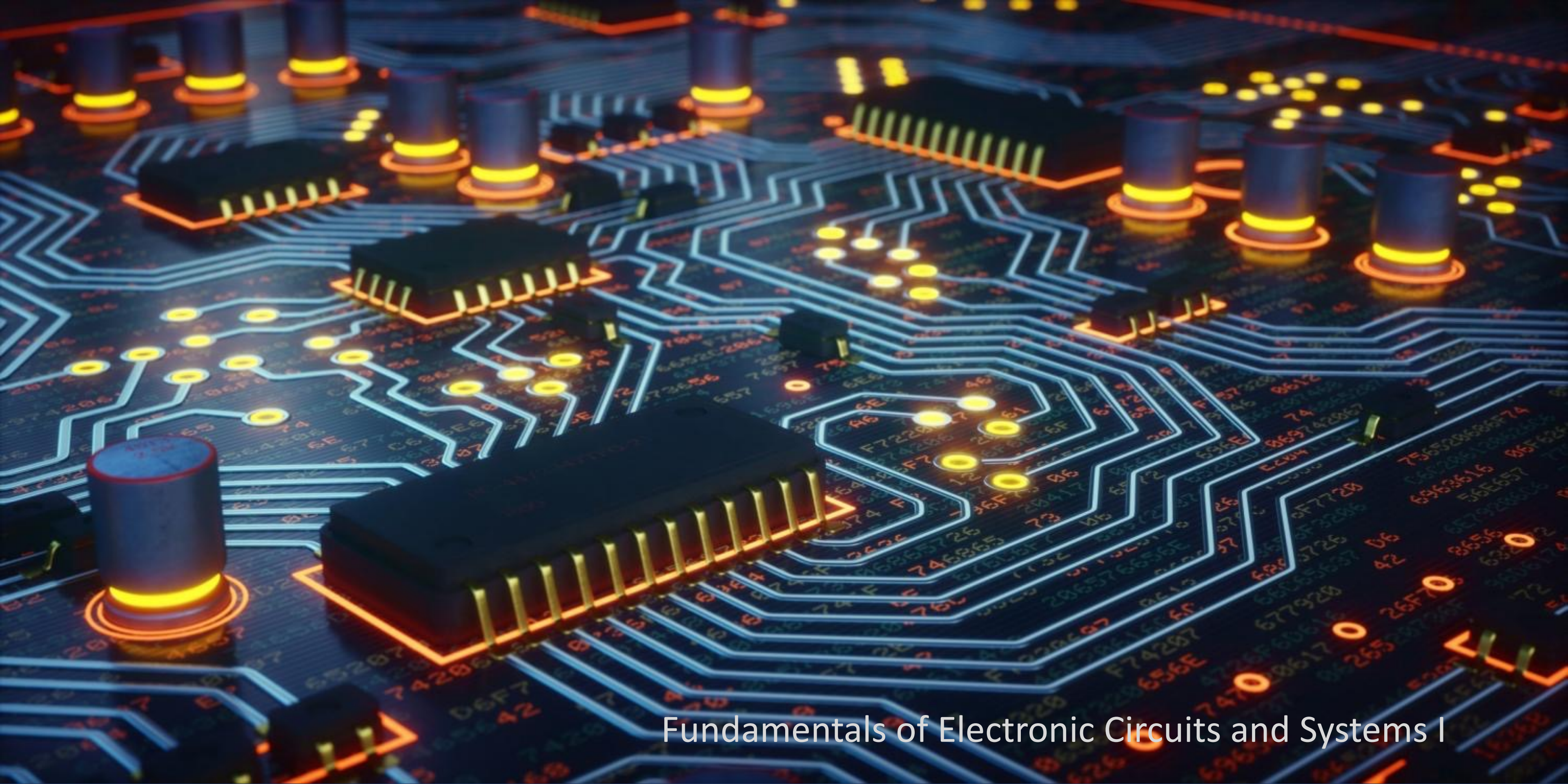


Fundamentals of Electronic Circuits and Systems I

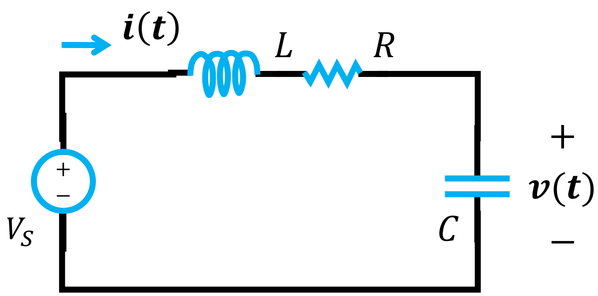
AC Steady-State Analysis

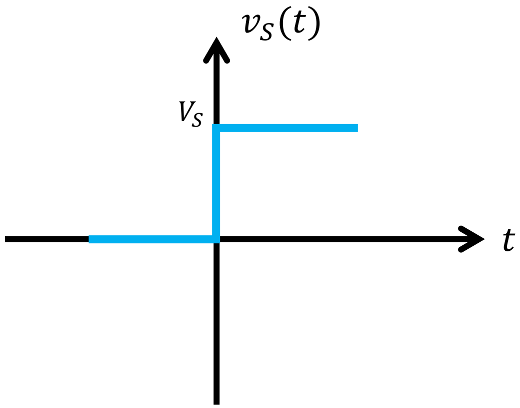
Milin Zhang
Dept of EE, Tsinghua University

Recall

- How to analyze 1st/2nd order circuit in time domain?
  - Write the circuit equation according to KVL/KCL
  - Solve the differential equation
    - Step 1a: Find the **particular integral solution** $v_p(t)$
    - Step 1b: substitute $v_p(t)$ to the equation to solve the unknown
    - Step 2a: find the **homogeneous equation**
    - Step 2b: find the **complementary solution** $v_c(t)$ to the homogeneous equation
    - Step 3a: find the initial voltage/current values
    - Step 3b: substitute the initials to the full solution to solve the unknown

Recall: Particular integral solution to step forcing func.





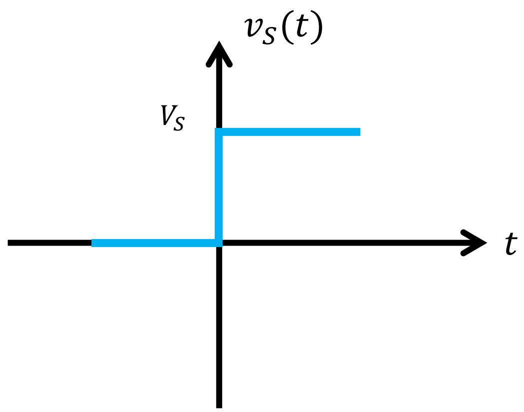
- Circuit equations according to KVL/KCL $\dfrac{d^2v(t)}{dt^2}+\dfrac{R}{L}\dfrac{dv(t)}{dt}+\dfrac{1}{LC}v(t)=\dfrac{V_S}{LC}$

- Particular integral solution $v_p(t)=V_S$

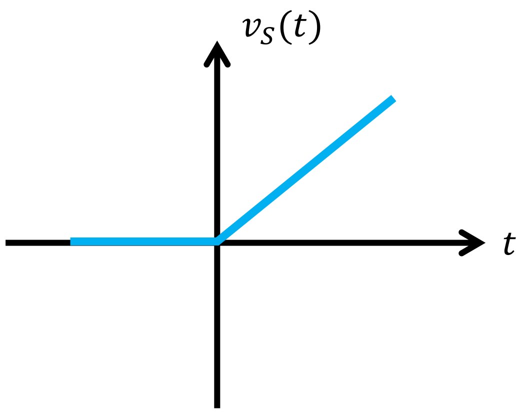
Outlines

- Response to different forcing function

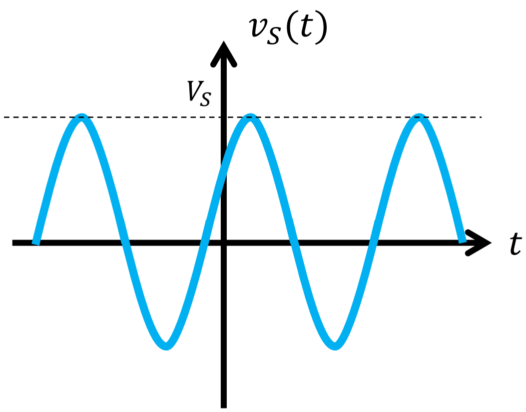
More forcing functions



Step function



Ramp function

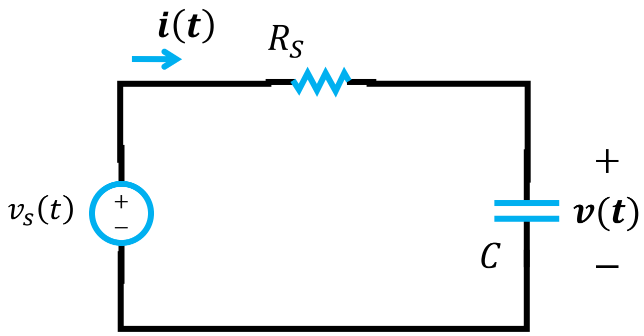


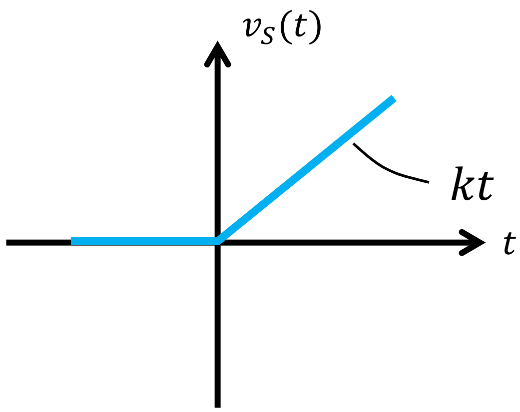
Sinusoidal function

Assume $x_p(t) = K$

Example 1: ramp forcing function

QUESTION: Assume there is no charge on the capacitor C at $t = -\infty$. Find the voltage response to the ramping forcing function.





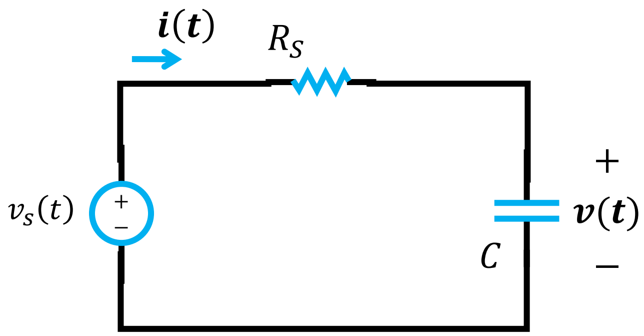
- According to KVL

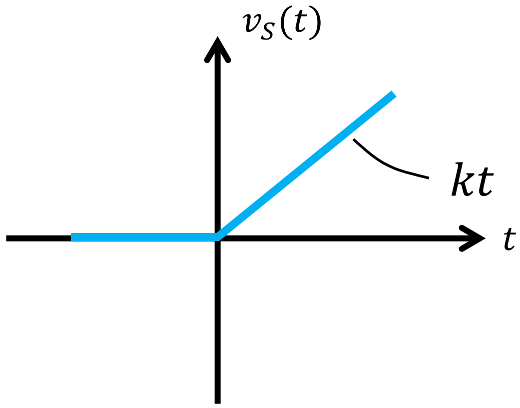
$$i_R R + v(t) = v_S(t)$$

$$\rightarrow \quad \frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}kt$$

Example 1: ramp forcing function

QUESTION: Assume there is no charge on the capacitor C at $t = -\infty$. Find the voltage response to the ramping forcing function.





$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}kt$$

- Step 1a: find the **particular integral solution $v_p(t)$**

Assume $v_p(t) = pt + q$

- Step 1b: substitute $v_p(t)$ to the equation

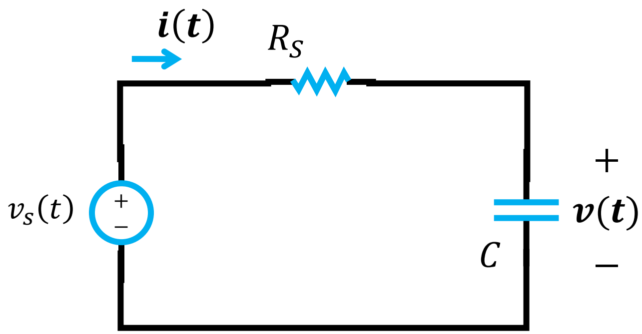
$$p + \frac{pt+q}{RC} = \frac{1}{RC}kt$$

→ $pt + (RCp + q) = kt$

→ $p = k, q = -RCk$

Example 1: ramp forcing function

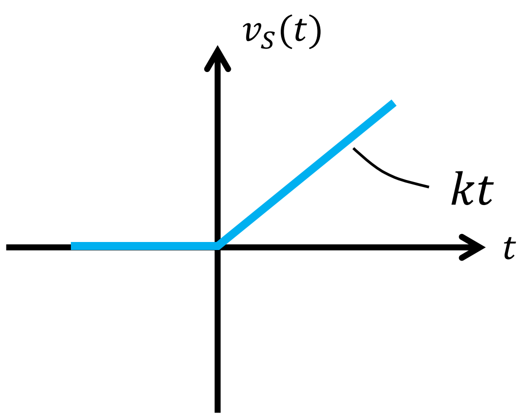
QUESTION: Assume there is no charge on the capacitor C at $t = -\infty$. Find the voltage response to the ramping forcing function.



$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}kt$$

- Step 1: find the **particular integral solution $v_p(t)$**

$$v_p(t) = kt - RCq$$

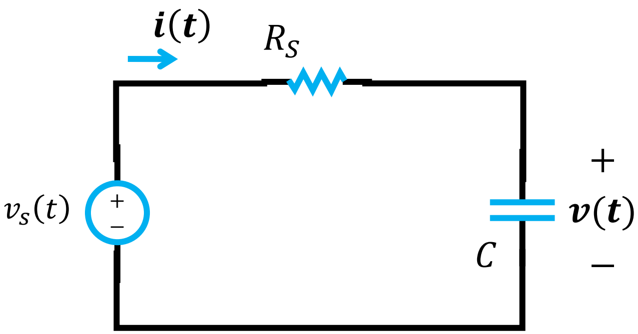


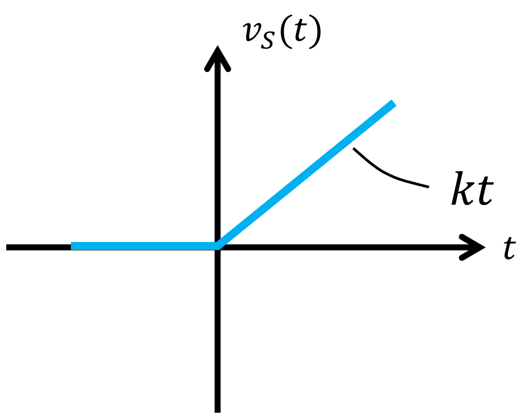
- Step 2: find the **complementary solution** $v_c(t)$

$$v_c(t) = K_2 e^{-a_1 t} \qquad \text{where } a_1 = \frac{1}{RC}$$

Example 1: ramp forcing function

QUESTION: Assume there is no charge on the capacitor C at $t = -\infty$. Find the voltage response to the ramping forcing function.





$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}kt$$

- Full solution

$$v(t) = kt - RCk + K_2 e^{-a_1 t} \quad \text{where } a_1 = \frac{1}{RC}$$

- Step 3a: find the value of $v(t)$ at one instant of time

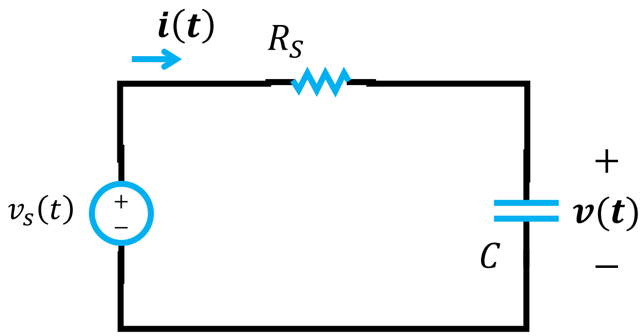
$$v(0) = 0$$

- Step 3b: substitute $v(0) = 0$ to $v(t)$

$$\rightarrow \quad K_2 = RCk$$

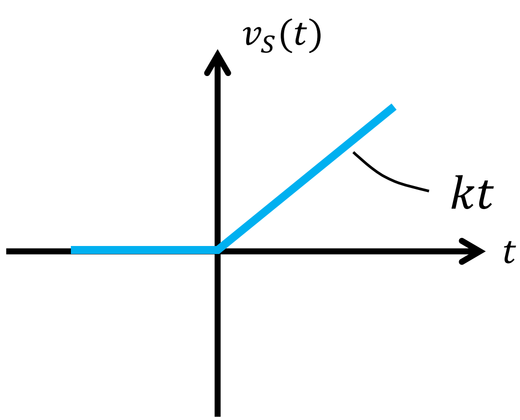
Example 1: ramp forcing function

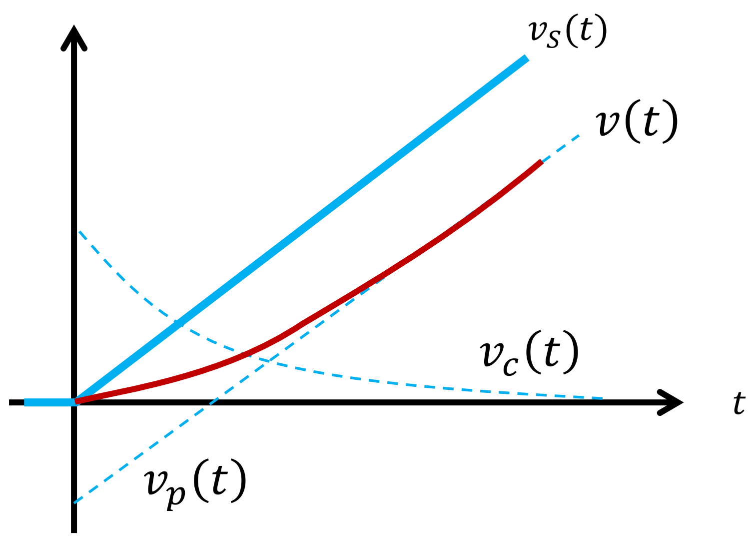
QUESTION: Assume there is no charge on the capacitor C at $t = -\infty$. Find the voltage response to the ramping forcing function.



$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}kt$$

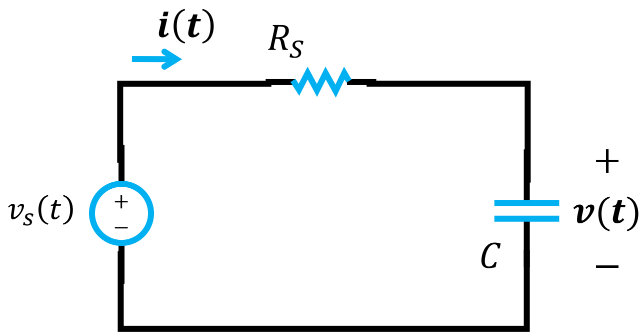
Solution $\quad v(t) = kt - RCk + RCke^{-\frac{1}{RC}}$



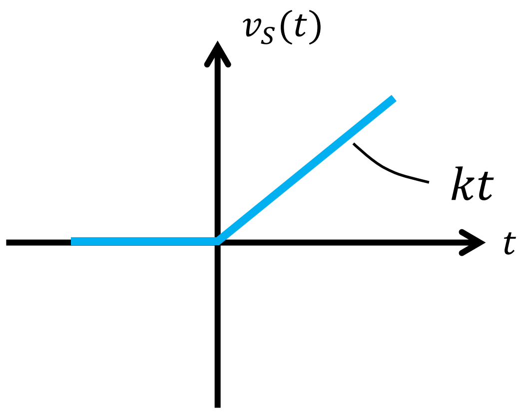


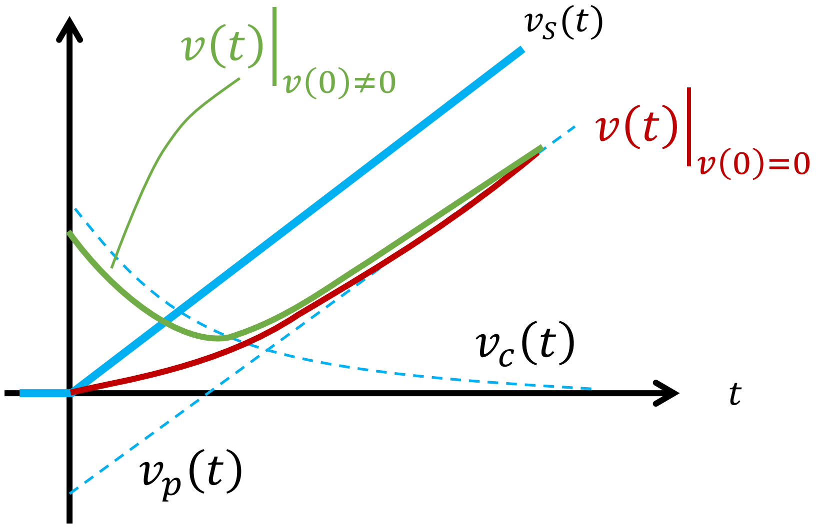
Example 1: ramp forcing function

QUESTION: Assume there is no charge on the capacitor C at $t = -\infty$. Find the voltage response to the ramping forcing function.

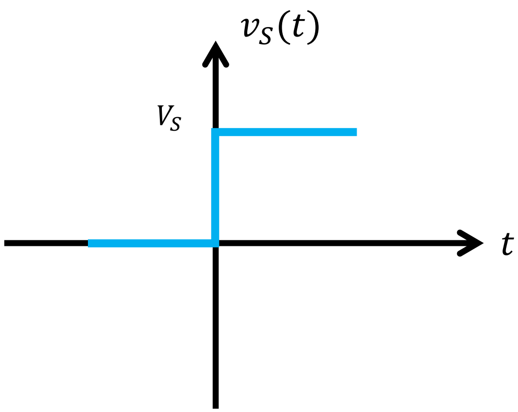


What if $v(0) \neq 0$?



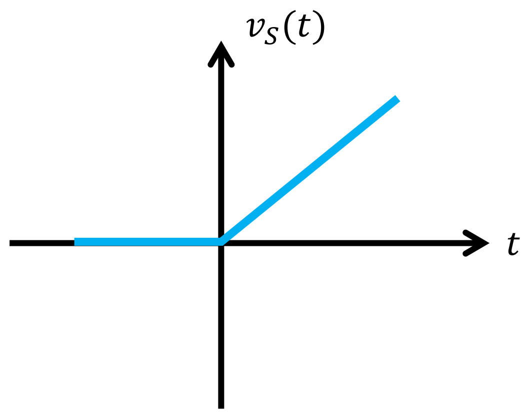


More forcing functions



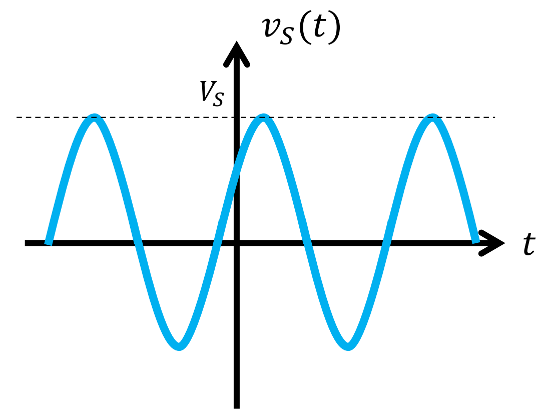
Step function

Assume $x_p(t) = K$



Ramp function

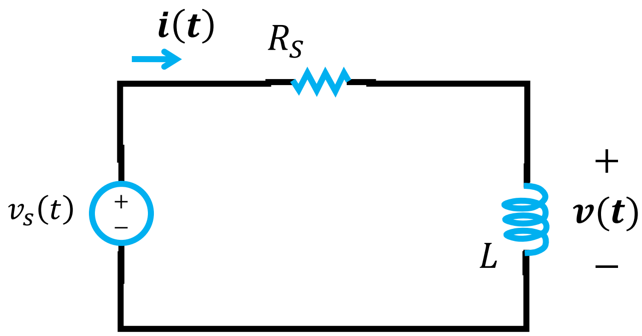
Assume $x_p(t) = pt + q$

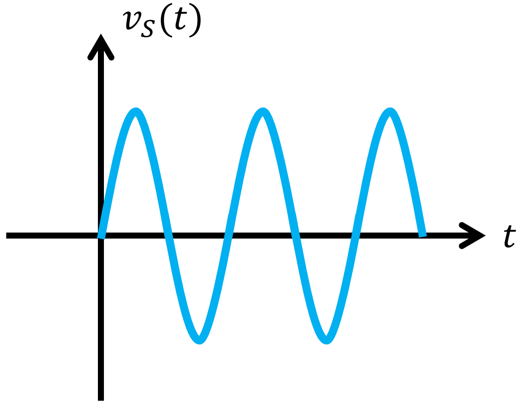


Sinusoidal function

Example 2: Sinusoidal forcing function

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to the sinusoidal forcing function.





$v_S(t) = V_S \sin(\omega t)$

- According to KVL

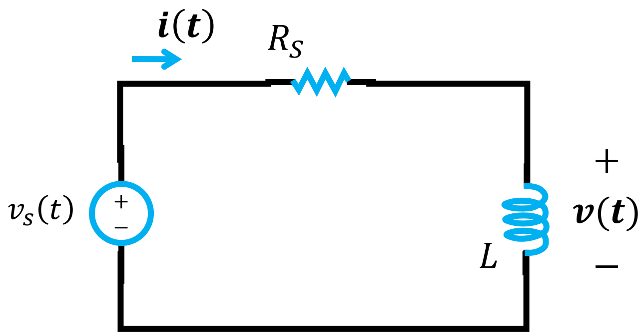
$$i(t)R + v_L(t) = v_S(t)$$

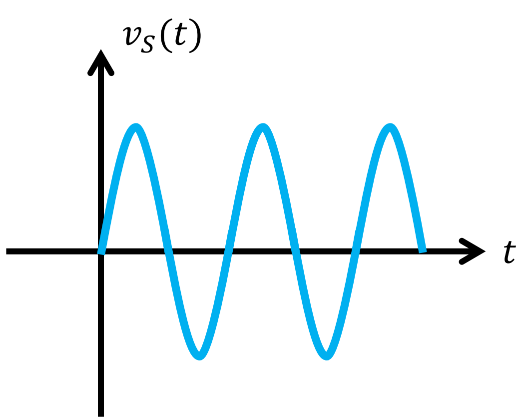
$$\rightarrow \quad i(t)R + L\frac{d}{dt}i(t) = v_S(t)$$

$$\rightarrow \quad \frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{1}{L}v_S(t)$$

Example 2: Sinusoidal forcing function

QUESTION: Assume there is no charge on the inductor L at $t=-\infty$. Find the voltage response to the sinusoidal forcing function.





$v_S(t) = V_S \sin(\omega t)$

$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{1}{L}v_S(t)$$

- Step 1a: find the **particular integral solution $i_p(t)$**

Assume $i_p(t) = K_1\cos(\omega t) + K_2\sin(\omega t)$

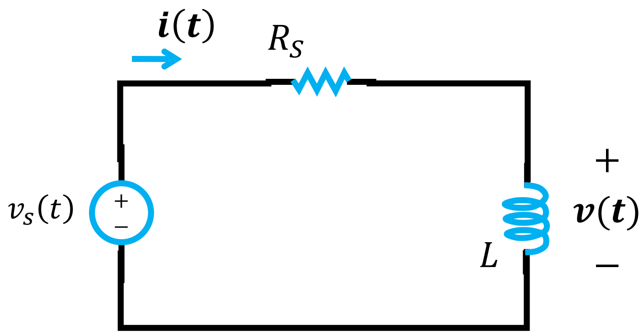
- Step 1b: substitute $i_p(t)$ to the equation

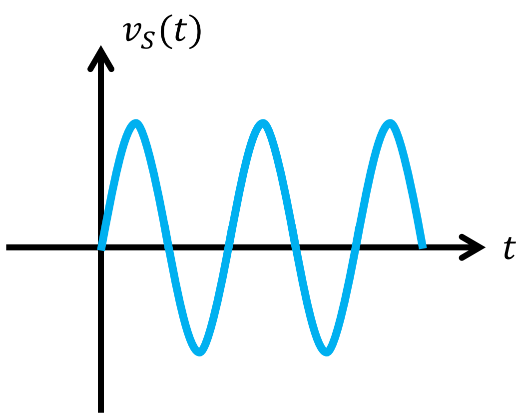
$$V_S\sin(\omega t) = K_1R\cos(\omega t) + K_2R\sin(\omega t) + L\omega K_2\cos(\omega t) - L\omega K_1\sin(\omega t)$$

$$\rightarrow \begin{cases} V_S = K_2R - L\omega K_1 \\ 0 = K_1R + L\omega K_2 \end{cases} \rightarrow \begin{cases} K_1 = V_S\dfrac{-\omega L}{R^2+\omega^2L^2} \\ K_2 = V_S\dfrac{R}{R^2+\omega^2L^2} \end{cases}$$

Example 2: Sinusoidal forcing function

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to the sinusoidal forcing function.





$v_S(t) = V_S \sin(\omega t)$

$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{1}{L}v_S(t)$$

- Step 1: find the **particular integral solution $i_p(t)$**

$$i_p(t) = K_1 \cos(\omega t) + K_2\, sin(\omega t)$$

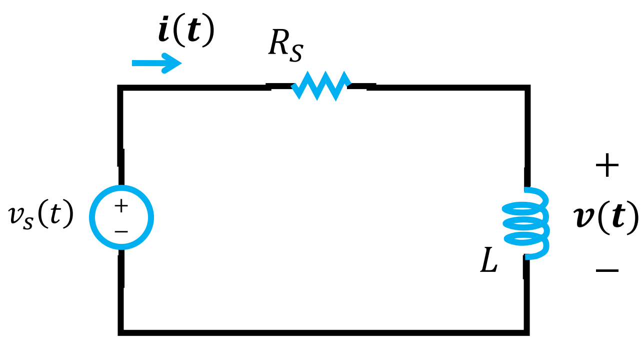
where $\begin{cases} K_1 = V_S \dfrac{-\omega L}{R^2 + \omega^2 L^2} \\ K_2 = V_S \dfrac{R}{R^2 + \omega^2 L^2} \end{cases}$

- Step 2: find the **complementary solution** $i_c(t)$

$$i_c(t) = K_3 e^{-\frac{R}{L}t}$$

Example 2: Sinusoidal forcing function

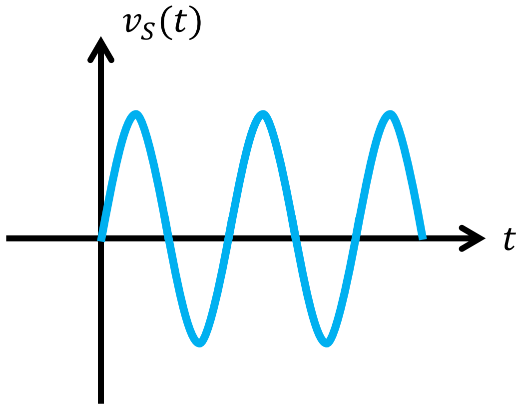
QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to the sinusoidal forcing function.



$$\frac{d}{dt} i(t) + \frac{R}{L} i(t) = \frac{1}{L} v_S(t)$$

- Full solution

$$i(t) = K_1 \cos(\omega t) + K_2 \sin(\omega t) + K_3 e^{-\frac{R}{L}t}$$



$v_S(t) = V_S \sin(\omega t)$

- Step 3a: find the value of $i(t)$ at one instant of time

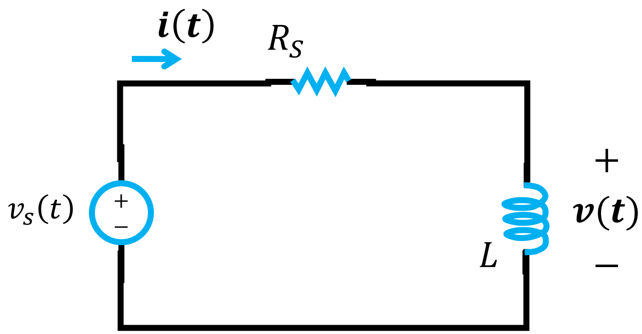
$$i(0) = 0$$

- Step 3b: substitute $i(0) = 0$ to $i(t)$

$$\rightarrow \quad K_3 = \frac{\omega L V_S}{R^2 + \omega^2 L^2}$$

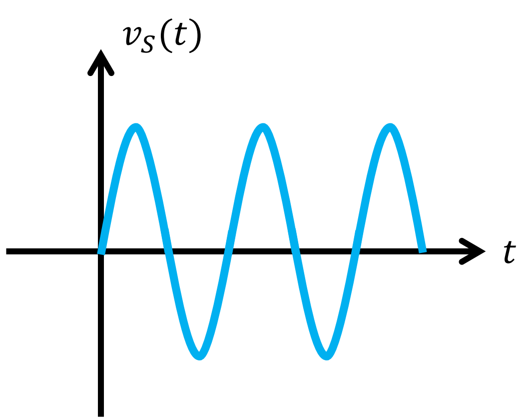
Example 2: Sinusoidal forcing function

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to the sinusoidal forcing function.

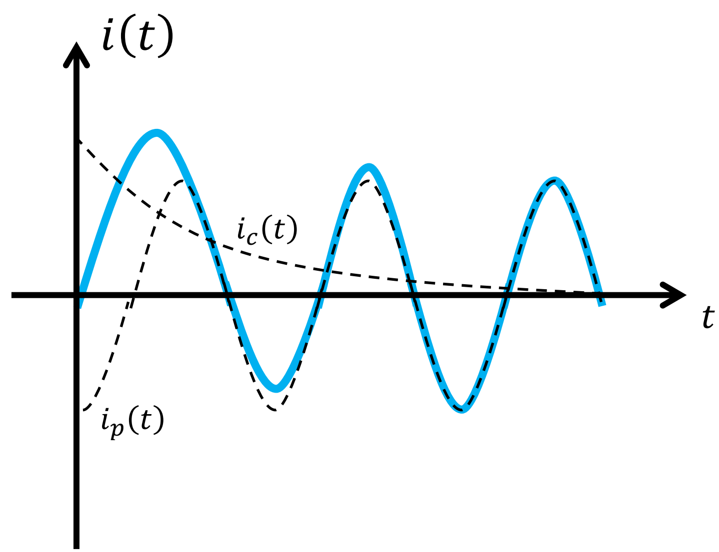


$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{1}{L}v_S(t)$$

Solution $\quad i(t) = K_1\cos(\omega t) + K_2\sin(\omega t) + K_3 e^{-\frac{R}{L}t}$



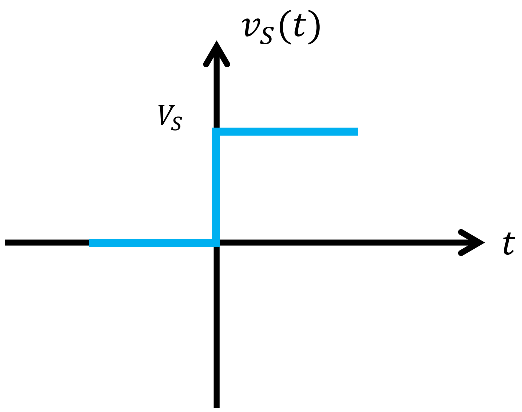
$v_S(t) = V_S\sin(\omega t)$



where

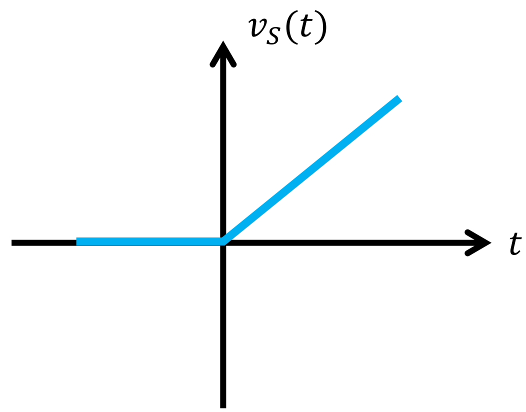
$$\begin{cases} K_1 = V_S\dfrac{-\omega L}{R^2 + \omega^2 L^2} \\ K_2 = V_S\dfrac{R}{R^2 + \omega^2 L^2} \\ K_3 = V_S\dfrac{\omega L}{R^2 + \omega^2 L^2} \end{cases}$$

More forcing functions



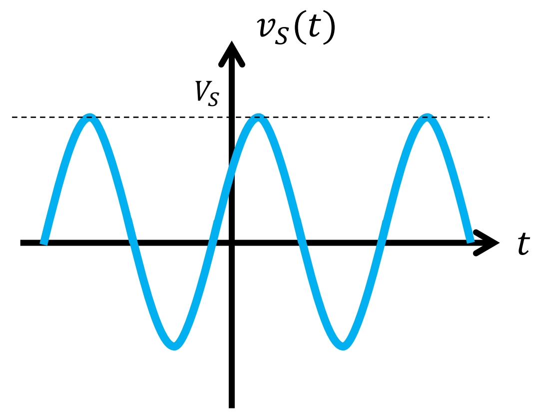
Step function

Assume $x_p(t) = K$



Ramp function

Assume $x_p(t) = pt + q$



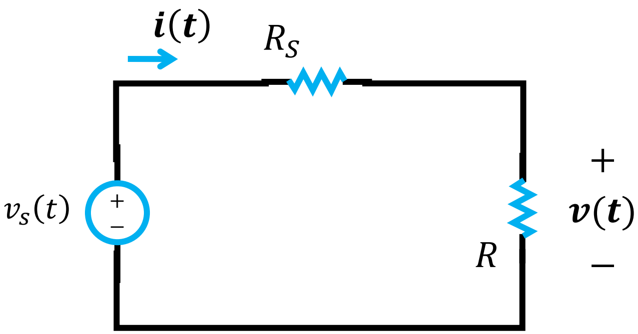
Sinusoidal function

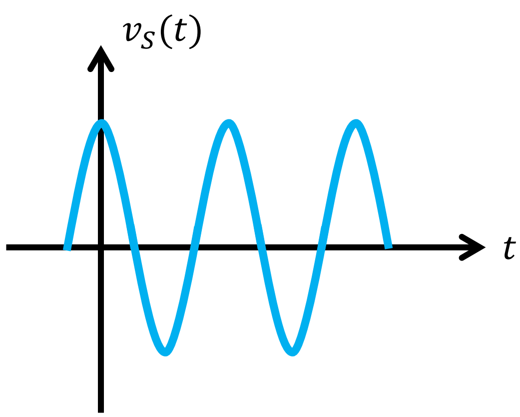
Assume $x_p(t) = K_1 \sin(\omega t)$
$+K_2 \cos(\omega t)$

Outlines

- Response to different forcing function
  - Ramping forcing function
  - Sinusoidal forcing function
- **Complex forcing function**

Resistors with a sin. forcing func.





$v_S(t) = V_S\cos(\omega t)$

With $v_{S1}(t) = V_S\,cos(\omega t)$

- According to KVL

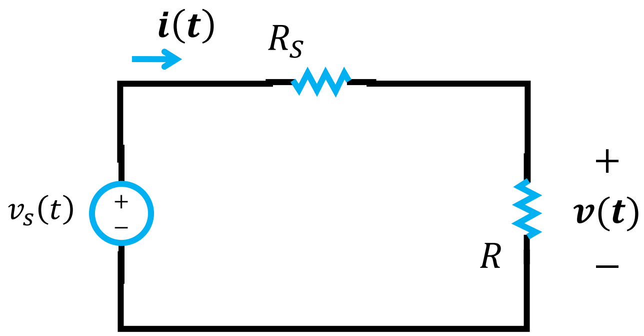
$$v(t) = \frac{R}{R+R_S} v_{s1}(t) = \frac{RV_S}{R+R_S}\cos(\omega t)$$

With $v_{S2}(t) = jV_S\,sin(\omega t)$

- According to KVL

$$v(t) = \frac{R}{R+R_S} v_{s2}(t) = j\frac{RV_S}{R+R_S}\sin(\omega t)$$

Resistors with a sin. forcing func.



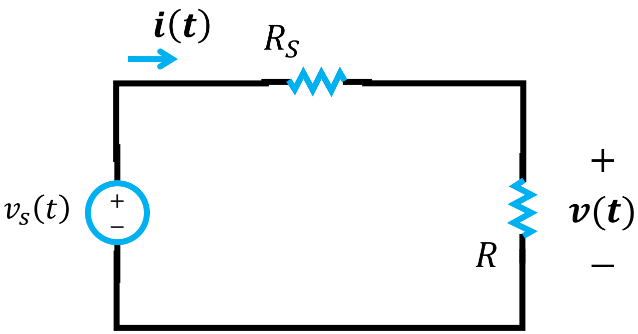
What if $v_S(t) = V_S\, cos(\omega t) + jV_S\, sin(\omega t)$?

- According to KVL

$$v(t) = \frac{R}{R + R_S} v_s(t)$$

$$= \frac{R}{R + R_S} (V_S\, cos(\omega t) + jV_S\, sin(\omega t))$$

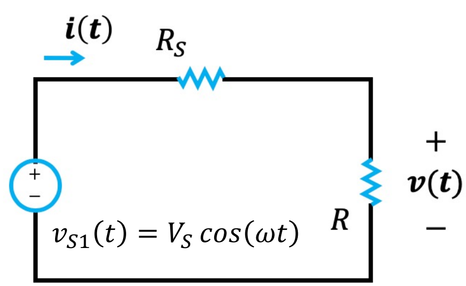
Resistors with a sin. forcing func.



What if $v_S(t) = V_S\cos(\omega t) + jV_S\sin(\omega t)$?

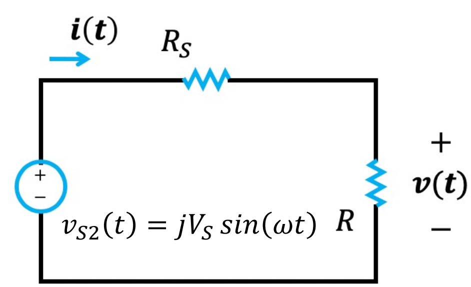
Solution: $v(t) = \dfrac{R}{R+R_S}\left(V_S\cos(\omega t) + jV_S\sin(\omega t)\right)$

=



$$v(t) = \frac{RV_S}{R+R_S}\cos(\omega t)$$

+



$$v(t) = j\frac{RV_S}{R+R_S}\sin(\omega t)$$

Recall: Euler's eq. & Complex num.

Phasor representation: $A\,cos(\omega t+\theta)\ =|z|\angle\theta$

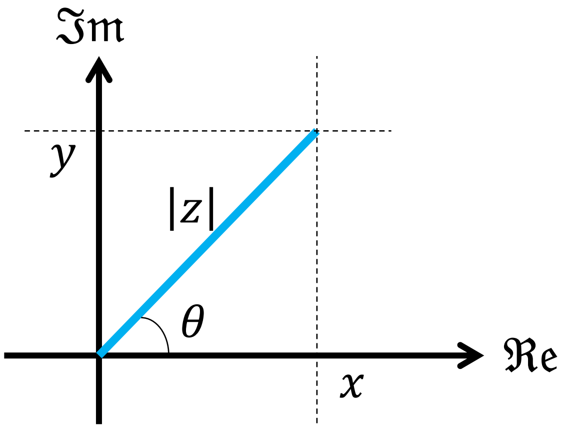
Euler's equation: $e^{j\omega t}=cos(\omega t)+j\,sin(\omega t)$

- Real part $\mathfrak{Re}\left[e^{j\omega t}\right]=\cos(\omega t)$
- imaginary part $\mathfrak{Im}\left[e^{j\omega t}\right]=\sin(\omega t)$

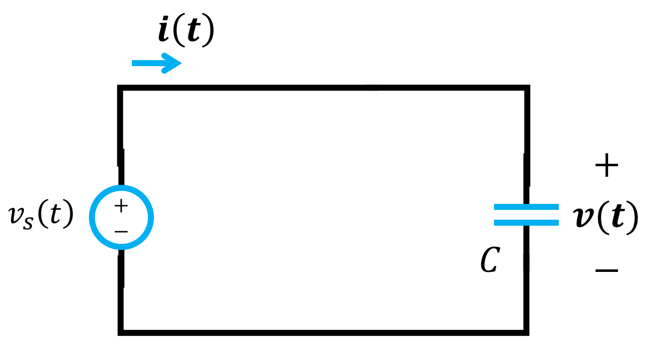
Complex numbers

- Rectangular coordinates $z=x+jy$
- Polar coordinates $z=|z|e^{j\theta}$

Where $\begin{cases} |z|=\sqrt{x^2+y^2} \\ \theta=tan^{-1}\left(\frac{y}{x}\right) \end{cases}$



Complex forcing function



- If a complex function is applied to the capacitor

$$v_S(t) = V_S e^{j\omega t}$$

- According to the $i - v$ relationship of capacitor

$$i(t) = C\frac{dv(t)}{dt}$$

$$= C\frac{d}{dt}\left(V_S e^{j\omega t}\right)$$

$$= j\omega C V_S e^{j\omega t}$$

Complex forcing function

- If a complex function is applied to the inductor

$$i_S(t) = I_S e^{j\omega t}$$

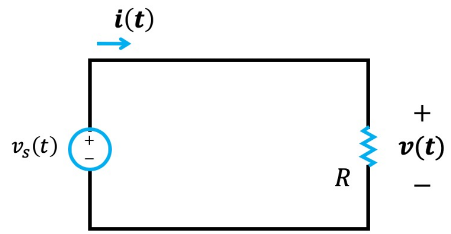
- According to the $i - v$ relationship of inductor

$$v(t) = L\frac{di(t)}{dt}$$

$$= L\frac{d}{dt}\left(I_S e^{j\omega t}\right)$$

$$= j\omega L I_S e^{j\omega t}$$

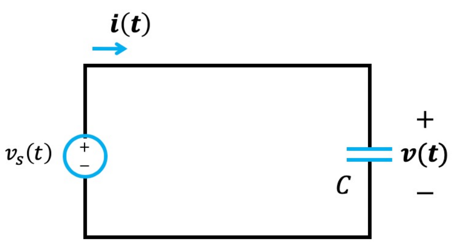
Complex forcing func. on passive dev.



$$v(t) = V_S e^{j\omega t}$$

$$i(t) = \frac{V_S}{R} e^{j\omega t}$$

$$\frac{v(t)}{i(t)} = R$$



$$v(t) = V_S e^{j\omega t}$$

$$i(t) = j\omega C V_S e^{j\omega t}$$

$$\frac{v(t)}{i(t)} = \frac{1}{j\omega C}$$

$$i(t) = I_S e^{j\omega t}$$

$$v(t) = j\omega L I_S e^{j\omega t}$$

$$\frac{v(t)}{i(t)} = j\omega L$$

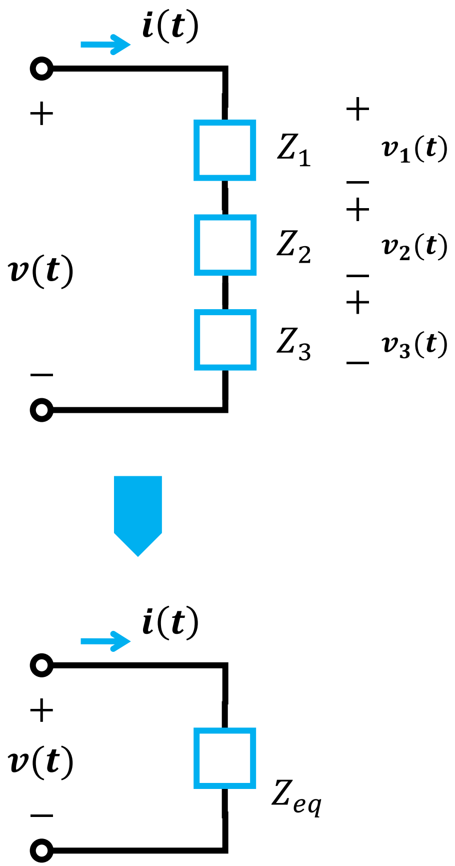
Impedance

Impedance, Z,

is defined as the ratio of the phasor voltage to the phasor current

			
i-v relation	$i=\frac{v}{R}$	$i(t)=C\frac{dv(t)}{dt}$	$i(t)=i(t_0)+\frac{1}{L}\int_{t_0}^{t}v(x)dx$
v-i relation	$v=iR$	$v(t)=v(t_0)+\frac{1}{C}\int_{t_0}^{t}i(x)dx$	$v(t)=L\frac{di(t)}{dt}$
Impedance	$\boldsymbol{R}$	$\boldsymbol{\frac{1}{j\omega C}}$	$\boldsymbol{j\omega L}$

Series connection



According to KVL

$$v(t) = v_1(t) + v_2(t) + v_3(t)$$

According i-v relationship

$$v(t) = i(t)Z_1 + i(t)Z_2 + i(t)Z_3$$

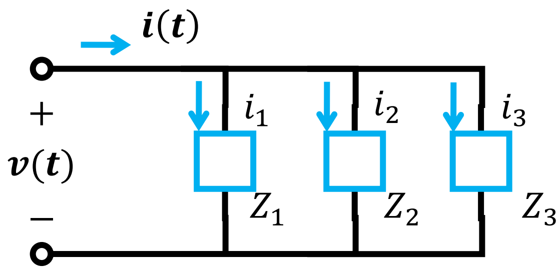
$$= i(t)(Z_1 + Z_2 + Z_3)$$

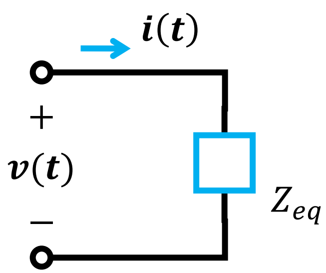
For the equivalent circuit

$$v(t) = i(t)Z_{eq}$$

$$\boldsymbol{Z_{eq} = Z_1 + Z_2 + Z_3}$$

Parallel connection





According i-v relationship

$$v(t) = i_1(t)Z_1 = i_2(t)Z_2 = i_3(t)Z_3$$

$$v(t) = i(t)Z_{eq}$$

According to KVL

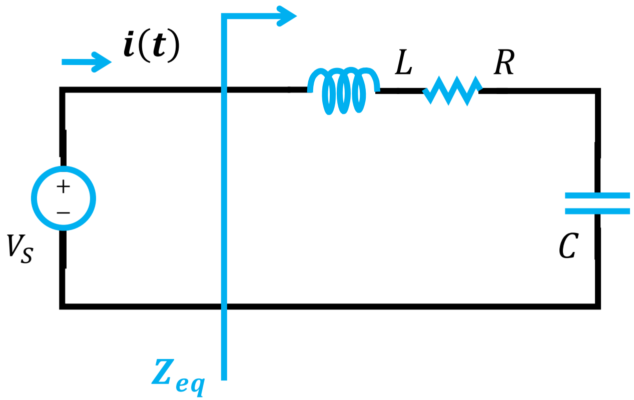
$$v(t) = Z_{eq}(i_1(t) + i_2(t) + i_3(t))$$

$$= Z_{eq}\left(\frac{v(t)}{Z_1} + \frac{v(t)}{Z_2} + \frac{v(t)}{Z_3}\right)$$

$$\frac{\mathbf{1}}{\boldsymbol{Z_{eq}}} = \frac{\mathbf{1}}{\boldsymbol{Z_1}} + \frac{\mathbf{1}}{\boldsymbol{Z_2}} + \frac{\mathbf{1}}{\boldsymbol{Z_3}}$$

Capacitive & Inductive

QUESTION: Find the value of the equivalent impedance, Z_{eq}

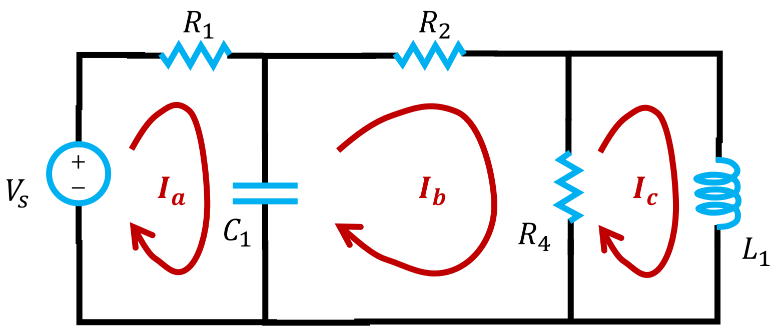


$$Z_{eq} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

- if $\omega L - \frac{1}{\omega C} > 0$ **the reactance is inductive**
- if $\omega L - \frac{1}{\omega C} < 0$ **the reactance is capacitive**

Example 4: mesh-current with imp.

QUESTION: Find the output current of the voltage source

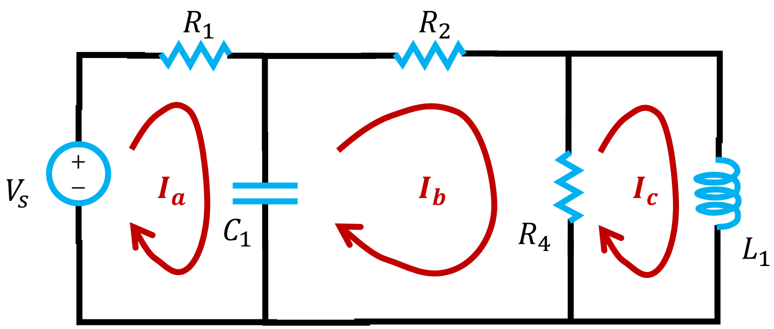


- According to KVL

$$\begin{cases} -V_s + I_a R_1 + (I_a - I_b)\dfrac{1}{j\omega C} = 0 \\ (I_b - I_a)\dfrac{1}{j\omega C} + I_b R_2 + (I_b - I_c) R_4 = 0 \\ (I_c - I_b) R_4 + j\omega L I_c = 0 \end{cases}$$

Example 4: mesh-current with imp.

QUESTION: Find the output current of the voltage source

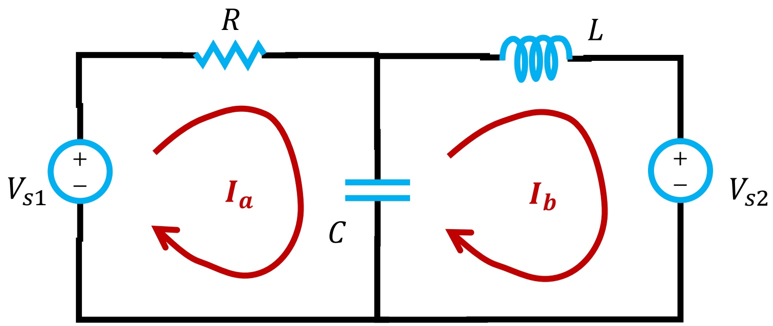


- According to KVL **3 equations in 3 unknowns**

$$\begin{cases} I_a\left(R_1+\frac{1}{j\omega C}\right)-I_b\frac{1}{j\omega C}=V_s \\ -I_a\frac{1}{j\omega C}+I_b\left(\frac{1}{j\omega C}+R_2+R_4\right)-I_cR_4=0 \\ -I_bR_4+I_c(R_4+j\omega L)=0 \end{cases}$$

Example 5: Superposition with imp.

QUESTION: Find the voltage on the capacitor C



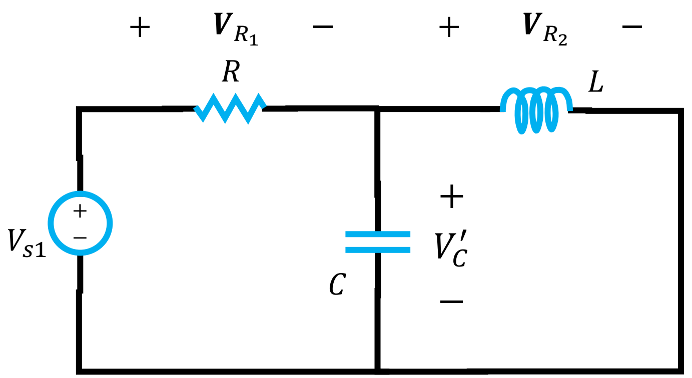
$$\begin{cases} I_a = \dfrac{\frac{1}{j\omega C}V_{s1} + \left(R + \frac{1}{j\omega C}\right)V_{s2}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}} \\ I_b = \dfrac{j\left(\omega L - \frac{1}{\omega C}\right)V_{s1} + \frac{1}{j\omega C}V_{s2}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}} \end{cases}$$

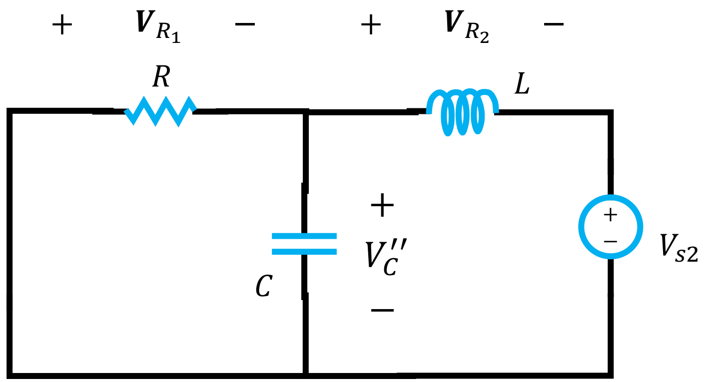
- According to KVL

$$\begin{cases} -V_{s1} + RI_a + \dfrac{1}{j\omega C}(I_a - I_b) = 0 \\ \dfrac{1}{j\omega C}(I_b - I_a) + j\omega L I_b - V_{s2} = 0 \end{cases}$$

$$V_C = \frac{1}{j\omega C}(I_a - I_b) = \frac{\frac{L}{C}V_{s1} - \frac{R}{j\omega C}V_{s2}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}}$$

Example 5: Superposition with imp.





- According to voltage division

$$V_C' = \frac{j\omega L || \frac{1}{j\omega C}}{R + j\omega L || \frac{1}{j\omega C}} V_{s1}$$

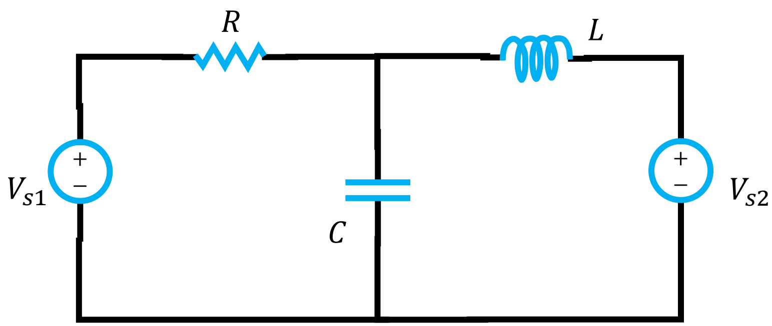
$$= \frac{\frac{L}{C}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}} V_{s1}$$

$$V_C'' = -\frac{R || \frac{1}{j\omega C}}{j\omega L + R || \frac{1}{j\omega C}} V_{s2}$$

$$= -\frac{\frac{R}{j\omega C}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}} V_{s2}$$

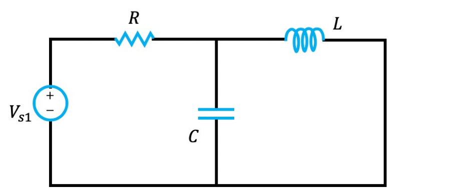
Example 5: Superposition with imp.

QUESTION: Find the voltage on the capacitor C



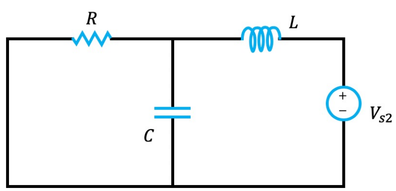
$$V_C = \frac{\frac{L}{C}V_{s1} - \frac{R}{j\omega C}V_{s2}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}}$$

=



$$V_C' = \frac{\frac{L}{C}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}}V_{s1}$$

+

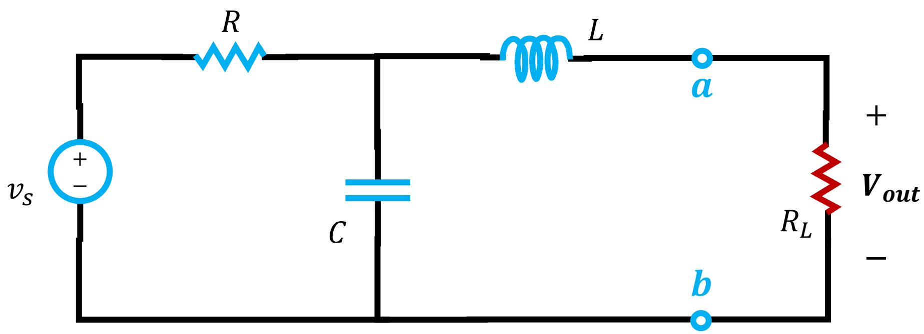


$$V_C'' = -\frac{\frac{R}{j\omega C}}{j\omega LR + \frac{R}{j\omega C} + \frac{L}{C}}V_{s2}$$

The superposition property works for ALL LINEAR circuit

Example 6: Thévenin equivalency with imp.

QUESTION: Find the Thévenin equivalent circuit of the network at the terminals a & b



Thévenin's theorem

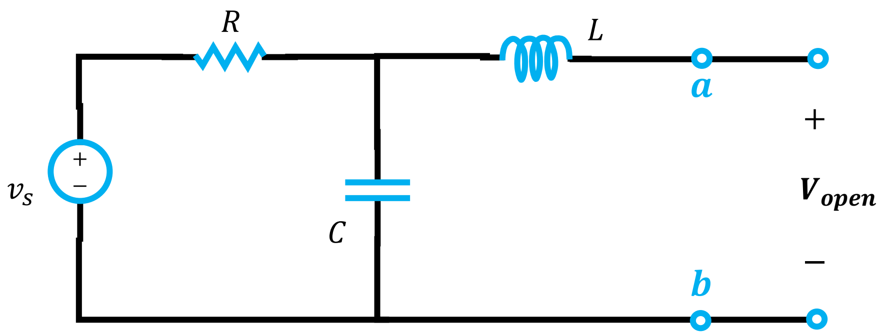
LINEAR two-terminal circuit can be replaced by an equivalent circuit composed of a voltage source and a series impedance

Norton's theorem

LINEAR two-terminal circuit can be replaced by an equivalent circuit composed of a current source and a parallel impedance

Example 6: Thévenin equivalency with imp.

QUESTION: Find the Thévenin equivalent circuit of the network at the terminals a & b



- Step 1: remove the load
- Step 2: find V_{open}

- According to voltage division

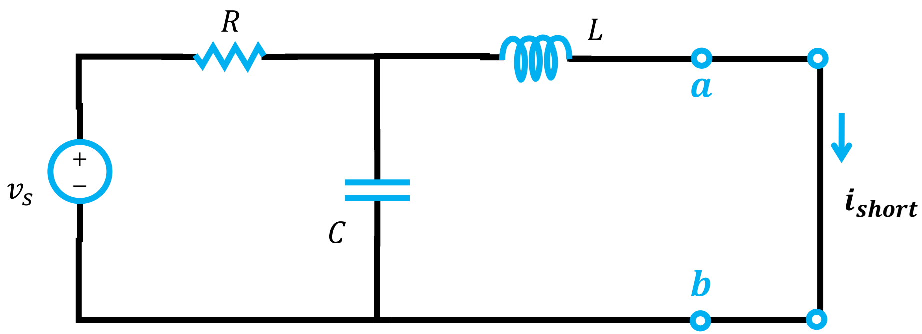
$$V_{open} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} v_s = \frac{1}{1 + j\omega CR} v_s$$

$\parallel$

V_{TH}

Example 6: Thévenin equivalency with imp.

QUESTION: Find the Thévenin equivalent circuit of the network at the terminals a & b



- Step 1: remove the load
- Step 2: find V_{open}
- Step 3: find i_{short}

- According to Ohm's law

$$i_{open} = \frac{v_s}{R + \frac{1}{j\omega C} || j\omega L} \cdot \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L}$$

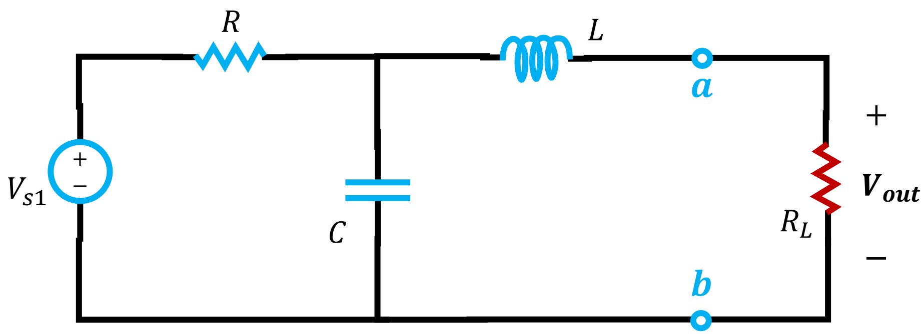
$$= \frac{v_s}{R + j\omega L - \omega^2 LC}$$

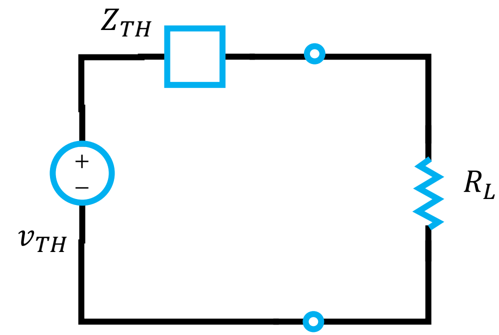
$\parallel$

i_N

Example 6: Thévenin equivalency with imp.

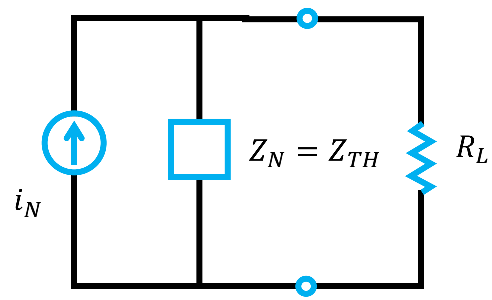
QUESTION: Find the Thévenin equivalent circuit of the network at the terminals a & b





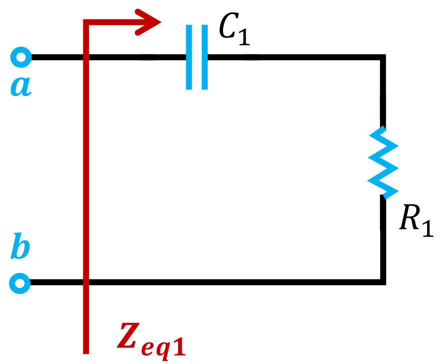
- Step 1: remove the load
- Step 2: find $V_{open} = v_{TH}$
- Step 3: find $i_{short} = i_N$
- Step 4: find Z_{TH}

$$Z_{TH} = \frac{v_{TH}}{i_N} = \frac{R + j\omega L - \omega^2 LC}{1 + j\omega CR}$$



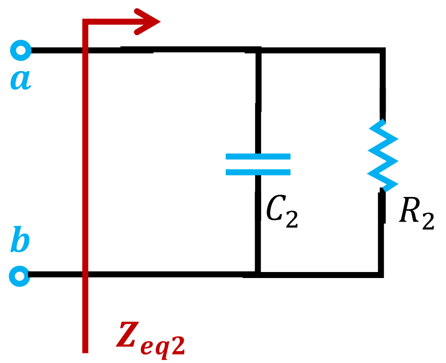
Example 7

QUESTION: Find the value of and , which makes the two circuit equivalent to each other



- The equivalent impedance of the upper circuit

$$Z_{eq1} = R_1 + \frac{1}{j\omega C_1}$$



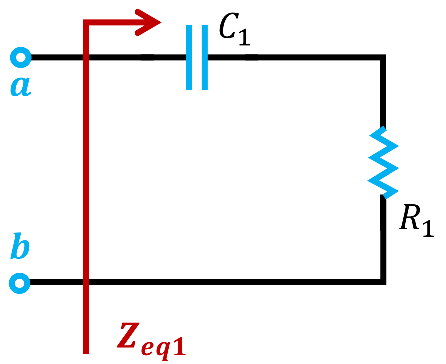
- The equivalent admittance of the lower circuit

$$G_{eq2} = \frac{1}{R_2} + j\omega C_2$$

$$Z_{eq1} = \frac{1}{G_{eq2}} \quad \text{when} \quad \begin{cases} \Re\mathfrak{e}[Z_{eq1}] = \Re\mathfrak{e}\left[\frac{1}{G_{eq2}}\right] \\ \Im\mathfrak{m}[Z_{eq1}] = \Im\mathfrak{m}\left[\frac{1}{G_{eq2}}\right] \end{cases}$$

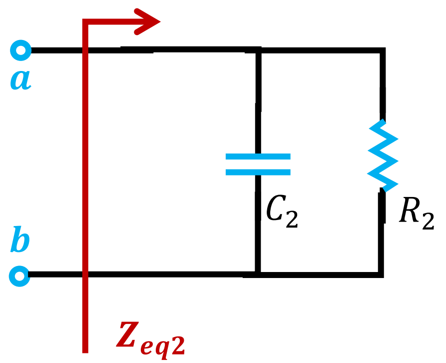
Example 7

QUESTION: Find the value of R_2 and C_2, which makes the two circuit equivalent to each other



$$G_{eq1} = \frac{1}{R_1 + \frac{1}{j\omega C_1}} = \frac{R_1 + j\frac{1}{\omega C_1}}{R_1^2 + \omega^2 C_1^2}$$

$$= \frac{R_1\omega^2 C_1^2}{R_1^2 + \omega^2 C_1^2} + j\frac{\omega C_1}{R_1^2 + \omega^2 C_1^2}$$

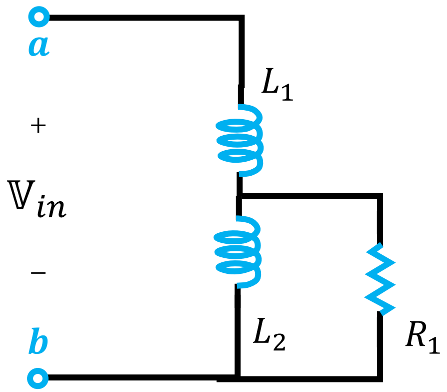


$$G_{eq2} = \frac{1}{R_2} + j\omega C_2$$

$$\begin{cases} R_2 = \dfrac{R_1^2 + \omega^2 C_1^2}{R_1\omega^2 C_1^2} \\ C_2 = \dfrac{C_1}{R_1^2 + \omega^2 C_1^2} \end{cases}$$

Example 8

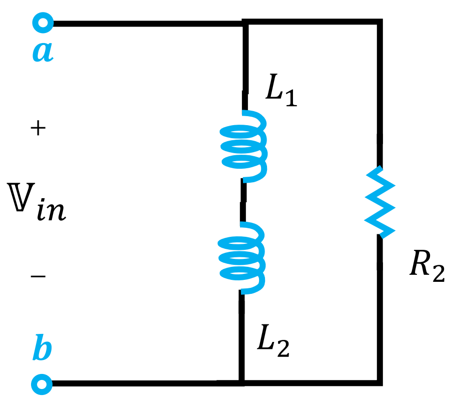
QUESTION: Find the value of R_2, which makes the power dissipated by R_2 is approximately equivalent to the dissipated by R_1, when the load resistance R_1 is very high



- According to KVL

$$\mathbb{V}_{R_1} = \mathbb{V}_{in}\frac{R_1||Z_{L_2}}{Z_{L_1} + R_1||Z_{L_2}}$$

$$\xrightarrow{R_1 \gg \omega L_2} \mathbb{V}_{in}\frac{Z_{L_2}}{Z_{L_1} + Z_{L_2}} = \mathbb{V}_{in}\frac{L_2}{L_1 + L_2}$$



- the power dissipated by R_1 and R_2

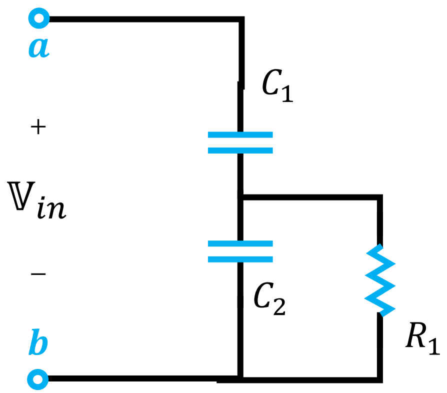
$$\begin{cases} P_{R_1} = \dfrac{\mathbb{V}_{R_1}^2}{R_1} \\ P_{R_2} = \dfrac{\mathbb{V}_{in}^2}{R_2} \end{cases}$$

$$R_2 = \frac{R_1}{p^2}$$

$$where\ \ p = \frac{L_2}{L_1 + L_2}$$

Example 9

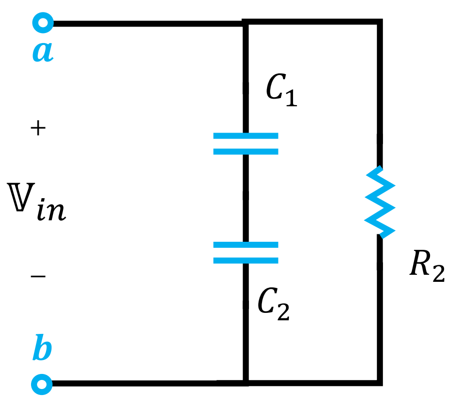
QUESTION: Find the value of R_2, which makes the power dissipated by R_2 is approximately equivalent to the dissipated by R_1, when the load resistance R_1 is very high



- According to KVL

$$\mathbb{V}_{R_1} = \mathbb{V}_{in} \frac{R_1 || Z_{C_2}}{Z_{C_1} + R_1 || Z_{C_2}}$$

$$\xrightarrow{R_1 \gg \frac{1}{\omega C_2}} \mathbb{V}_{in} \frac{Z_{C_2}}{Z_{C_1} + Z_{C_2}} = \mathbb{V}_{in} \frac{C_1}{C_1 + C_2}$$



- the power dissipated by R_1 and R_2

$$\begin{cases} P_{R_1} = \dfrac{\mathbb{V}_{R_1}^2}{R_1} \\ P_{R_2} = \dfrac{\mathbb{V}_{in}^2}{R_2} \end{cases}$$

$$R_2 = \frac{R_1}{p^2}$$

$$where \ \ p = \frac{C_1}{C_1 + C_2}$$

Outlines

- Response to different forcing function
  - Ramping forcing function
  - Sinusoidal forcing function
- Complex forcing function
  - Impedance & admittance
  - KVL&KCL / Superposition / Equivalency

i-v relation	$i = \frac{v}{R}$	$i(t) = C\frac{dv(t)}{dt}$	$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(x)dx$
v-i relation	$v = iR$	$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)dx$	$v(t) = L\frac{di(t)}{dt}$
Impedance	$\boldsymbol{R}$	$\boldsymbol{\frac{1}{j\omega C}}$	$\boldsymbol{j\omega L}$

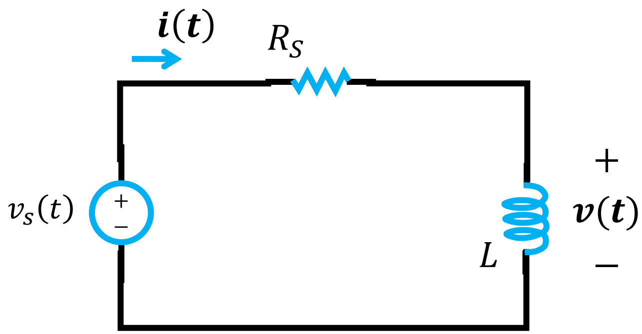
Outlines

- Response to different forcing function
  - Ramping forcing function
  - Sinusoidal forcing function
- Complex forcing function
  - Impedance & admittance
  - KVL&KCL / Superposition / Equivalency
- **AC steady-state analysis**

i-v relation	$i=\frac{v}{R}$	$i(t)=C\frac{dv(t)}{dt}$	$i(t)=i(t_0)+\frac{1}{L}\int_{t_0}^{t}v(x)dx$
v-i relation	$v=iR$	$v(t)=v(t_0)+\frac{1}{C}\int_{t_0}^{t}i(x)dx$	$v(t)=L\frac{di(t)}{dt}$
Impedance	R	$\frac{1}{j\omega C}$	$j\omega L$

Example 10

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to a complex forcing function $\mathbb{V}_S = V_S e^{j\omega t}$.



METHODS 1

- According to KVL

$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{1}{L}v_S(t)$$

- The current through L must be in the form of

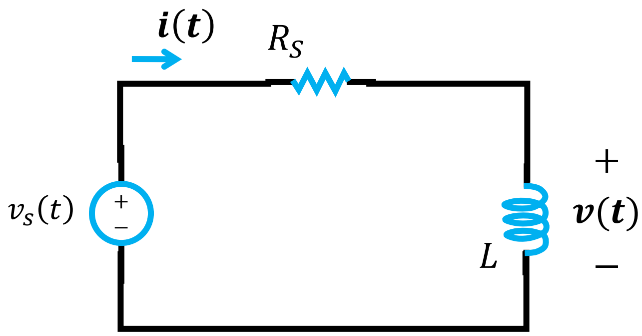
$$i_S(t) = I_S e^{j(\omega t+\varphi)}$$

- Substitute $i_S(t)$ to the equation

$$\frac{d}{dt}\left(I_S e^{j(\omega t+\varphi)}\right) + \frac{R}{L}I_S e^{j(\omega t+\varphi)} = \frac{1}{L}V_S e^{j\omega t}$$

Example 10

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to a complex forcing function $\mathbb{V}_S = V_S e^{j\omega t}$.



$$j\omega I_S e^{j(\omega t+\varphi)} + \frac{R}{L} I_S e^{j(\omega t+\varphi)} = \frac{1}{L} V_S e^{j\omega t}$$

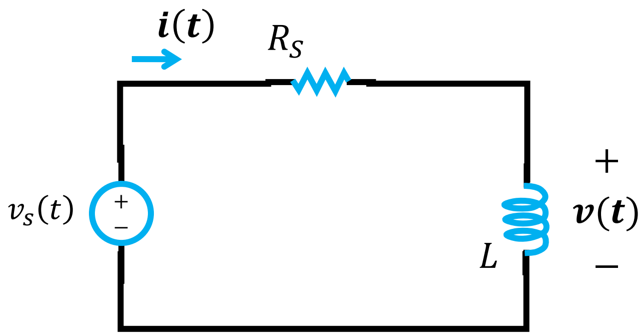
$$j\omega I_S e^{j\varphi} + \frac{R}{L} I_S e^{j\varphi} = \frac{1}{L} V_S$$

$$I_S e^{j\varphi} = \frac{V_S}{R + j\omega L}$$

$$\Rightarrow \begin{cases} I_S = \dfrac{V_S}{\sqrt{R^2 + \omega^2 L^2}} \\ \varphi = -tan^{-1} \dfrac{\omega L}{R} \end{cases}$$

Example 10

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to a complex forcing function $\mathbb{V}_S = V_S e^{j\omega t}$.



$$i_S(t) = I_S e^{j(\omega t+\varphi)}$$

$$= I_S \cos(\omega t + \varphi) + jI_S \sin(\omega t + \varphi)$$

$$\Re\mathrm{e}[i_S(t)] = I_S \cos(\omega t + \varphi)$$

$$= I_S(\cos \omega t \cos \varphi + \sin \omega t \sin \varphi)$$

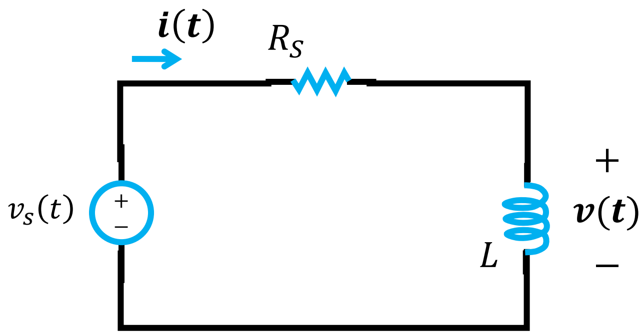
$$= K_1 \cos(\omega t) + K_2 \sin(\omega t)$$

where $\begin{cases} K_1 = V_S \dfrac{R}{R^2 + \omega^2 L^2} \\ K_2 = V_S \dfrac{-\omega L}{R^2 + \omega^2 L^2} \end{cases}$

$$\begin{cases} I_S = \dfrac{V_S}{\sqrt{R^2 + \omega^2 L^2}} \\ \varphi = -tan^{-1} \dfrac{\omega L}{R} \end{cases}$$

Example 10

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to a complex forcing function $\mathbb{V}_S = V_S e^{j\omega t}$.



Solution from the differential equation

$$i_S(t) = I_S e^{j(\omega t + \varphi)}$$

$$\begin{cases} I_S = \dfrac{V_S}{\sqrt{R^2 + \omega^2 L^2}} \\ \varphi = -tan^{-1}\dfrac{\omega L}{R} \end{cases}$$

METHODS 2

- Consider the impedance circuit
- According to KVL

$$\mathbb{V}_S = R\mathbb{I}_S + j\omega L \mathbb{I}_S \quad \Rightarrow \quad \mathbb{I}_S = \frac{\mathbb{V}_S}{R + j\omega L}$$

- The real part of $\mathbb{I}_S$

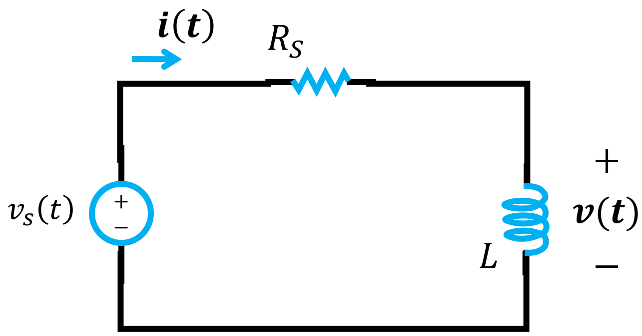
$$\Re\mathrm{e}[\mathbb{I}_S] = \frac{V_S}{R^2 + \omega^2 L^2}[R\cos(\omega t) + \omega L \sin(\omega t)]$$

$\|$

$\Re\mathrm{e}[i_S(t)]$

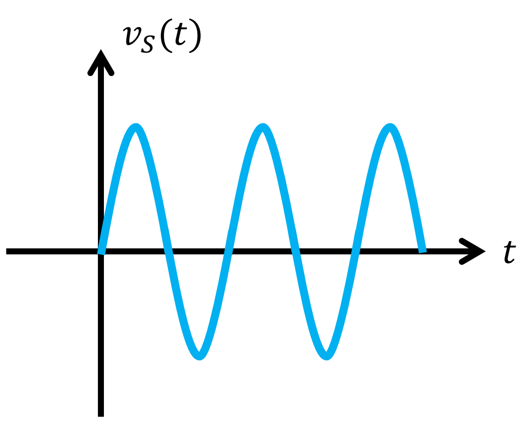
Recall: Sinusoidal forcing function

QUESTION: Assume there is no charge on the inductor L at $t = -\infty$. Find the voltage response to the sinusoidal forcing function.

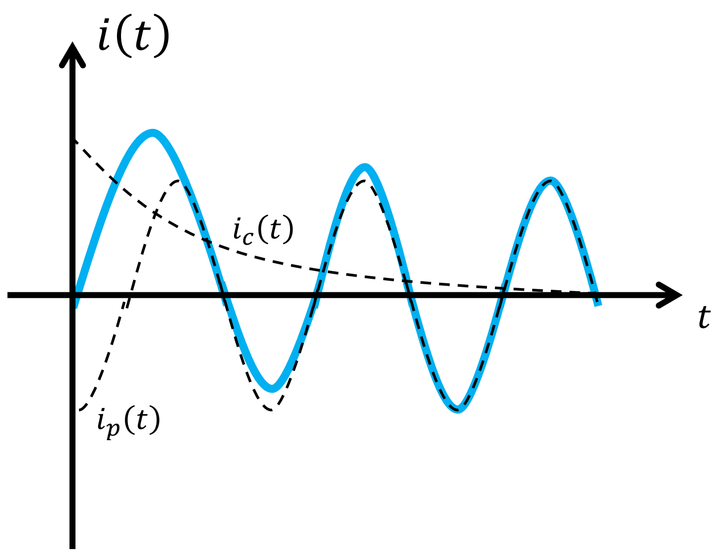


$$\frac{d}{dt} i(t) + \frac{R}{L} i(t) = \frac{1}{L} v_S(t)$$

Solution $\quad i(t) = K_1 \cos(\omega t) + K_2 \sin(\omega t) + K_3 e^{-\frac{R}{L}t}$



$v_S(t) = V_S \sin(\omega t)$



where

$$\begin{cases} K_1 = V_S \dfrac{-\omega L}{R^2 + \omega^2 L^2} \\ K_2 = V_S \dfrac{R}{R^2 + \omega^2 L^2} \\ K_3 = V_S \dfrac{\omega L}{R^2 + \omega^2 L^2} \end{cases}$$

Circuit analysis in TIME domain

Method can be applied to ALL circuit

Problem in physical world ➤ Circuit model ➤ (KVL, KCL) Differential equation → (Homogeneous solution / Particular solution) → Initial conditions ➤ Full solution

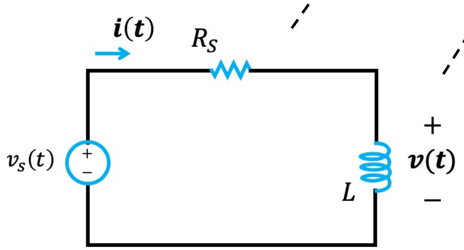
$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = \frac{1}{L}v_S(t)$$

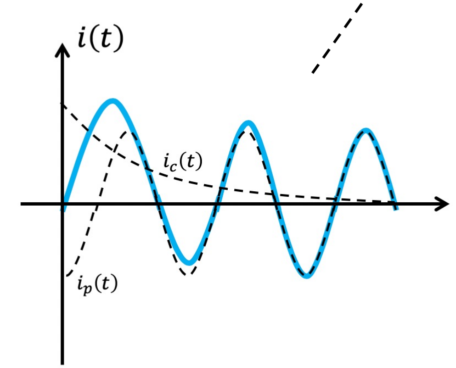
→ $v_c(t)$

→ $v_p(t)$

$$i(0) = 0$$

KVL
KCL





Circuit analysis in TIME domain

Method for ALL circuit

Problem in physical world ➧ Circuit model ➧ (KVL, KCL) Differential equation → (Homogeneous solution / Particular solution) → Initial conditions ➧ Full solution

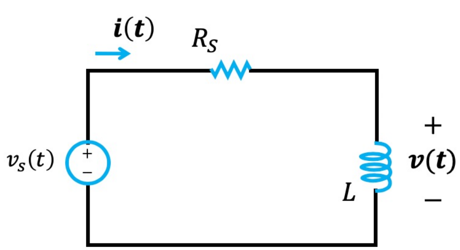
Method ONLY for LINEAR circuit

Circuit Impedance model ➧ (KVL, KCL) Algebraic equation

$\Re\mathrm{e}[*]$

$$\Re\mathrm{e}[\mathbb{I}_S] = \frac{V_S}{R^2 + \omega^2 L^2}[R\cos(\omega t) - \omega L \sin(\omega t)]$$

$$\mathbb{V}_S = R\mathbb{I}_S + j\omega L\mathbb{I}_S$$



$$Z_L = j\omega L$$

Outlines

- Response to different forcing function
  - Ramping forcing function
  - Sinusoidal forcing function
- Complex forcing function
  - Impedance & admittance
  - KVL&KCL / Superposition / Equivalency

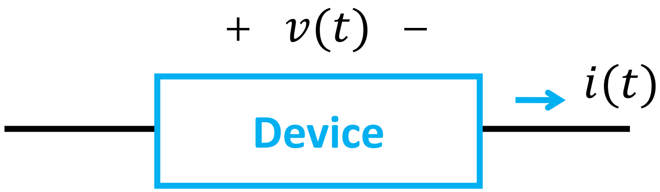
i-v relation	$i = \frac{v}{R}$	$i(t) = C\frac{dv(t)}{dt}$	$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(x)dx$
v-i relation	$v = iR$	$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)dx$	$v(t) = L\frac{di(t)}{dt}$
Impedance	R	$\frac{1}{j\omega C}$	$j\omega L$

- AC steady-state analysis
  - Impedance circuit analysis for steady-state solution in LINEAR circuit
- **Complex Power analysis**

Recall: Inst./Avg. power & eff. v/i

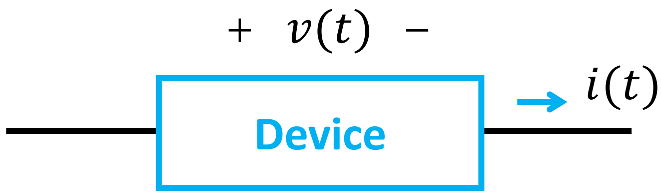
Given the voltage $v(t) = V_P \cos(\omega t + \varphi_V)$

Given the current $i(t) = I_P \cos(\omega t + \varphi_I)$



- Instantaneous power $$p(t) = \frac{1}{2} V_P I_P \cos(\varphi_V - \varphi_I) + \frac{1}{2} V_P I_P \cos(2\omega t + \varphi_V + \varphi_I)$$
- Average power $$\bar{P} = \frac{V_P}{\sqrt{2}} \frac{I_P}{\sqrt{2}} \cos(\varphi_V - \varphi_I) = v_{rms} i_{rms} \cos(\varphi_V - \varphi_I)$$
- Effective voltage $$v_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} v^2(t) dt} = \frac{V_P}{\sqrt{2}}$$
- Effective current $$i_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} i^2(t) dt} = \frac{I_P}{\sqrt{2}}$$

Complex power



Given $v(t) = V_s \cos(\omega t + \varphi_V)$ → $V_s \angle \varphi_V$ → $\mathbb{V} = V_s e^{j\varphi_V}$

Given $i(t) = I_s \cos(\omega t + \varphi_I)$ → $I_s \angle \varphi_I$ → $\mathbb{I} = I_s e^{j\varphi_I}$

DEFINE COMPLEX POWER $\mathbb{S} = \frac{1}{2}\mathbb{V}\mathbb{I}^*$

$$\mathbb{S} = \frac{1}{2}\mathbb{V}\mathbb{I}^* = \frac{1}{2}\left(V_s e^{j\varphi_V}\right)\left(I_s e^{-j\varphi_I}\right) = \frac{1}{2} V_s I_s e^{j(\varphi_V - \varphi_I)}$$

$$= \frac{1}{2} V_s I_s [\cos(\varphi_V - \varphi_I) + j \sin(\varphi_V - \varphi_I)]$$

Complex power

$$\mathbb{S} = \frac{1}{2} V_s I_s \cos(\varphi_V - \varphi_I) + j\frac{1}{2} V_s I_s \sin(\varphi_V - \varphi_I)$$

Average power $\overline{P}$ + Reactive power Q = COMPLEX POWER $\mathbb{S}$

- **Average power** $\overline{\boldsymbol{P}} = \Re\mathfrak{e}\{\mathbb{S}\} = V_{rms} I_{rms} \cos(\varphi_V - \varphi_I) = I_{rms}^2 R \cos(\varphi_V - \varphi_I)$

  (unit: watt/W)

- **Reactive power** $\boldsymbol{Q} = \Im\mathfrak{m}\{\mathbb{S}\} = V_{rms} I_{rms} \sin(\varphi_V - \varphi_I) = I_{rms}^2 X \sin(\varphi_V - \varphi_I)$

  (unit: Volt-Ampere Reactive/VAR)

- **COMPLEX POWER** $\mathbb{S} = \overline{\boldsymbol{P}} + j\boldsymbol{Q}$

  (unit: Volt-Amp/VA)

Complex power

- **COMPLEX POWER** $\mathbb{S} = \overline{\boldsymbol{P}} + j\boldsymbol{Q}$

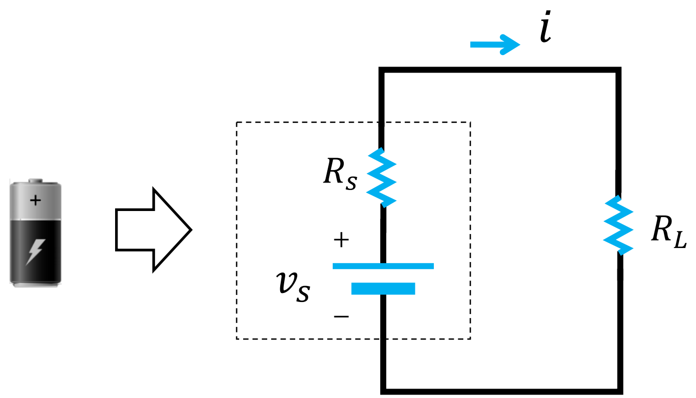
$$\overline{\boldsymbol{P}} = \Re\{\mathbb{S}\} = V_{rms} I_{rms} \cos(\varphi_V - \varphi_I)$$

$$\boldsymbol{Q} = \Im\{\mathbb{S}\} = jV_{rms} I_{rms} \sin(\varphi_V - \varphi_I)$$

- Define **power factor** $pf = \dfrac{P}{V_{rms} I_{rms}} = \cos(\varphi_V - \varphi_I)$

- Define **power angle** $\tan(\varphi_V - \varphi_I) = \dfrac{Q}{P}$

Recall: Max. Power Trans. in DC Circ.



Practical voltage source

- Voltage of R_L

$$v_{R_L} = v_s - iR_s$$

- Power at the load R_L

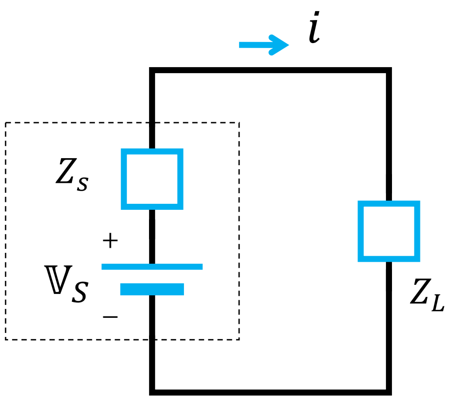
$$P_L = v_{R_L} i = (v_s - iR_s)i \ = -R_s\left(i^2 - \frac{v_s}{R_s} i\right) = -R_s\left(i - \frac{1}{2}\frac{v_s}{R_s}\right)^2 + \frac{1}{4}\frac{v_s^2}{R_s}$$

$$\leq \frac{1}{4}\frac{v_{s,rms}^2}{R_s}$$

The maximum power being absorbed by the load

- When $R_s = R_L$ $\quad P_L = P_{L,max} = \dfrac{1}{4}\dfrac{v_{s,rms}^2}{R_s}$

Maximum Average Power Transfer



Assume

$$\mathbb{V}_S = V_S \angle \theta_{v_S}$$

$$Z_s = R_s + jX_s$$

$$Z_L = R_L + jX_L$$

- The current through Z_L is $\mathbb{I}_L = I_L \angle \theta_{i_L}$

$$\mathbb{I}_L = \frac{\mathbb{V}_S}{Z_s + Z_L} = \frac{\mathbb{V}_S}{(R_s + jX_s) + (R_L + jX_L)}$$

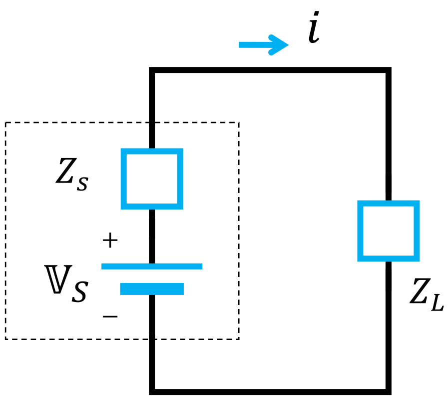
$$\rightarrow \quad I_L = \frac{V_S}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

- The voltage on Z_L is $\mathbb{V}_L = V_L \angle \theta_{v_L}$

$$\mathbb{V}_L = \frac{\mathbb{V}_S Z_L}{Z_s + Z_L} = \frac{\mathbb{V}_S (R_L + jX_L)}{(R_s + jX_s) + (R_L + jX_L)}$$

$$\rightarrow \quad V_L = \frac{V_S \sqrt{R_L^2 + X_L^2}}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

Maximum Average Power Transfer



$$\begin{cases} I_L = \dfrac{V_S}{\sqrt{(R_s+R_L)^2+(X_s+X_L)^2}} \\ V_L = \dfrac{V_S\sqrt{R_L^2+X_L^2}}{\sqrt{(R_s+R_L)^2+(X_s+X_L)^2}} \end{cases}$$

- The average power at the load Z_L

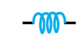
$$P_L = V_{rms}I_{rms}\cos(\theta_{v_L}-\theta_{i_L}) \qquad \text{where} \quad \theta_{v_L}-\theta_{i_L} = cos^{-1}\left(\frac{R_L}{\sqrt{R_L^2+X_L^2}}\right)$$

$$= \frac{V_{rms}^2 R_L}{(R_s+R_L)^2+(X_s+X_L)^2} \le \frac{V_{rms}^2 R_L}{(R_s+R_L)^2} \le \frac{1}{4}\frac{V_{rms}^2}{R_S}$$

- when $\begin{cases} R_s = R_L \\ X_s = -X_L \end{cases} \rightarrow \boldsymbol{Z_s = Z_L^*} \qquad P_{L,max} = \dfrac{1}{4}\dfrac{V_{rms}^2}{R_S}$

Outlines

- Response to different forcing function
  - Ramping forcing function
  - Sinusoidal forcing function
- Complex forcing function
  - Impedance & admittance
  - KVL&KCL / Superposition / Equivalency
- AC steady-state analysis
  - Impedance circuit analysis for steady-state solution in LINEAR circuit
- Complex Power analysis
  - Definition of complex power/average power/reactive power
  - Maximum average power transfer

			
i-v relation	$i=\frac{v}{R}$	$i(t)=C\frac{dv(t)}{dt}$	$i(t)=i(t_0)+\frac{1}{L}\int_{t_0}^{t}v(x)dx$
v-i relation	$v=iR$	$v(t)=v(t_0)+\frac{1}{C}\int_{t_0}^{t}i(x)dx$	$v(t)=L\frac{di(t)}{dt}$
Impedance	$\boldsymbol{R}$	$\boldsymbol{\frac{1}{j\omega C}}$	$\boldsymbol{j\omega L}$

Reading tasks & learning goals

- Reading tasks
  - Basic Engineering Circuit Analysis, 10th edition
    - Chapter 8.1-8.8 & 9.1-9.6
- Learning goals
  - Know how to solve response with **ramp/sin. forcing functions**
  - Understand the basic characteristics of sinusoidal functions
  - Know how to calculate **impedance/admittance**
  - Know how to perform **AC steady-state analysis**
  - Know how to calculate **real power**/**reactive power**/**complex power**/**power factor** in AC circuits
  - Be able to calculate the **maximum average power transfer** for a load in an AC circuit