_2}\mathbb{V}_{in}$$

- If consider the parasitic cap, according to KVL

$$\mathbb{V}_{out} = \frac{R_2||Z_{C_2}}{R_1 + R_2||Z_{C_2}}\mathbb{V}_{in}$$

$$\mathbb{G}_2(\omega) = \frac{\mathbb{V}_{out}}{\mathbb{V}_{in}} = \frac{R_2}{R_1 + R_2 + j\omega R_1 R_2 C_2}$$

Example 3

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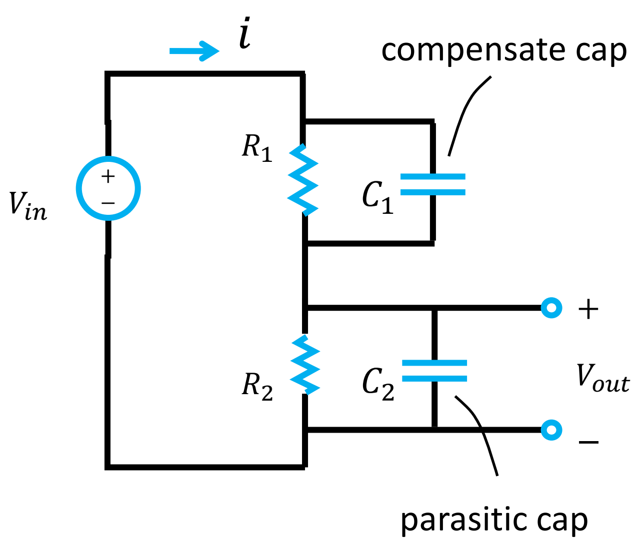

- By applying the compensate cap, according to KVL

$$\mathbb{V}_{out} = \frac{R_2||Z_{C_2}}{R_1||Z_{C_1} + R_2||Z_{C_2}}\mathbb{V}_{in}$$

- The transfer function

$$\mathbb{G}_2(\omega) = \frac{\mathbb{V}_{out}}{\mathbb{V}_{in}}$$

$$= \frac{R_2(1 + j\omega R_1 C_1)}{R_1(1 + j\omega R_2 C_2) + R_2(1 + j\omega R_1 C_1)}$$

Example 3

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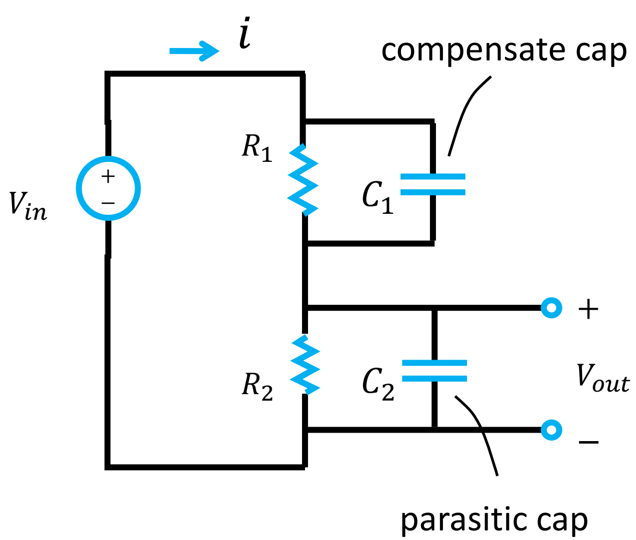


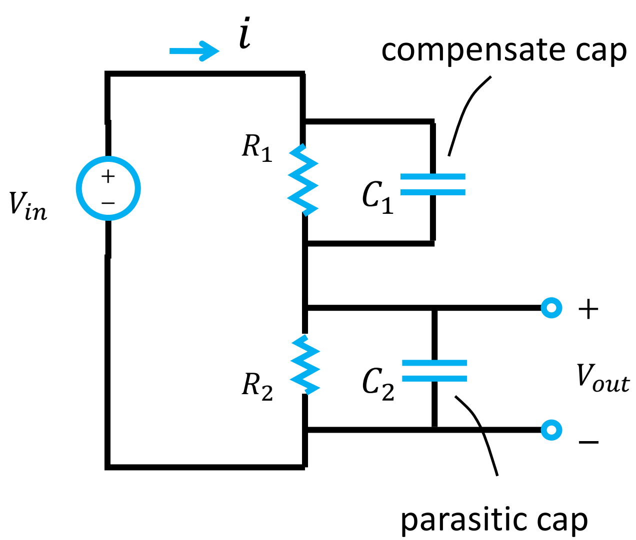
$$\mathbb{G}_2(\omega) = \frac{R_2(1 + j\omega R_1 C_1)}{R_1(1 + j\omega R_2 C_2) + R_2(1 + j\omega R_1 C_1)}$$

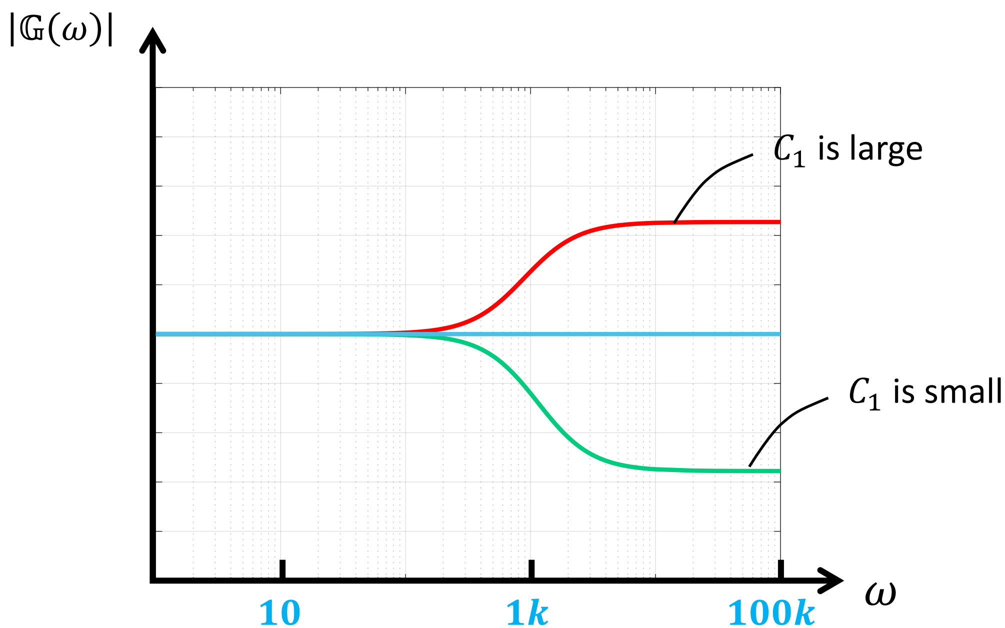
If $R_1 C_1 = R_2 C_2 \quad \rightarrow \quad \mathbb{G}_2(\omega) = \frac{R_2}{R_1 + R_2}$

Independent of frequency

Example 3

QUESTION: a voltage divider usually does work well at high frequency due to the parasitic capacitance. In order to remedy this capacitance issue, a small capacitor in parallel with the series resistor is applied. This capacitor is called compensate capacitor. Find the proper value of the compensate capacitor.





$$\mathbb{G}_2(\omega) = \frac{R_2(1 + j\omega R_1 C_1)}{R_1(1 + j\omega R_2 C_2) + R_2(1 + j\omega R_1 C_1)}$$

Example 3

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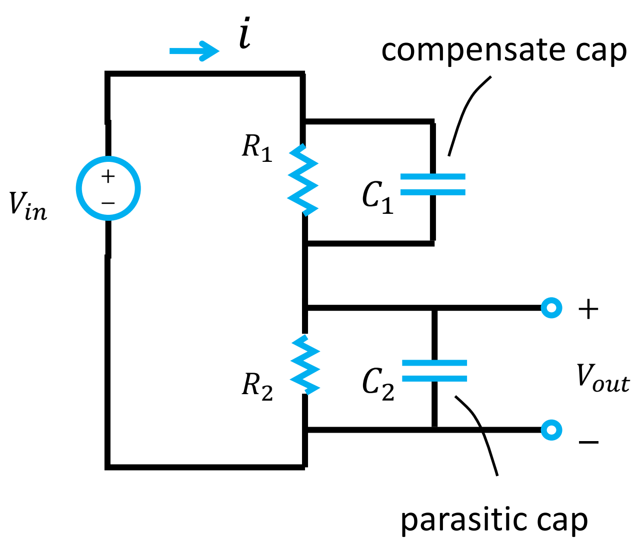
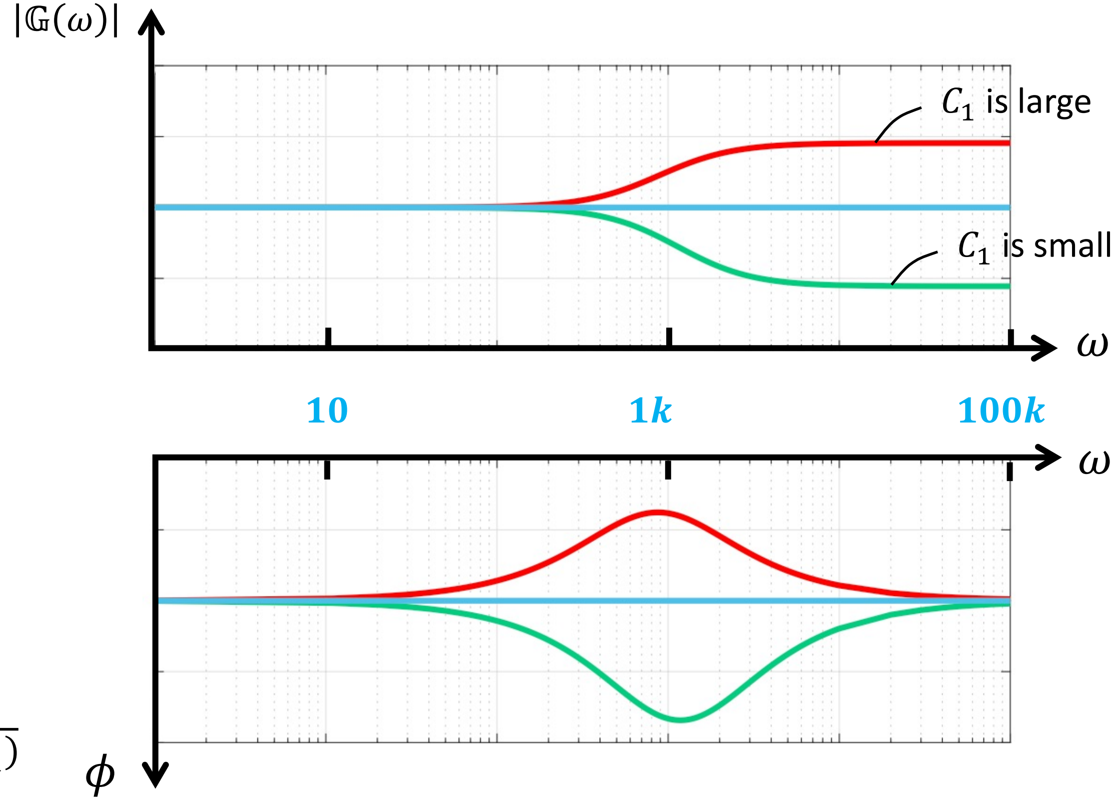
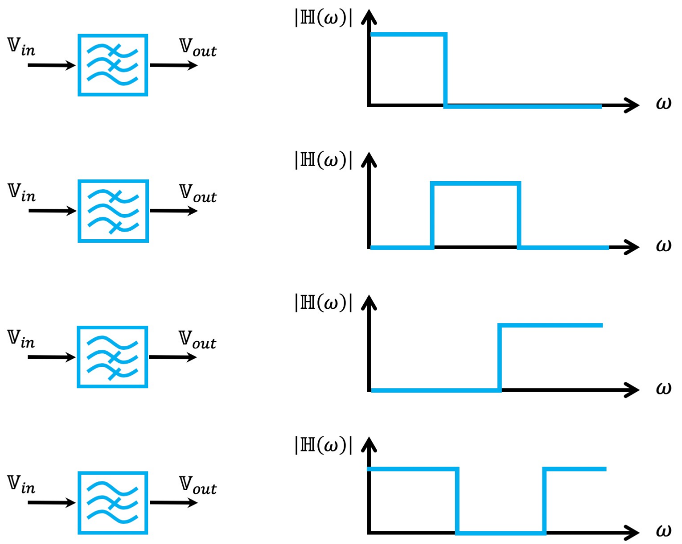




$$\mathbb{G}_2(\omega) = \frac{R_2(1 + j\omega R_1 C_1)}{R_1(1 + j\omega R_2 C_2) + R_2(1 + j\omega R_1 C_1)}$$

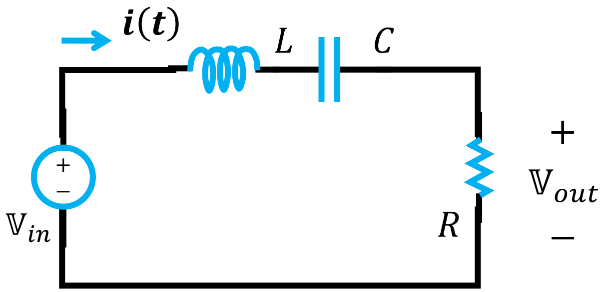
Outlines

- Transfer function
- Filters
  - Common ideal filters
  - Actual filters
    - High pass filters / low pass filters
    - Band pass filters / band stop filters
    - **Quality factor**



Example 4

QUESTION: calculate the voltage transfers from input to output based on varying R



- According to KVL

$$\mathbb{V}_{out} = \mathbb{I}R = \frac{R}{R + Z_L + Z_C}\mathbb{V}_{in}$$

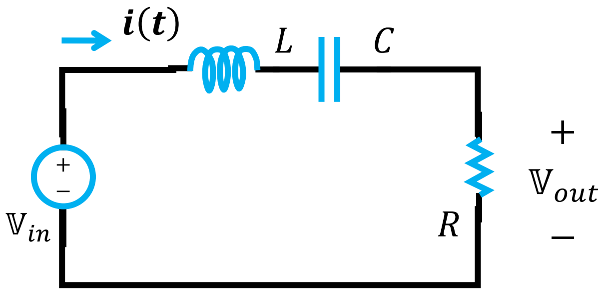
- The transfer function

$$\mathbb{G}(\omega) = \frac{\mathbb{V}_{out}}{\mathbb{V}_{in}} = \frac{R}{R + Z_L + Z_C}$$

$$= \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

Example 4

QUESTION: calculate the voltage transfers from input to output based on varying R



$$\mathbb{G}(\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\omega = 2\pi f$$

- If frequency is very low , $\omega \to 0$

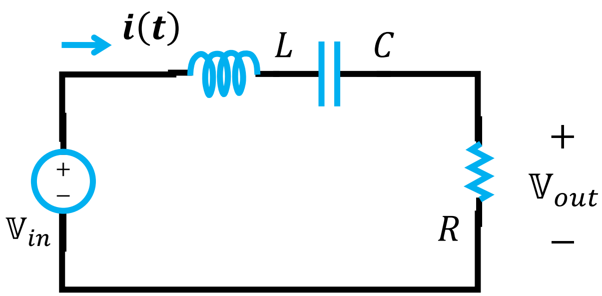
$$\mathbb{G}(\omega) = \frac{R}{R + j\left(\underset{\searrow 0}{\omega L} - \underset{\searrow \infty}{\frac{1}{\omega C}}\right)} \to 0$$

- If frequency is very high, $\omega \to \infty$

$$\mathbb{G}(\omega) = \frac{R}{R + j\left(\underset{\searrow \infty}{\omega L} - \underset{\searrow 0}{\frac{1}{\omega C}}\right)} \to 0$$

Example 4

QUESTION: calculate the voltage transfers from input to output based on varying R



$$\mathbb{G}(\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\omega = 2\pi f$$

- If frequency is very low, $\mathbb{G}(\omega) \to 0$
- If frequency is very high, $\mathbb{G}(\omega) \to 0$
- What if $\omega L - \dfrac{1}{\omega C} = 0$

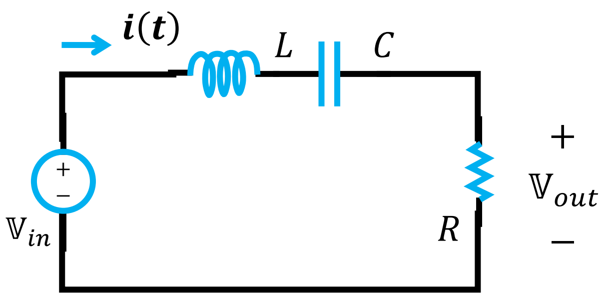
$\to$ $\boldsymbol{\omega_0} = \dfrac{1}{\sqrt{LC}}$ **RESONANCE FREQUENCY**

$\mathbb{G}(\omega_0) = 1$ thus, $\mathbb{V}_{out} = \mathbb{V}_{in}$

The RESONANCE FREQUENCY, ω_0, at which the impedance of the circuit is purely real. The circuit itself at ω_0 is called IN RESONANCE.

Example 4

QUESTION: calculate the voltage transfers from input to output based on varying R



- Find the cutoff frequency ω_c

$$|\mathbb{G}(\omega)| = \frac{1}{\sqrt{1+\left(\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right)\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\mathbb{G}(\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

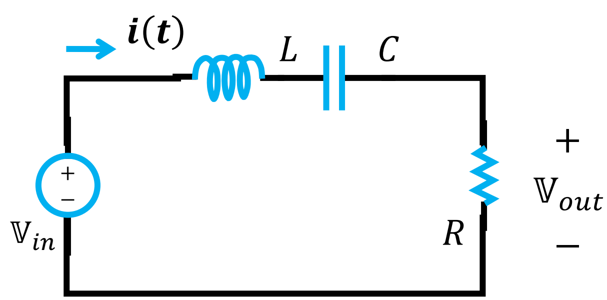
$$|\mathbb{G}(\omega_c)| = \frac{1}{\sqrt{2}}$$

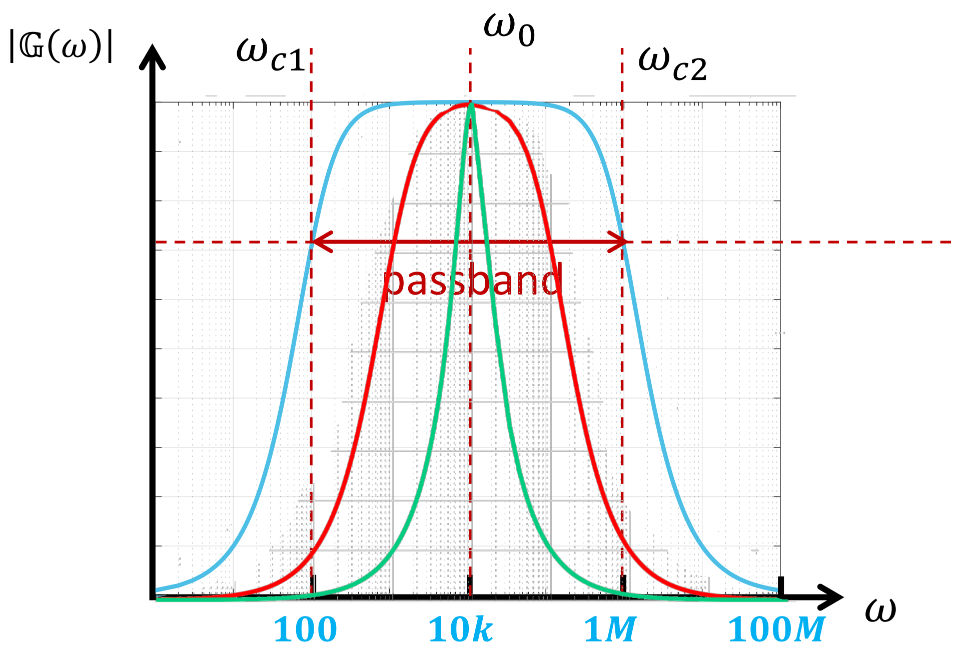
$$\rightarrow \begin{cases} \omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{\sqrt{LC}}\right)^2} \\ \omega_{c2} = +\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{\sqrt{LC}}\right)^2} \end{cases}$$

Note: ω_{c1} and ω_{c2} are not symmetric around ω_0

Example 4

QUESTION: calculate the voltage transfers from input to output based on varying R





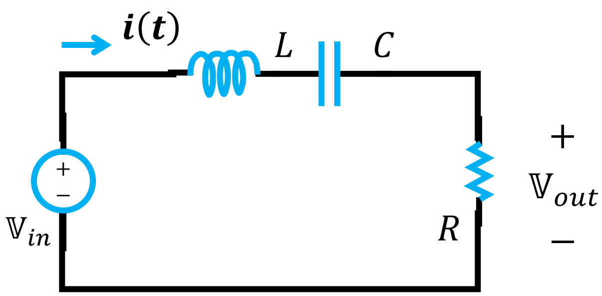
- The bandwidth

$$BW = \omega_{c2} - \omega_{c1} = \frac{R}{L}$$

This is a BAND PASS FILTER
which passes some particular band of frequencies and reject all frequencies outside the range

Quality factor

QUESTION: calculate the voltage transfers from input to output based on varying R



$$\mathbb{G}(\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$= \frac{1}{1 + j\frac{1}{R}\left(\frac{\omega L}{\omega_0\sqrt{LC}} - \frac{\omega_0\sqrt{LC}}{\omega C}\right)}$$

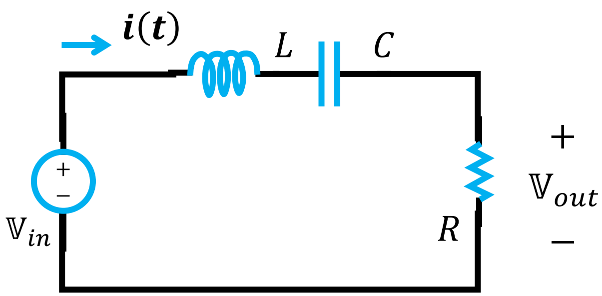
$$= \frac{1}{1 + j\frac{1}{R}\sqrt{\frac{L}{C}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

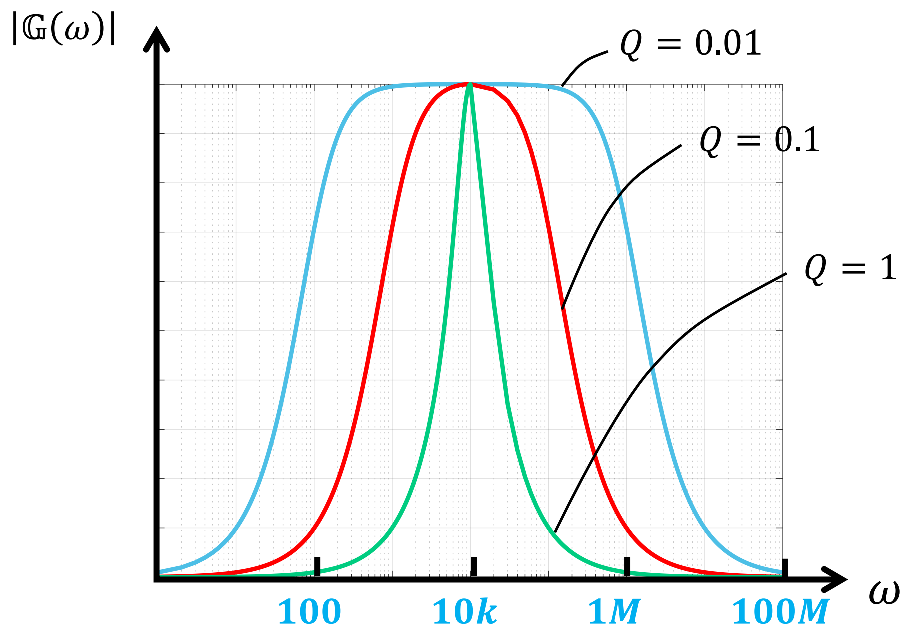
$$\left[\begin{array}{l} \mathbb{G}(\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \\ \omega_0 = \frac{1}{\sqrt{LC}} \\ |\mathbb{G}(\omega_c)| = \frac{1}{\sqrt{2}} \end{array}\right]$$

Define QUALITY FACTOR $\boldsymbol{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$

Quality factor

QUESTION: calculate the voltage transfers from input to output based on varying R





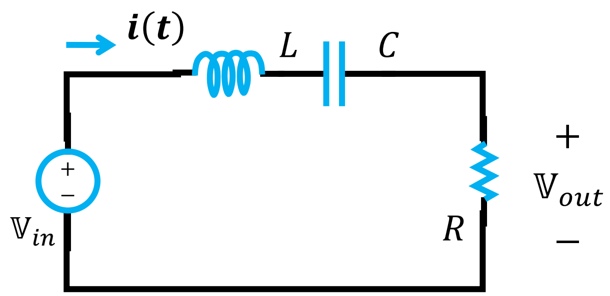
$$\mathbb{G}(\omega) = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Quality factor

QUESTION: calculate the voltage transfers from input to output based on varying R



$$\mathbb{G}(\omega) = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Let's recalculate the bandwidth BW with Q

- The magnitude $|\mathbb{G}(\omega)| = \dfrac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}$

- According to $|\mathbb{G}(\omega_c)| = \dfrac{1}{\sqrt{2}}$

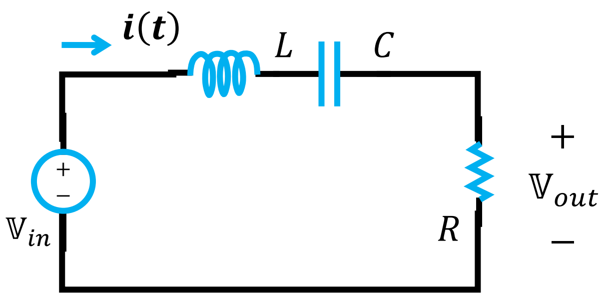
$$\rightarrow \begin{cases} \omega_{c1} = \omega_0\left[-\frac{1}{2Q} + \sqrt{\frac{1}{4Q^2+1}}\right] \\ \omega_{c2} = \omega_0\left[+\frac{1}{2Q} + \sqrt{\frac{1}{4Q^2+1}}\right] \end{cases}$$

$$BW = \frac{\omega_0}{Q}$$

A relationship between BW, Q and ω_0

Quality factor

QUESTION: calculate the voltage transfers from input to output based on varying R



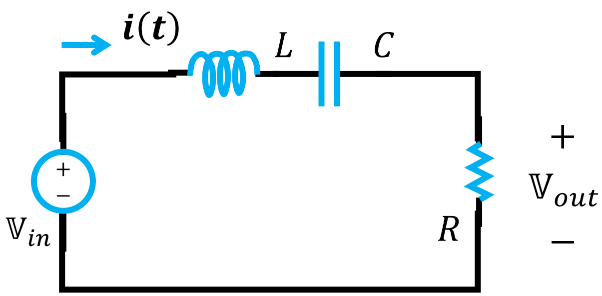
Let's go back to time domain @ resonant freq $\boldsymbol{\omega_0}$

- Assume $\boldsymbol{v_{in}(t)} = V_0 \cos(\omega_0 t)$
- The equivalent impedance @ω_0, $Z_{eq} = R$

- According to KVL, the current of the circuit $\boldsymbol{i(t)} = \frac{V_0}{R}\cos(\omega_0 t)$
- Energy stored in L, $\boldsymbol{w_L(t)} = \frac{1}{2}Li^2(t) = \frac{V_0^2 L}{2R^2}\cos^2(\omega_0 t)$
- Energy stored in C, $\boldsymbol{w_C(t)} = \frac{1}{2}Cv_C^2(t) = \frac{1}{2}C\left(\frac{1}{C}\int i dt\right)^2 = \frac{V_0^2}{2\omega_0^2 R^2 C}\sin^2(\omega_0 t)$

Quality factor

QUESTION: calculate the voltage transfers from input to output based on varying R



$$\boldsymbol{w_L(t)} = \frac{V_0^2 L}{2R^2}\cos^2(\omega_0 t)$$

$$\boldsymbol{w_C(t)} = \frac{V_0^2}{2\omega_0^2 R^2 C}\sin^2(\omega_0 t)$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Let's go back to time domain @ resonant freq ω_0

- Total stored energy

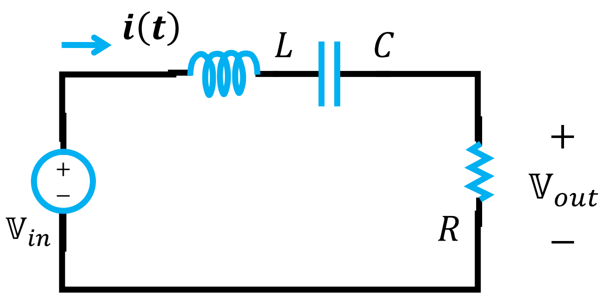
$$w_s = w_L(t) + w_C(t) = \frac{V_0^2 L}{2R^2}$$

- Total dissipated energy per cycle

$$w_D = \int_0^T i^2(t) R dt = \frac{V_0^2 T}{2R}$$

Quality factor

QUESTION: calculate the voltage transfers from input to output based on varying R



$$w_s = w_L(t) + w_C(t) = \frac{V_0^2 L}{2R^2}$$

$$w_D = w_R(t) = \frac{V_0^2 T}{2R}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Let's go back to time domain @ resonant freq ω_0

- The ratio of w_s and w_D

$$\frac{w_s}{w_D} = \frac{\frac{V_0^2 L}{2R^2}}{\frac{V_0^2 T}{2R}} = \frac{L}{RT} = \frac{\omega_0 L}{2\pi R}$$

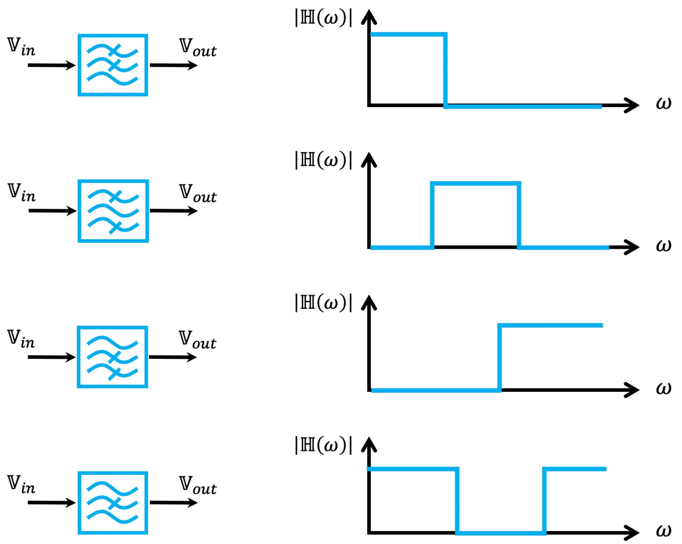
- Recall $Q = \frac{1}{R}\sqrt{L/C}$

$$Q = 2\pi \frac{w_s}{w_D}$$

***Q* is the ratio of energy stored to energy lost at the resonant frequency**

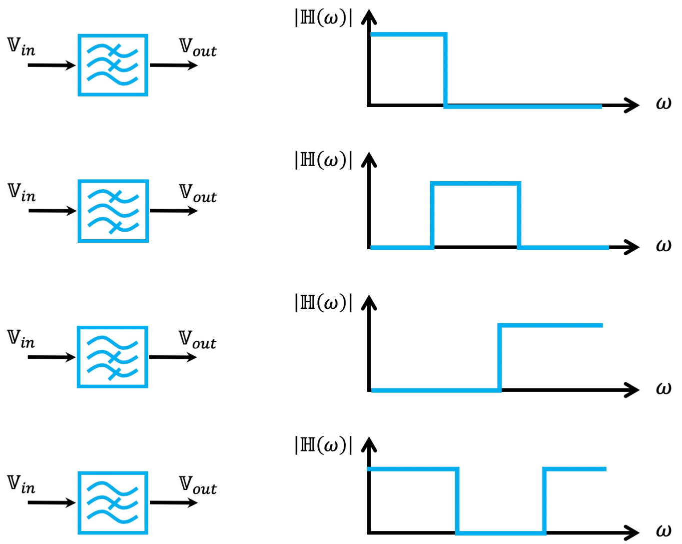
Outlines

- Transfer function
- Filters
  - Common ideal filters
  - Actual filters
    - High pass filters / low pass filters
    - Band pass filters / band stop filters
    - Quality factor $Q = 2\pi \frac{w_s}{w_D}$



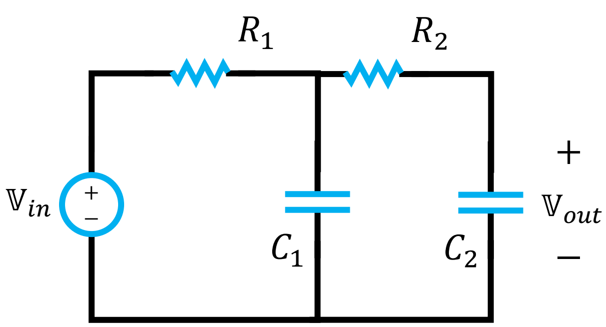
Outlines

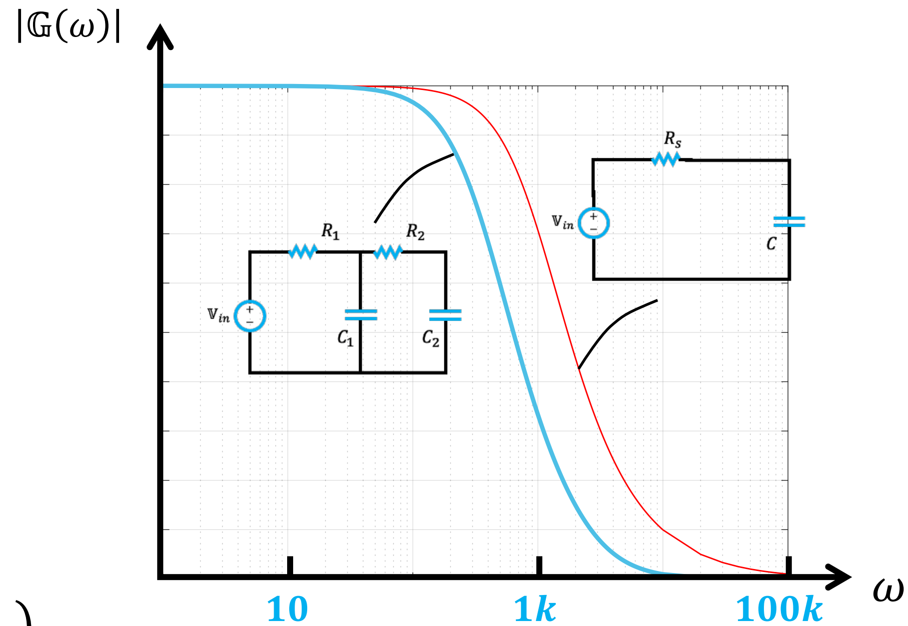
- Transfer function
- Filters
  - Common ideal filters
  - Actual filters
    - High pass filters / low pass filters
    - Band pass filters / band stop filters
    - Quality factor $Q = 2\pi \frac{w_s}{w_D}$
    - **Cascading filters**



Example 5: Cascading filters

QUESTION: find the transfer function of the circuit below





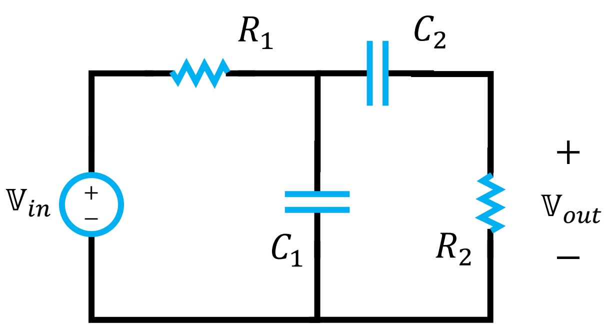
- According to KVL

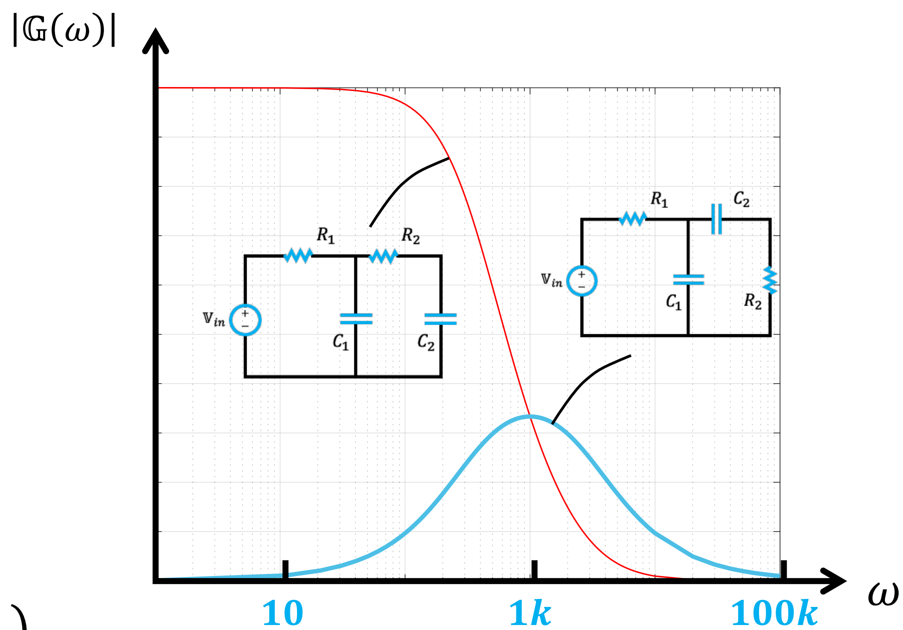
$$\mathbb{V}_{out} = \frac{Z_{C_2}}{R_2 + Z_{C_2}} \cdot \frac{Z_{C_1}||(R_2 + Z_{C_2})}{R_1 + Z_{C_1}||(R_2 + Z_{C_2})} \mathbb{V}_{in}$$

- The transfer function $\mathbb{G}(\omega) = \frac{Z_{C_2}}{R_2 + Z_{C_2}} \cdot \frac{Z_{C_1}||(R_2 + Z_{C_2})}{R_1 + Z_{C_1}||(R_2 + Z_{C_2})}$

Example 6

QUESTION: find the transfer function of the circuit below





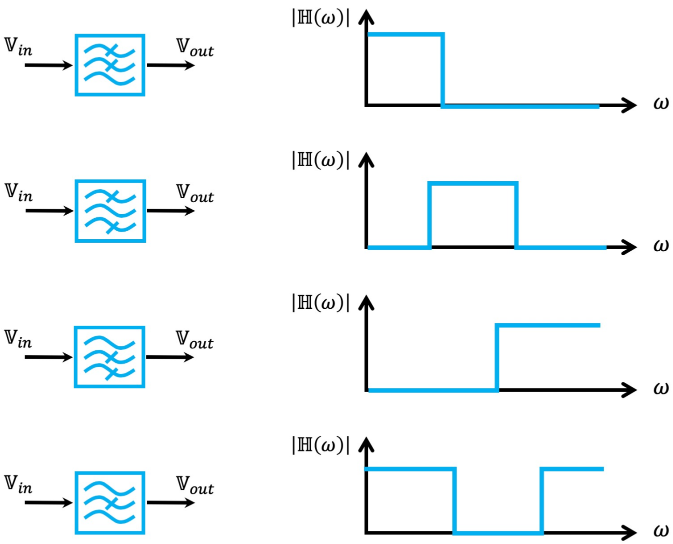
- According to KVL

$$\mathbb{V}_{out} = \frac{R_2}{R_2 + Z_{C_2}} \cdot \frac{Z_{C_1}||(R_2 + Z_{C_2})}{R_1 + Z_{C_1}||(R_2 + Z_{C_2})} \mathbb{V}_{in}$$

- The transfer function $\mathbb{G}(\omega) = \frac{R_2}{R_2 + Z_{C_2}} \cdot \frac{Z_{C_1}||(R_2 + Z_{C_2})}{R_1 + Z_{C_1}||(R_2 + Z_{C_2})}$

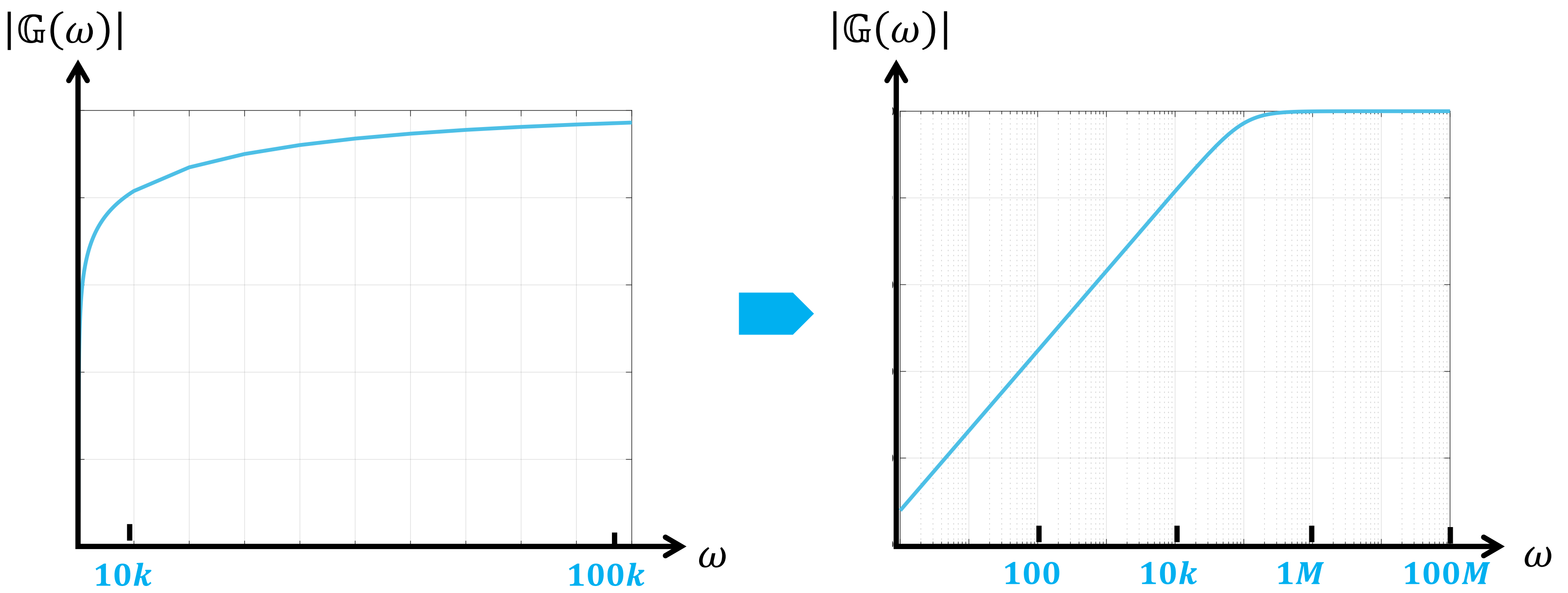
Outlines

- Transfer function
- Filters
  - Common ideal filters
  - Actual filters
    - High pass filters / low pass filters
    - Band pass filters / band stop filters
    - Quality factor $Q = 2\pi \frac{w_s}{w_D}$
    - Cascading filters
- **Bode plot**



Recall: logarithmic scale

Let's try to plot the transfer function $|\mathbb{G}(\omega)| = \dfrac{1}{\sqrt{1 + \left(\dfrac{R_1}{\omega L}\right)^2}}$

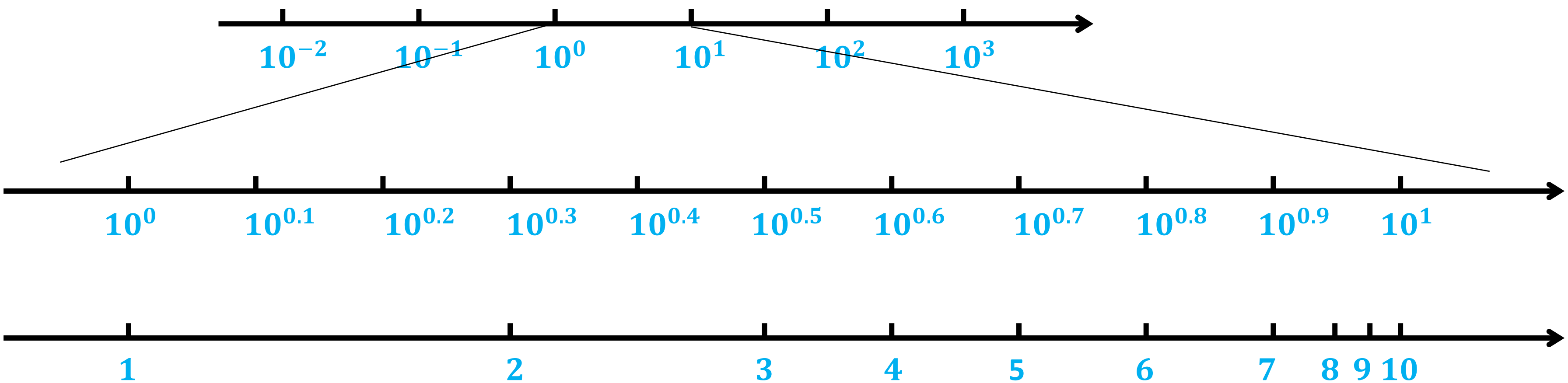


While plot the transfer function, the x-axis is usually plot in logarithmic scale

Logarithms v.s. Decibels

- Linear scale

- Powers of ten scale



Logarithms scale is a non-linear scale

Recall: Power in dB

Decibel (dB) is a unitless measurement for expressing ratios

- For power

$$10log_{10}\frac{P}{1W}$$ ➔dB

$$10log_{10}\frac{P}{1mW}$$ ➔dBm

- For voltage

$$20log_{10}\frac{v}{1V}$$ ➔dBV

$$20log_{10}\frac{v}{1mV}$$ ➔dBmV

Power gain in dB

- Define power gain as $\frac{P_{out}}{P_{in}}$

- Power gain in dB $10log_{10}\frac{P_{out}}{P_{in}}$

- Special cases

If $P_{out} = P_{in}$ $Power\ Gain\ (dB) = 10log_{10}\frac{P_{out}}{P_{in}} = 0dB$

If $P_{out} = \frac{1}{2}P_{in}$ $Power\ Gain\ (dB) = 10log_{10}\frac{P_{out}}{P_{in}} = -3dB$

If $P_{out} = 2P_{in}$ $Power\ Gain\ (dB) = 10log_{10}\frac{P_{out}}{P_{in}} = 3dB$

Voltage gain in dB

$$Power\ Gain\ (dB) = 10log_{10}\frac{P_{out}}{P_{in}}$$

$$= 10log_{10}\left(\frac{V_o^2}{R_o}\Big/\frac{V_i^2}{R_i}\right)$$

$$= 10log_{10}\left(\frac{V_o}{V_i}\right)^2 + 10log_{10}\left(\frac{R_i}{R_o}\right)$$

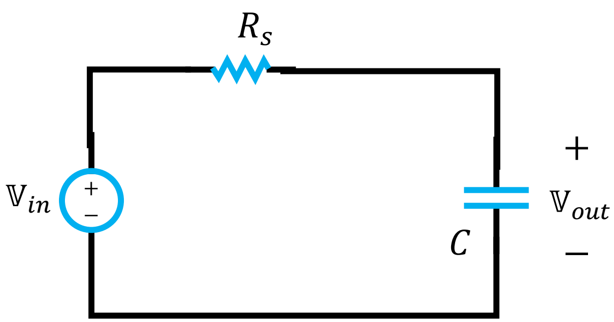
If $R_i = R_o$

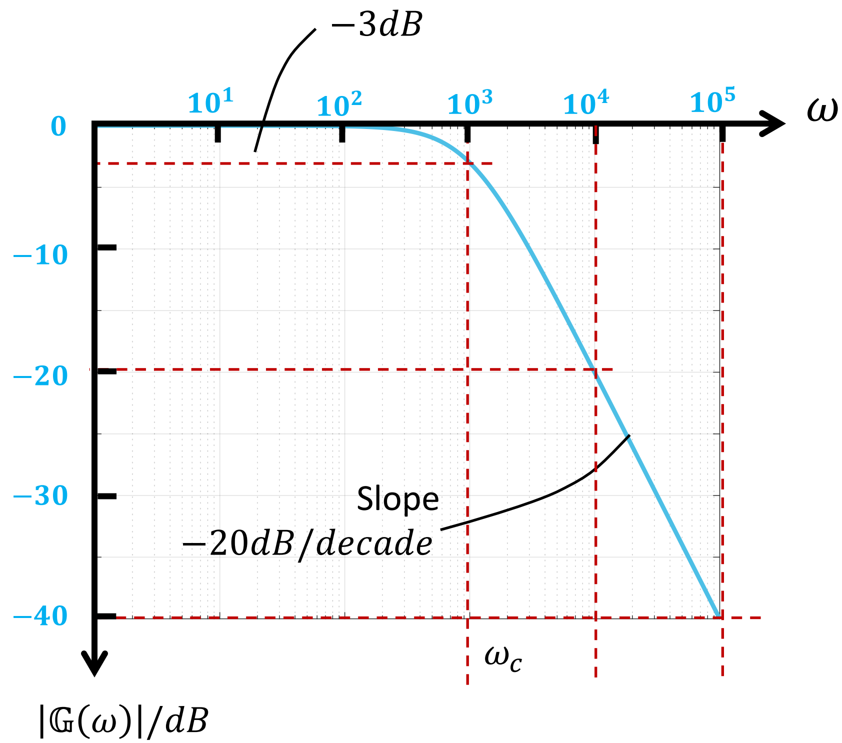
$$= 20log_{10}\left(\frac{V_o}{V_i}\right) + 10log_{10}(1)$$

If $R_i = R_o$

$$Power\ Gain\ (dB) = 10log_{10}\frac{P_{out}}{P_{in}} = 20log_{10}\left(\frac{V_o}{V_i}\right) = Voltage\ Gain$$

Transfer function with logs & dB plot





$$|\mathbb{G}(\omega)| = \frac{1}{\sqrt{1+(\omega C R_s)^2}}$$

$$|\mathbb{G}(10\omega_c)| = \frac{1}{\sqrt{1+(10\omega_c C R_s)^2}}$$

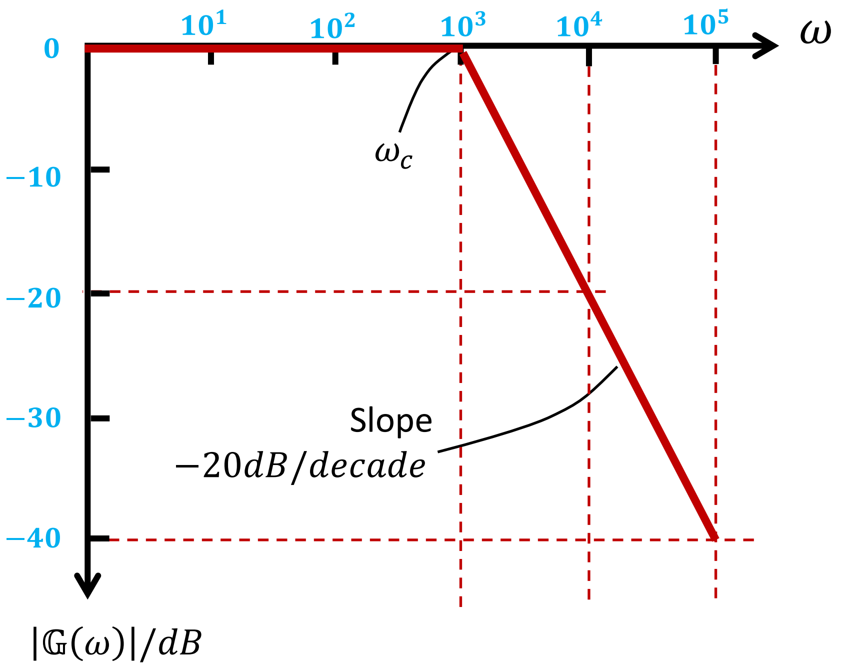
$$\approx \frac{1}{\sqrt{(10\omega_c C R_s)^2}} = \frac{1}{10\sqrt{(\omega_c C R_s)^2}}$$

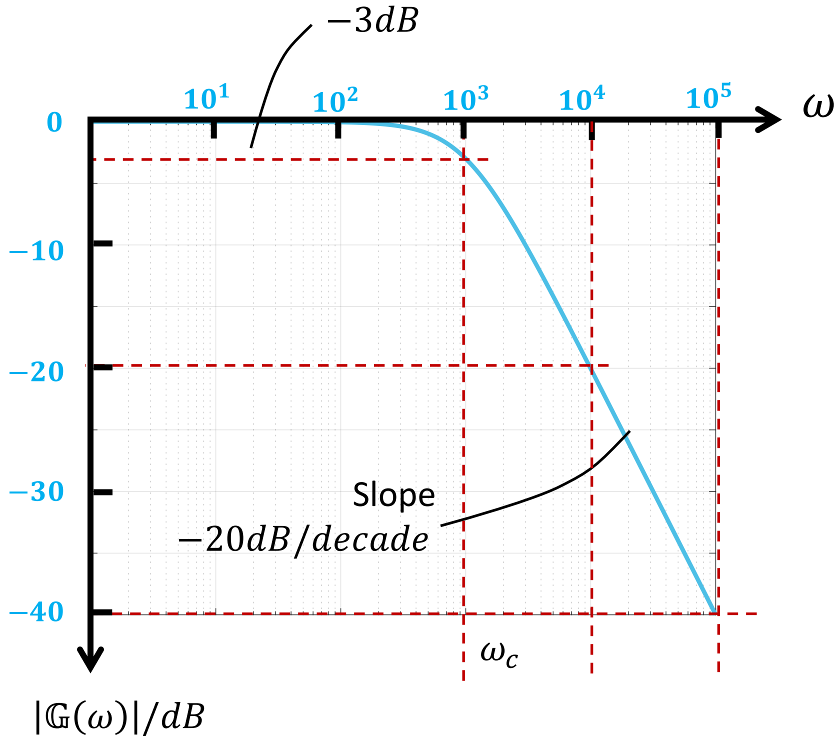
$$\approx \frac{1}{10\sqrt{1+(\omega_c C R_s)^2}} = \frac{1}{10}|\mathbb{G}(\omega_c)|$$

$$|\mathbb{G}(\omega_c)|(dB) - |\mathbb{G}(10\omega_c)|(dB) = 20log_{10}\left(\frac{|\mathbb{G}(\omega_c)|}{|\mathbb{G}(10\omega_c)|}\right) \approx \mathbf{20dB}$$

Bode plot

BODE PLOT generates a "straight-line" approximation of the transfer function





The slope of the dB plot is called the "roll-off" rate
The amplitude of the output signal is attenuated faster with higher roll-off

Bode plot

Transfer function in general form

$$\mathbb{H}(\omega) = H_0 \frac{\left(1+\frac{j\omega}{\omega_{z1}}\right)\left(1+\frac{j\omega}{\omega_{z2}}\right)\cdots\left(1+\frac{j\omega}{\omega_{zm}}\right)}{\left(1+\frac{j\omega}{\omega_{p1}}\right)\left(1+\frac{j\omega}{\omega_{p2}}\right)\cdots\left(1+\frac{j\omega}{\omega_{pn}}\right)}$$

$$\omega_{z1} < \omega_{z2} < \cdots < \omega_{zm}$$

$$\omega_{p1} < \omega_{p2} < \cdots < \omega_{pn}$$

$$= H_0 \frac{A_{z1}\angle\phi_{z1} \cdot A_{z2}\angle\phi_{z2} \cdots A_{zm}\angle\phi_{zm}}{A_{p1}\angle\phi_{p1} \cdot A_{p2}\angle\phi_{p2} \cdots A_{pn}\angle\phi_{pn}}$$

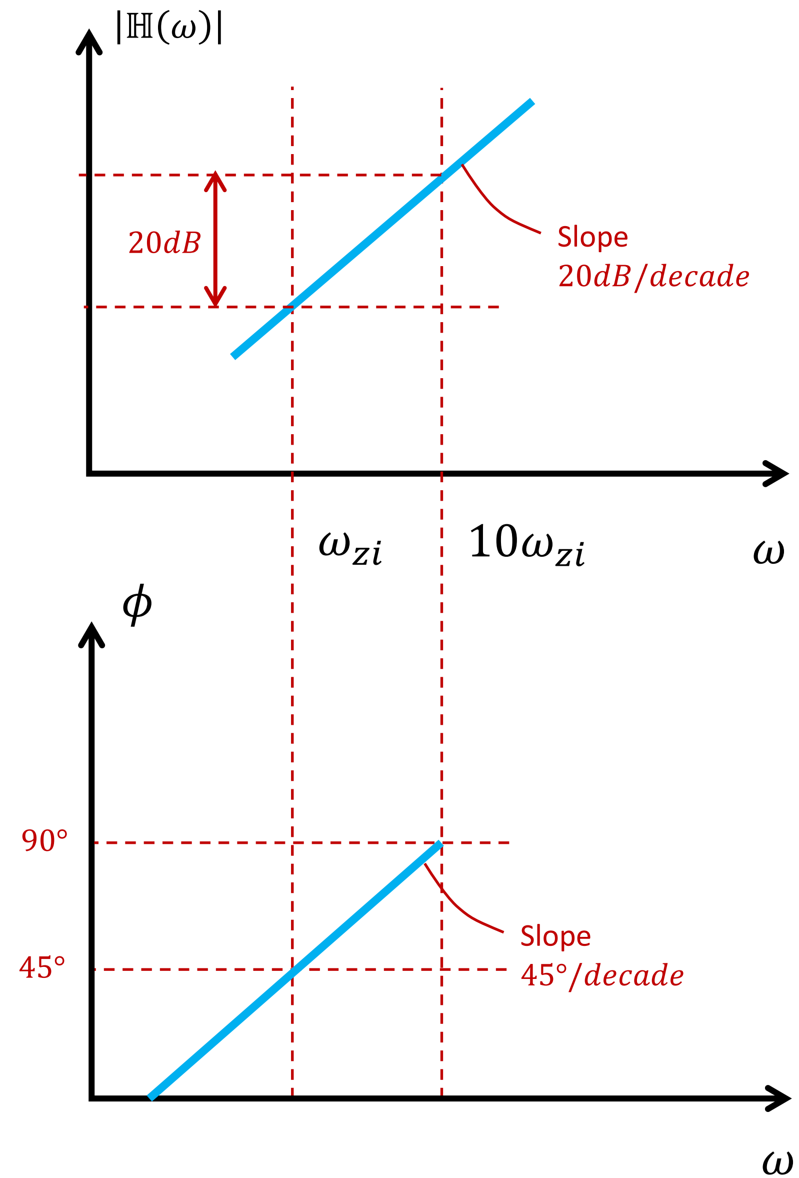
$$\begin{cases} |\mathbb{H}(\omega)| = |H_0| \dfrac{A_{z1} \cdot A_{z2} \cdots A_{zm}}{A_{p1} \cdot A_{p2} \cdots A_{pn}} \\ \angle\mathbb{H}(\omega) = \phi_{z1} + \phi_{z2} + \cdots + \phi_{zm} - \phi_{p1} - \phi_{p2} - \cdots - \phi_{pn} \end{cases}$$

Bode plot

- When is ω very low $\qquad \mathbb{H}(\omega) \xrightarrow{\omega \to 0} H_0$

- Consider $\qquad \mathbb{H}_{zi}(\omega) = 1 + \dfrac{j\omega}{\omega_{zi}}$

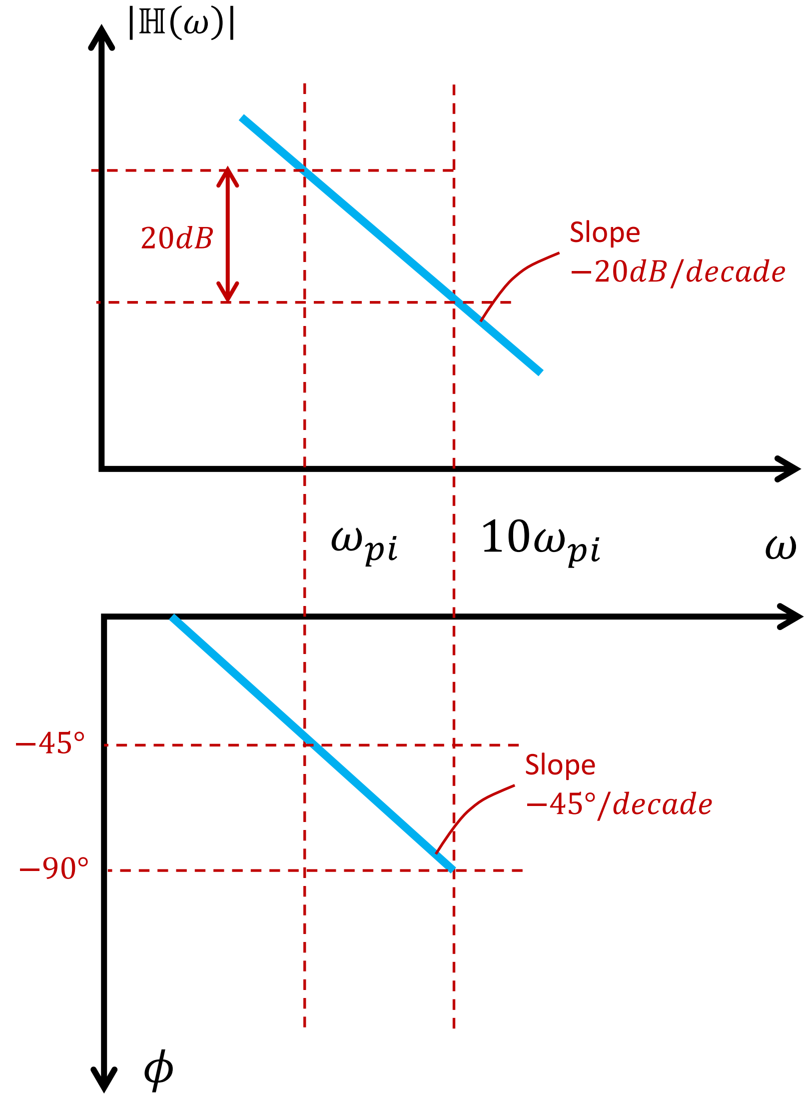
$$\begin{cases} \mathbb{H}_{zi}(\omega) \xrightarrow{\omega \ll \omega_{zi}} 1 \\ \mathbb{H}_{zi}(\omega) \xrightarrow{\omega = \omega_{zi}} 1 + j \\ \mathbb{H}_{zi}(\omega) \xrightarrow{\omega = 10\omega_{zi}} 10j \approx 10\mathbb{H}_{zi}(\omega_{zi}) \\ \mathbb{H}_{zi}(\omega) \xrightarrow{\omega \gg \omega_{zi}} \dfrac{j\omega}{\omega_{zi}} \end{cases}$$



Bode plot

- Consider $\mathbb{H}_{pi}(\omega) = \dfrac{1}{1 + \dfrac{j\omega}{\omega_{pi}}}$

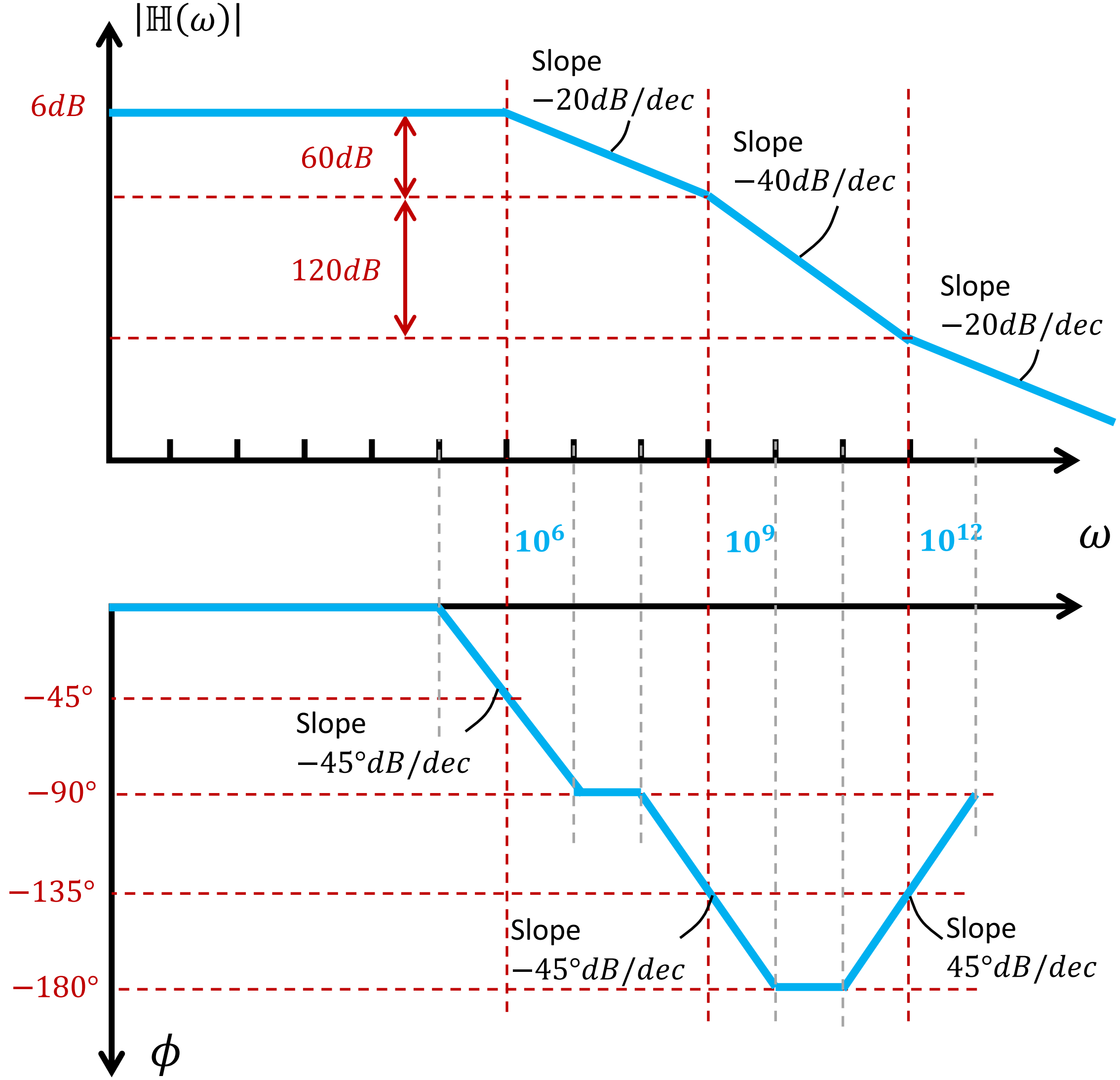
$$\begin{cases} \mathbb{H}_{pi}(\omega) \xrightarrow{\omega \ll \omega_{pi}} 1 \\ \mathbb{H}_{pi}(\omega) \xrightarrow{\omega = \omega_{pi}} \dfrac{1}{1+j} \\ \mathbb{H}_{pi}(\omega) \xrightarrow{\omega = 10\omega_{pi}} \dfrac{1}{10j} \approx \dfrac{1}{10} \mathbb{H}_{pi}(\omega_{pi}) \\ \mathbb{H}_{pi}(\omega) \xrightarrow{\omega \gg \omega_{pi}} \dfrac{\omega_{pi}}{j\omega} \end{cases}$$



Example 7

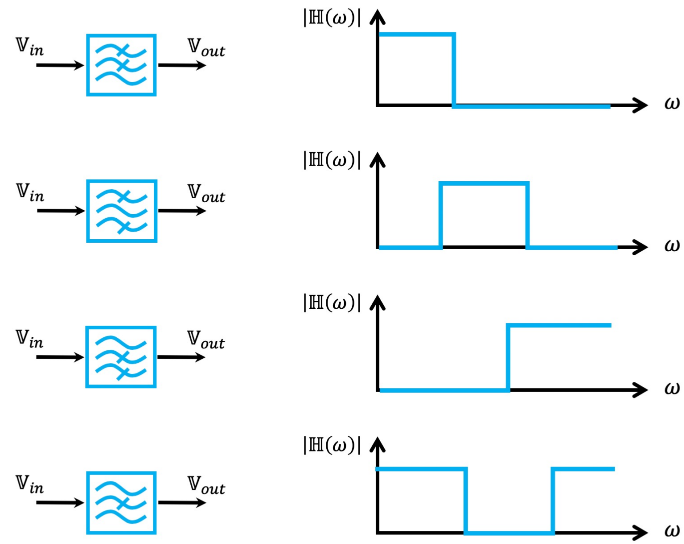
$$\mathbb{H}(\omega) = 2\frac{\left(1+\frac{j\omega}{10^{12}}\right)}{\left(1+\frac{j\omega}{10^{6}}\right)\left(1+\frac{j\omega}{10^{9}}\right)}$$

$$|\mathbb{H}(\omega=0)| = 20log_{10}2 = 6dB$$



Outlines

- Transfer function
- Filters
  - Common ideal filters
  - Actual filters
    - High pass filters / low pass filters
    - Band pass filters / band stop filters
    - Quality factor $Q = 2\pi \frac{w_s}{w_D}$
    - Cascading filters
- Bode plot
- Circuit element models in s-domain



The Laplace transform

YOU WILL LEARN IT LATER IN SIGNAL & SYSTEM

The Laplace transform of a function is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \qquad \text{where } s = \sigma + j\omega$$

The inverse Laplace transform is defined as

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{-st}dt$$

The useful Laplace transform pairs

$$\delta(t) \leftrightarrow 1 \qquad u(t) \leftrightarrow \frac{1}{s} \qquad e^{-at} \leftrightarrow \frac{1}{s+a} \qquad t \leftrightarrow \frac{1}{s^2}$$

Circuit element models

			
i-v characteristic	$i = \frac{v}{R}$	$i(t) = C\frac{dv(t)}{dt}$	$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(x)dx$
v-i characteristic	$v = iR$	$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)dx$	$v(t) = L\frac{di(t)}{dt}$
Impedance	R	$\frac{1}{j\omega C}$	$j\omega L$
s–domain model	$\boldsymbol{R}$	$\boldsymbol{\frac{1}{sC}}$	$\boldsymbol{sL}$
s–domain i-v characteristic	$\boldsymbol{I(s) = \frac{V(s)}{R}}$	$\boldsymbol{I(s) = sCV(s)}$	$\boldsymbol{I(s) = \frac{V(s)}{sL}}$

Outlines

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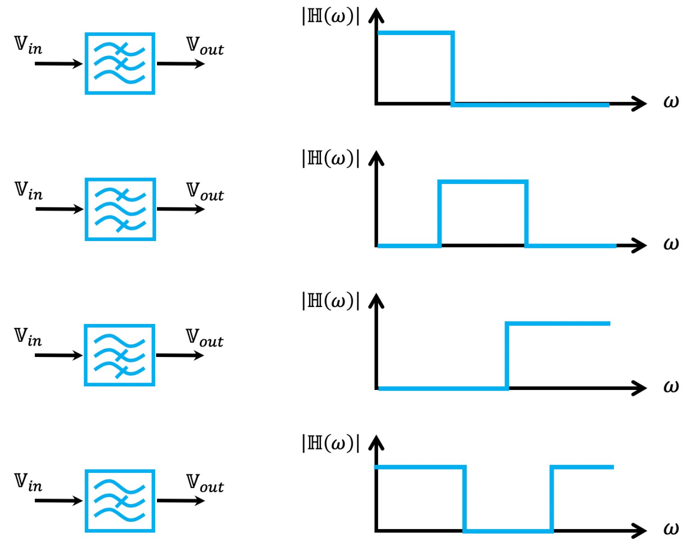

i-v relation	$i = \frac{v}{R}$	$i(t) = C\frac{dv(t)}{dt}$	$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(x)dx$
v-i relation	$v = iR$	$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)dx$	$v(t) = L\frac{di(t)}{dt}$
Impedance	R	$\frac{1}{j\omega C}$	$j\omega L$
s–domain model	R	$\frac{1}{sC}$	sL
s–domain i-v relation	$I(s) = \frac{V(s)}{R}$	$I(s) = sCV(s)$	$I(s) = \frac{V(s)}{sL}$

Reading tasks & learning goals

- Reading tasks
  - Basic Engineering Circuit Analysis, 10^{th} edition
    - Chapter 12.1-12.4, 12.5 before active filters
- Learning goals
  - Understand the variable-frequency performance of R, L and C
  - Be able to sketch and to understand a **Bode plot**
  - Know how to analyze series and parallel **resonant circuits**
  - Understand the concept of **magnitude**/**frequency**/**Quality factor**
  - Understand the characteristics of **basic filters**
  - Understand s–domain representations of basic circuit elements