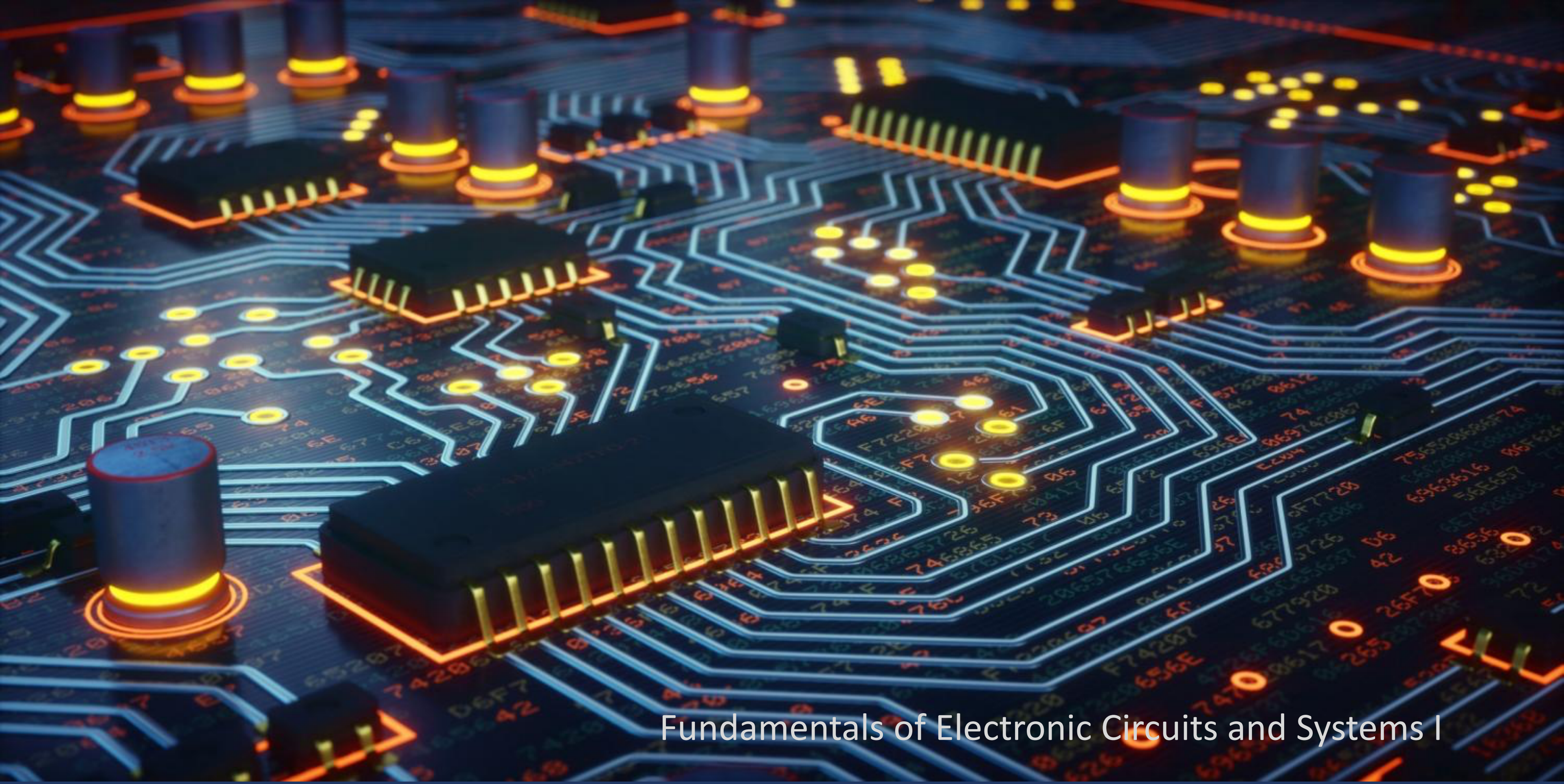

Fundamentals of Electronic Circuits and Systems I

Transient Circuit Analysis

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Outlines

- 1^{st} order circuit

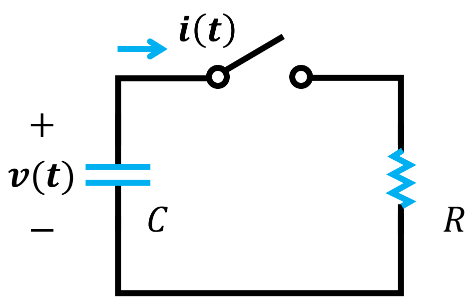
- 2^{nd} order circuits

Review: Capacitor & Inductor

			
i-v relation	$i = \frac{v}{R}$	$i(t) = C\frac{dv(t)}{dt}$	$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(x)dx$
v-i relation	$v = iR$	$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)dx$	$v(t) = L\frac{di(t)}{dt}$
p (power transferred in)	$p = vi$	$p = vi$	$p = vi$
w (stored energy)	0	$w = \frac{1}{2}Cv^2(t) = \frac{Q^2(t)}{2C}$	$w = \frac{1}{2}Li^2(t) = \frac{\lambda^2(t)}{2L}$
Series combination	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel combination	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
DC behavior	NO	open circuit	short circuit
Instantaneous change of v	√	×	√
Instantaneous change of i	√	√	×

Example 1: Source free RC circuit

QUESTION: Assume the capacitor C has been charged to V_0 before the switch is turned on. Find the response after the switch is turned on.



- Assume the switch is turned on $@t = 0$
- According to KCL

$$i_C = i_R$$

$$\rightarrow \quad C\frac{d}{dt}v(t) + \frac{v(t)}{R} = 0$$

$$\rightarrow \quad \frac{d}{dt}v(t) + \frac{v(t)}{RC} = 0$$

1st order differential equation

Review: Calculus

QUESTION: Find the solution $x(t)$ of the following equation

$$\frac{d}{dt}x(t) + a_1 x(t) = f(t)$$

General form of 1st order differential equation

- Step 1: Find any solution to the original equation $x(t) = x_p(t)$
- Step 2: Find a solution $x(t) = x_c(t)$ to the homogeneous equation

$$\frac{d}{dt}x(t) + a_1 x(t) = 0$$

- Step 3: the solution to the original equation can be written as

$$x(t) = x_p(t) + x_c(t)$$

$x_c(t)$: Complementary solution

$x_p(t)$: Particular integral solution

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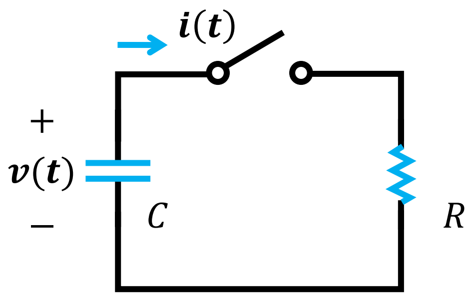

$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = 0$$

- Step 1a: find the **particular integral solution $v_p(t)$**

Assume $v_p(t) = K_1$

- Step 1b: substitute $i_p(t)$ to the equation

$$\frac{K_1}{RC} = 0 \quad \rightarrow \quad K_1 = 0$$

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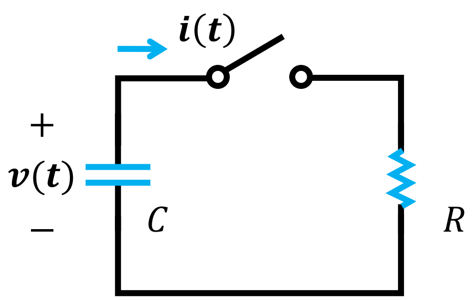

$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = 0$$

- Step 2a: find the **homogeneous equation**

$$\frac{d}{dt}v(t) + a_1 v(t) = 0 \quad \text{where } a_1 = \frac{1}{RC}$$

- Step 2b: find the **complementary solution** $v_c(t)$ to the homogeneous equation

$$v_c(t) = K_2 e^{-st}$$

[Particular integral solution

$v_p(t) = 0$]

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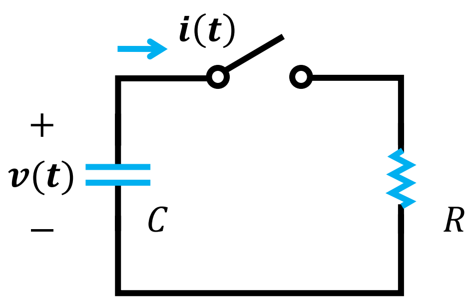
QUESTION: Assume the capacitor C has been charged to V_0 before the switch is turned on. Find the response after the switch is turned on.



Particular integral solution

$v_p(t) = 0$

$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = 0$$

- Step 2c: substitute $v_c(t)$ to the homogeneous equation

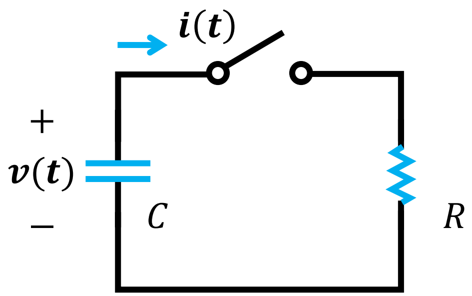
$$\frac{d}{dt}(K_2 e^{-st}) + a_1(K_2 e^{-st}) = 0$$

$$\rightarrow \quad -sK_2 e^{-st} + a_1 K_2 e^{-st} = 0$$

$$\rightarrow \quad s = a_1 \qquad \text{where } a_1 = \frac{1}{RC}$$

Example 1: Source free RC circuit

QUESTION: Assume the capacitor C has been charged to V_0 before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = 0$$

- Step 3a: find the value of $v(t)$ at one instant of time

$$v(0) = V_0$$

- Step 3b: substitute $v(0) = V_0$ to $v(t)$

$$v(t) = K_2 e^{-a_1 t}$$

$$\rightarrow \quad K_2 = V_0$$

Particular integral solution

$$v_p(t) = 0$$

Complementary solution

$$v_c(t) = K_2 e^{-a_1 t}$$

Example 1: Source free RC circuit

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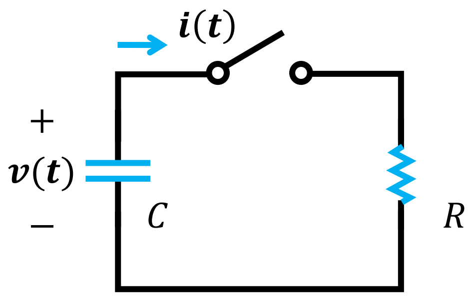


$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = 0$$

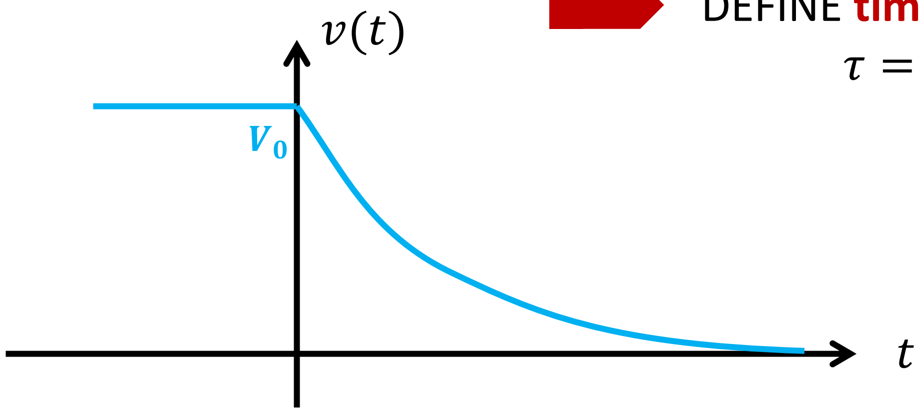
Solution: $v(t) = V_0 e^{-\frac{1}{RC}t}$

"−" indicates that the voltage across the capacitor is decaying exponentially

DEFINE **time constant**

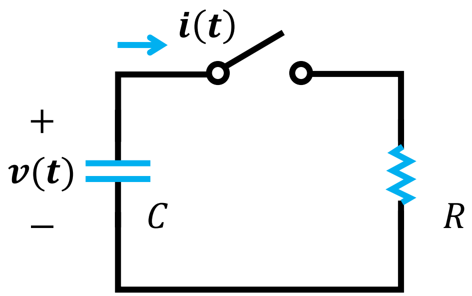
$$\tau = RC$$

The circuit reaches steady-state faster, if a smaller time constant is applied



Example 1: Source free RC circuit

QUESTION: Assume the capacitor C has been charged to V_0 before the switch is turned on. Find the response after the switch is turned on.



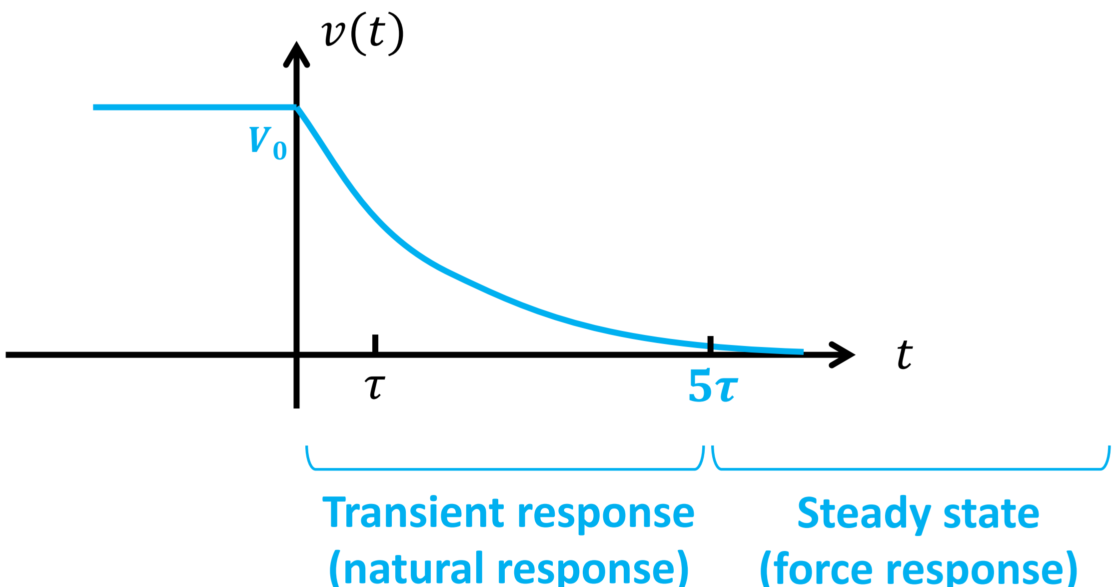
If $R = 1\text{k}\Omega$, $C = 1\mu F$

- When $t = \tau$, $v(t) = 0.368V_0$
- When $t = 5\tau$, $v(t) = 0.0067V_0$

$$\frac{d}{dt}v(t) + \frac{v(t)}{RC} = 0$$

$$v(t) = V_0 e^{-\frac{1}{\tau}t}$$

$$\tau = RC$$



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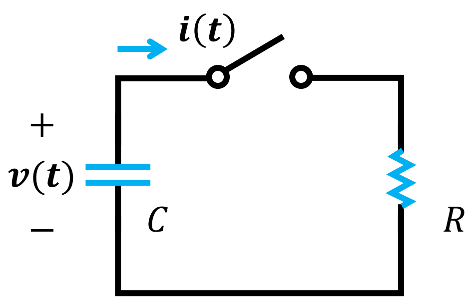
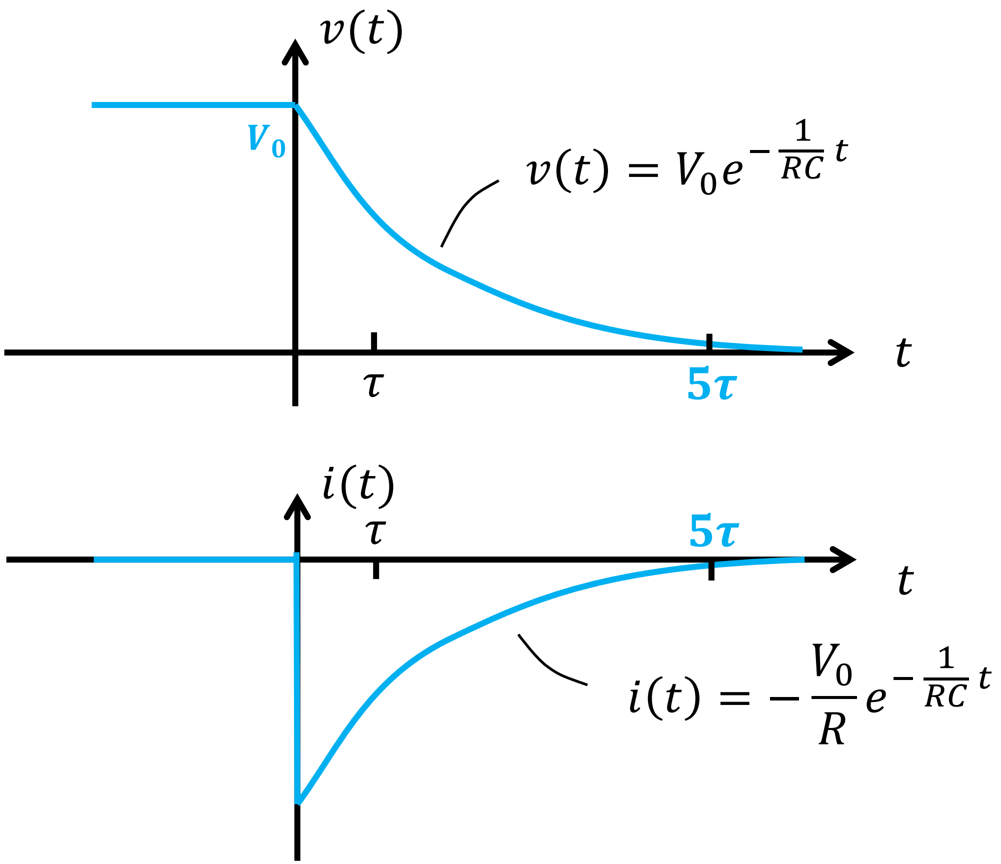
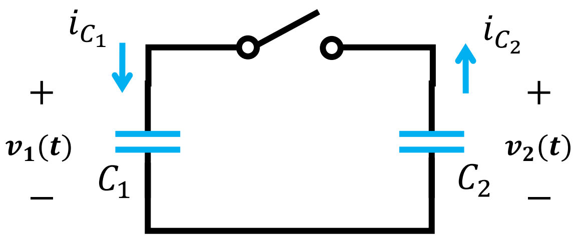




Voltage on capacitor CANNOT change abruptly, but current can.

Review: an old exmple

QUESTION: Find the total energy stored in the two capacitors before & after the switch is turned on, if the charges on the two capacitors at $t = 0$ are $q_1(0) = Q_1$ and $q_2(0) = Q_2$, respectively.



- BEFORE the switch is turned on ($t < 0$)

$$w(t<0) = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2}$$

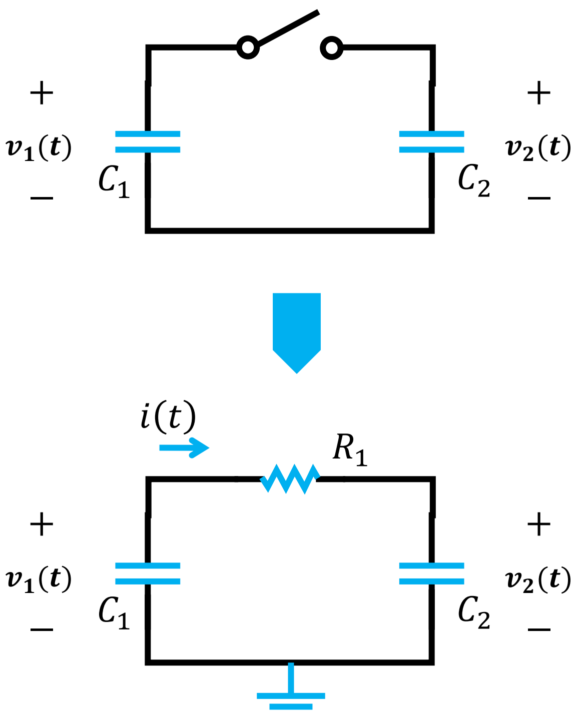
- AFTER the switch is turned on ($t \geq 0$). Assume the circuit is in steady state @ $t = t_1$

$$w(t_1) = \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}$$

WHY $w(t<0) \neq w(t_1)$?

Example 2

QUESTION: Find the total energy stored in the two capacitors before & after the switch is turned on, if the charges on the two capacitors at $t = 0$ are $q_1(0) = Q_1$ and $q_2(0) = Q_2$, respectively. The resistance of the switch is R_1 when it is turned on.



- After the switch is turned on, according to KVL

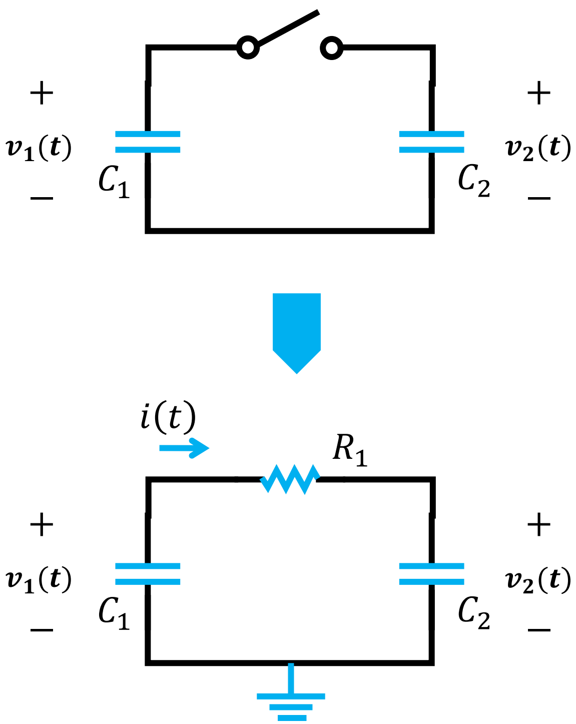
$$\frac{1}{C_1}\int_{-\infty}^{t} i(t)dt + i(t)R_1 + \frac{1}{C_2}\int_{-\infty}^{t} i(t)dt = 0$$

$$\rightarrow \quad \frac{d}{dt}i(t) + \frac{1}{R_1}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)i(t) = 0$$

Define $\quad a_1 = \frac{1}{R_1}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$

Example 2

QUESTION: Find the total energy stored in the two capacitors before & after the switch is turned on, if the charges on the two capacitors at $t = 0$ are $q_1(0) = Q_1$ and $q_2(0) = Q_2$, respectively. The resistance of the switch is R_1 when it is turned on.



$$\frac{d}{dt}i(t) + a_1 i(t) = 0$$

- Step 1: find the particular integral solution

$$i_p(t) = 0$$

- Step 2: find the complementary solution

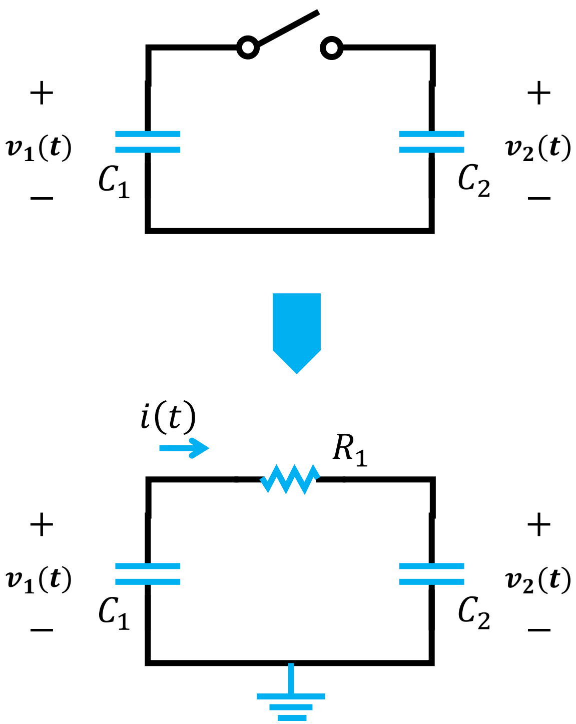
$$i_c(t) = K_2 e^{-a_1 t}$$

- Step 3: find the value of $i(t)$ at one instant of time

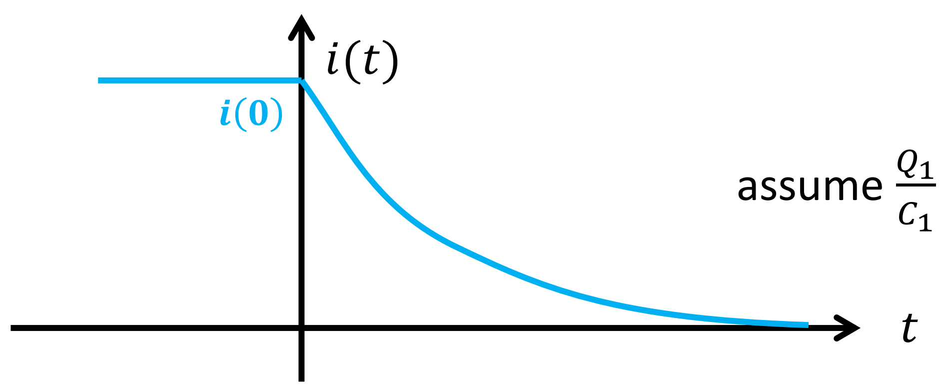
$$i(0) = \frac{1}{R_1}\left(\frac{Q_1}{C_1} - \frac{Q_2}{C_2}\right) \quad \rightarrow K_2 = i(0)$$

Example 2

QUESTION: Find the total energy stored in the two capacitors before & after the switch is turned on, if the charges on the two capacitors at $t = 0$ are $q_1(0) = Q_1$ and $q_2(0) = Q_2$, respectively. The resistance of the switch is R_1 when it is turned on.



$$i(t) = i(0)e^{-a_1 t} \quad \text{where} \quad \begin{cases} a_1 = \dfrac{1}{R_1}\left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right) \\ i(0) = \dfrac{1}{R_1}\left(\dfrac{Q_1}{C_1} - \dfrac{Q_2}{C_2}\right) \end{cases}$$



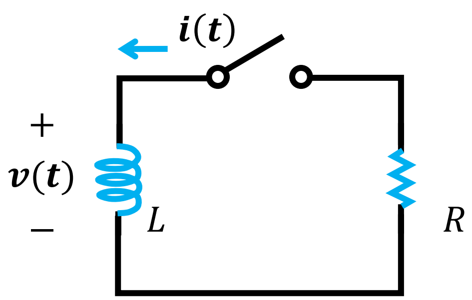
assume $\dfrac{Q_1}{C_1} > \dfrac{Q_2}{C_2}$

Outlines

- 1^{st} order circuit
  - Source free RC circuit
  - **Source free RL circuit**

Example 3: Source free RL circuit

QUESTION: Assume the capacitor L has been charged to I_0 before the switch is turned on. Find the response after the switch is turned on.



- Assume the switch is turned on $@t = 0$
- According to KVL

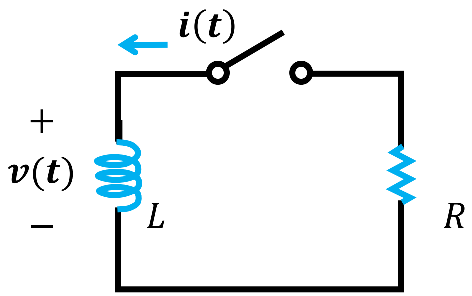
$$L\frac{d}{dt}i(t) + i(t)R = 0$$

$$\rightarrow \quad \frac{d}{dt}i(t) + \frac{R}{L}i(t) = 0$$

1st order differential equation

Example 3: Source free RL circuit

QUESTION: Assume the capacitor L has been charged to I_0 before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = 0$$

- Step 1a: find the **particular integral solution $i_p(t)$**

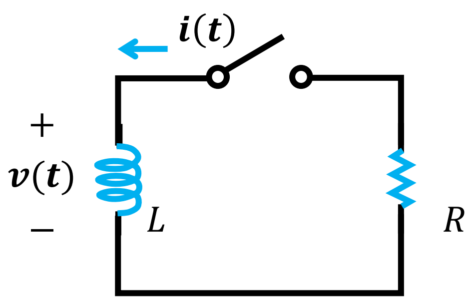
Assume $i_p(t) = K_1$

- Step 1b: substitute $i_p(t)$ to the equation

$$\frac{RK_1}{L} = 0 \quad \rightarrow \quad K_1 = 0$$

Example 3: Source free RL circuit

QUESTION: Assume the capacitor L has been charged to I_0 before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = 0$$

- Step 2a: find the **homogeneous equation**

$$\frac{d}{dt}i(t) + a_1 i(t) = 0 \qquad \text{where} \;\; a_1 = \frac{R}{L}$$

- Step 2b: find the **complementary solution** $i_c(t)$ to the homogeneous equation

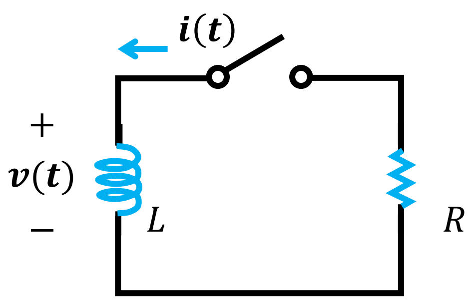
$$v_c(t) = K_2 e^{-st}$$

[Particular integral solution

$i_p(t) = 0$]

Example 3: Source free RL circuit

QUESTION: Assume the capacitor L has been charged to I_0 before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = 0$$

Particular integral solution

$$i_p(t) = 0$$

- Step 2c: substitute $v_c(t)$ to the homogeneous equation

$$\frac{d}{dt}(K_2e^{-st}) + a_1(K_2e^{-st}) = 0$$

$$\rightarrow \quad -sK_2e^{-st} + a_1K_2e^{-st} = 0$$

$$\rightarrow \quad s = a_1 \qquad \text{where } a_1 = \frac{R}{L}$$

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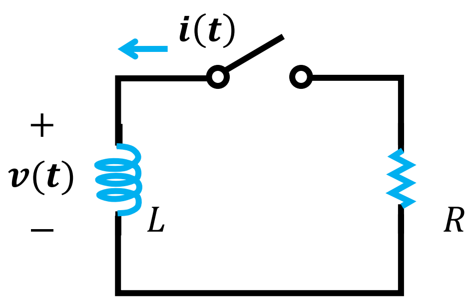


$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = 0$$

- Step 3a: find the value of $i(t)$ at one instant of time

$$i(0) = I_0$$

- Step 3b: substitute $i(0) = I_0$ to $i(t)$

$$i(t) = K_2 e^{-a_1 t}$$

$$\rightarrow \quad K_2 = I_0$$

Particular integral solution

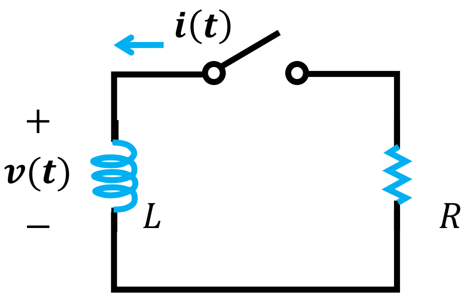
$$i_p(t) = 0$$

Complementary solution

$$i_c(t) = K_2 e^{-a_1 t}$$

Example 3: Source free RL circuit

QUESTION: Assume the capacitor L has been charged to I_0 before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}i(t) + \frac{R}{L}i(t) = 0$$

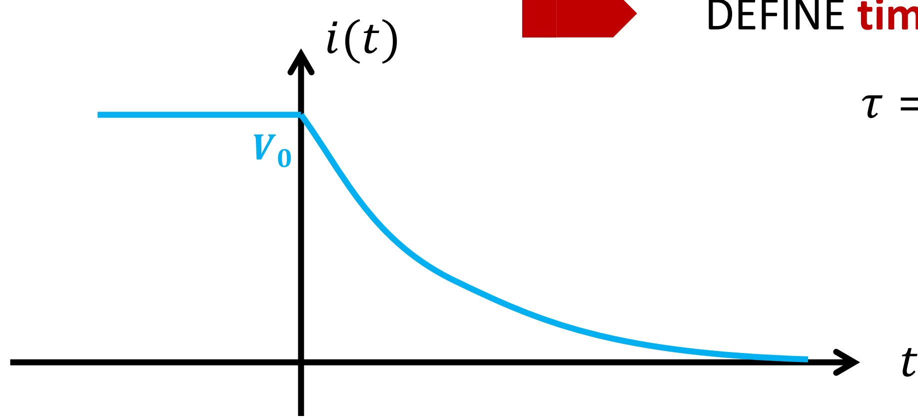
"−" indicates that the current through the capacitor is decaying exponentially

Solution: $i(t) = I_0 e^{-\frac{R}{L}t}$

DEFINE **time constant**

$$\tau = \frac{L}{R}$$

The circuit reaches steady-state faster, if a smaller time constant is applied



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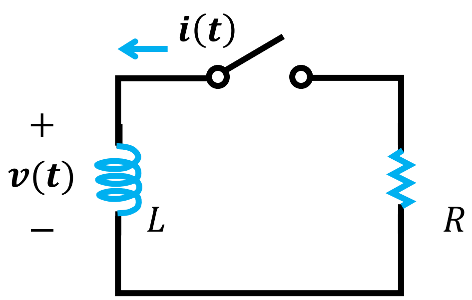
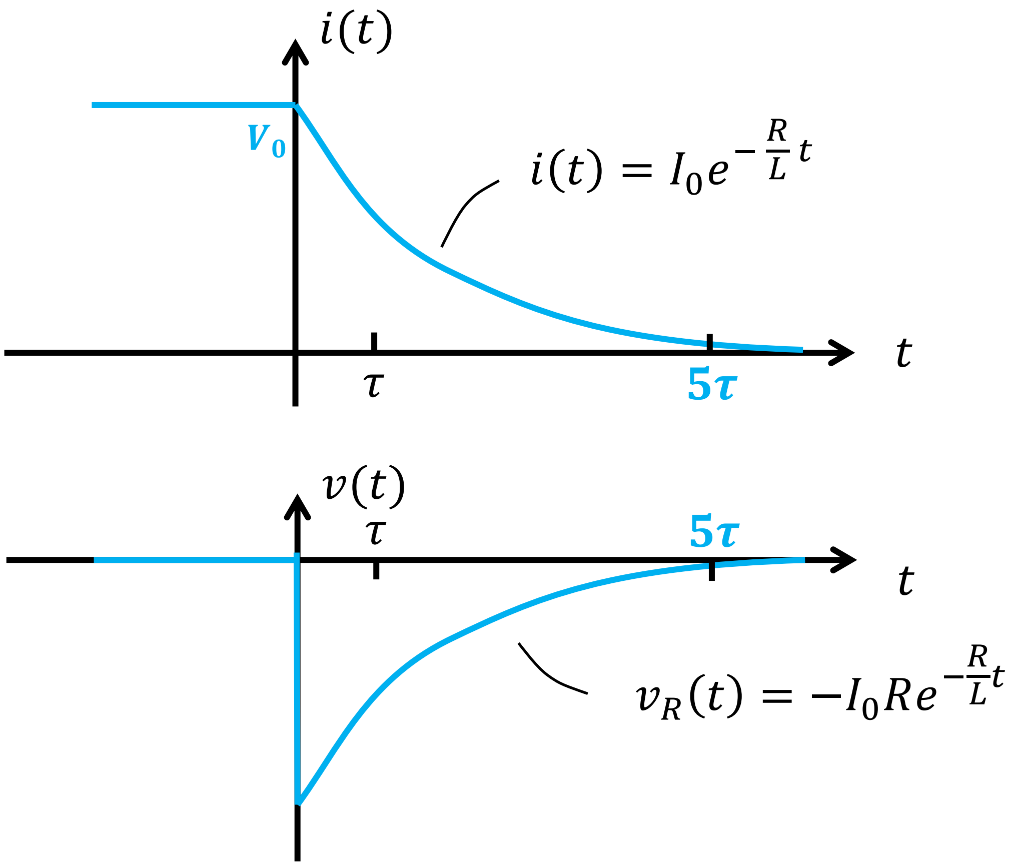




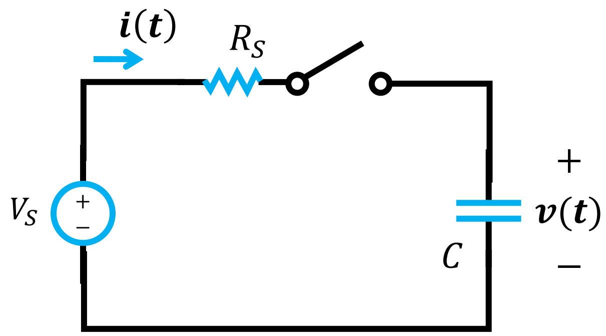
Current through inductor CANNOT change abruptly, but voltage can.

Outlines

- 1^{st} order circuit
  - Source free RC circuit
  - Source free RL circuit
  - **Pulse response**

Example 4: charging of a capacitor

QUESTION: Assume there is no charge on the capacitor C before the switch is turned on. Find the response after the switch is turned on.



- Assume the switch is turned on $@t = 0$
- According to KCL

$$i_C = i_R = i(t) = C\frac{d}{dt}v(t)$$

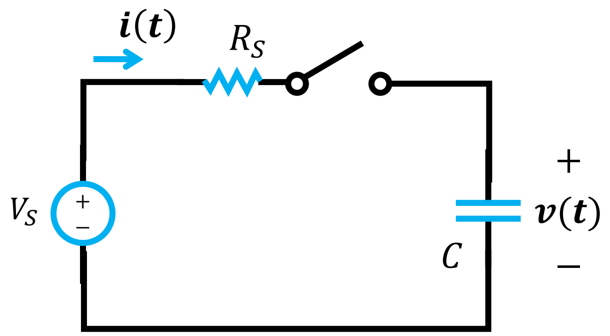
- According to KVL

$$i_R R + v(t) = V_S$$

$$\rightarrow \quad RC\frac{d}{dt}v(t) + v(t) = V_S$$

Example 4: charging of a capacitor

QUESTION: Assume there is no charge on the capacitor C before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}V_S$$

- Step 1a: find the **particular integral solution $x_p(t)$**

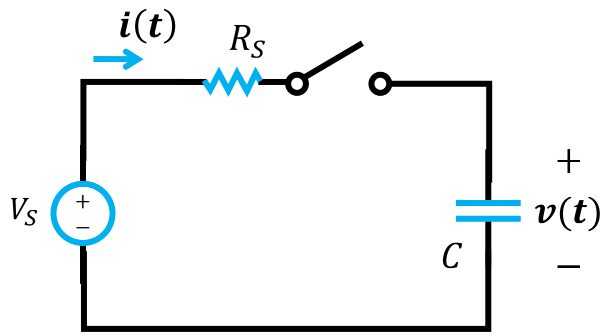
Assume $x_p(t) = K_1$

- Step 1b: substitute $x_p(t)$ to the equation

$$\frac{K_1}{RC} = \frac{1}{RC}V_S \quad \rightarrow \quad K_1 = V_S$$

Example 4: charging of a capacitor

QUESTION: Assume there is no charge on the capacitor C before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}V_S$$

Particular integral solution

$$v_p(t) = V_S$$

- Step 2a: find the **homogeneous equation**

$$\frac{d}{dt}v(t) + a_1 v(t) = 0 \qquad \text{where } a_1 = \frac{1}{RC}$$

- Step 2b: find the **complementary solution** $v_c(t)$ to the homogeneous equation

$$v_c(t) = K_2 e^{-a_1 t}$$

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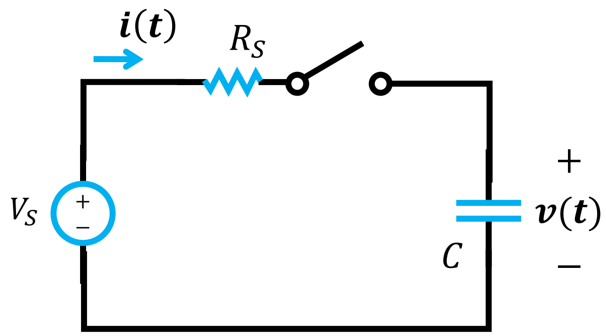
QUESTION: Assume there is no charge on the capacitor C before the switch is turned on. Find the response after the switch is turned on.



$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}V_S$$

Particular integral solution

$$v_p(t) = V_S$$

Complementary solution

$$v_c(t) = K_2 e^{-a_1 t}$$

Full solution $v(t) = v_p(t) + v_c(t)$

- Step 3a: find the value of $v(t)$ at one instant of time

$$v(0) = 0$$

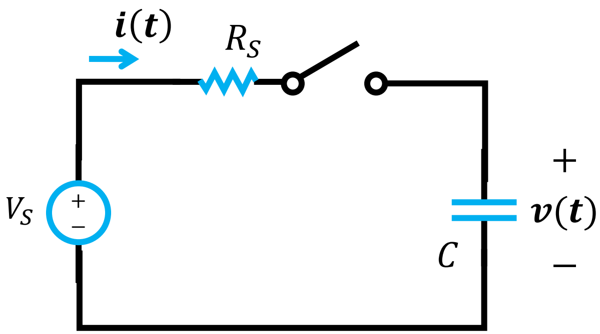
- Step 3b: substitute $v(0) = 0$ to $v(t)$

$$v(t) = V_S + K_2 e^{-a_1 t}$$

$$\rightarrow \quad K_2 = -V_S$$

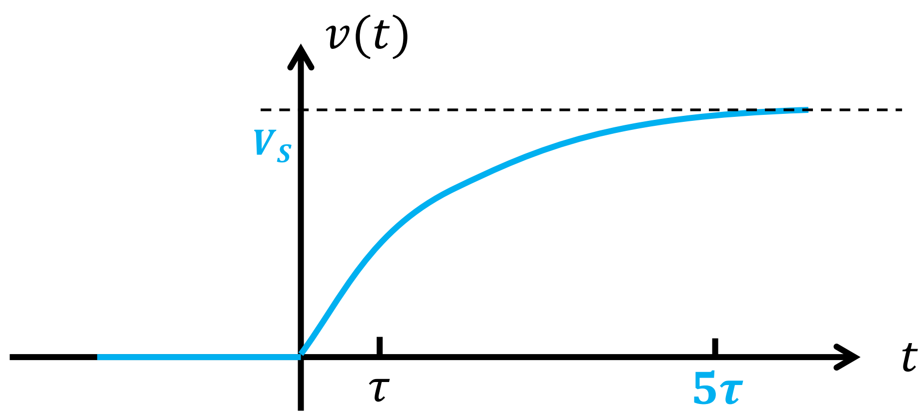
Example 4: charging of a capacitor

QUESTION: Assume there is no charge on the capacitor C before the switch is turned on. Find the response after the switch is turned on.



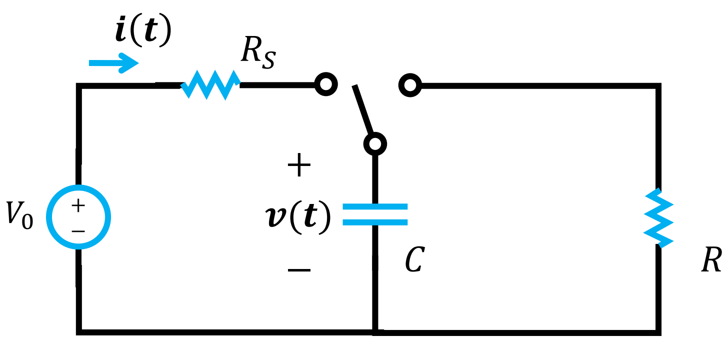
$$\frac{d}{dt}v(t) + \frac{1}{RC}v(t) = \frac{1}{RC}V_S$$

Solution: $v(t) = V_S - V_S e^{-\frac{1}{RC}t}$

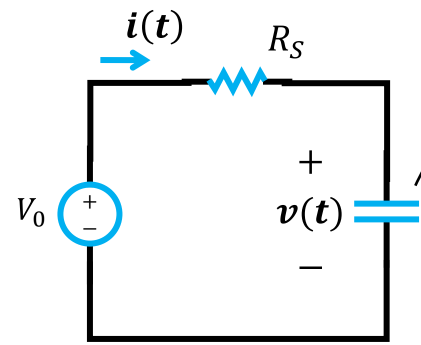


Charging & discharging of a cap

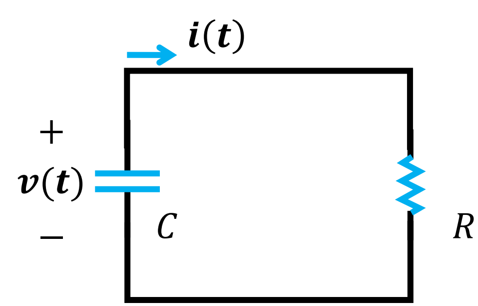
A more practical circuit



- PHASE 1



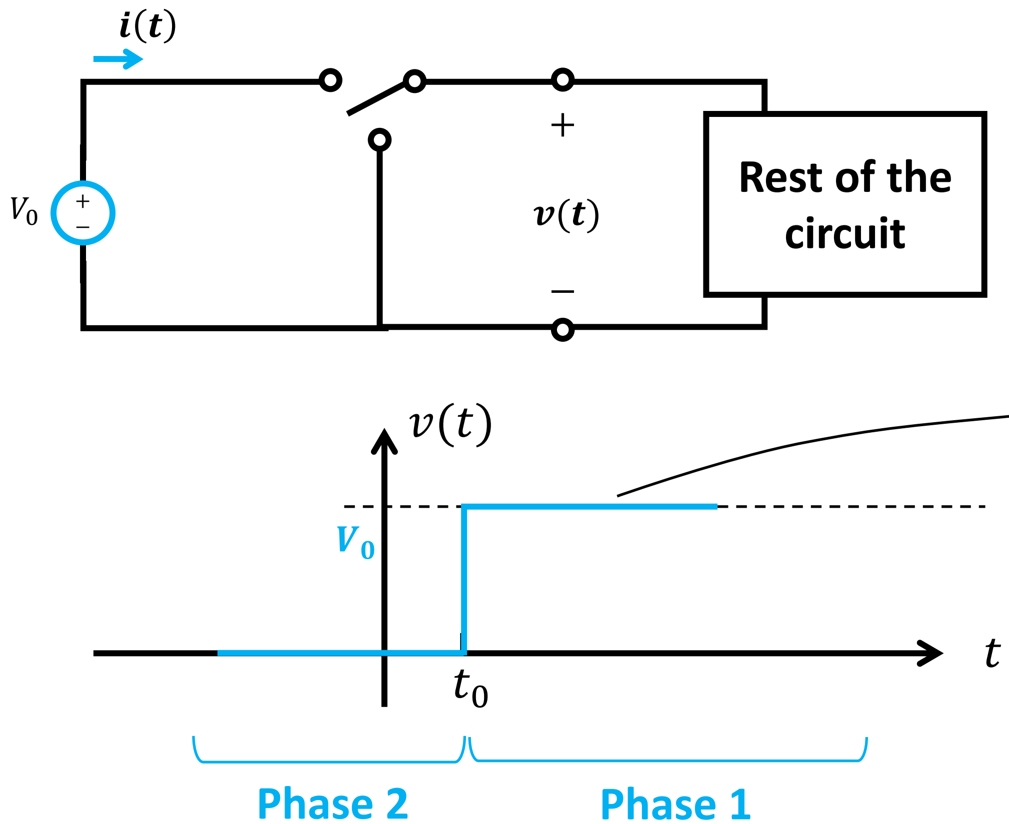
- PHASE 2



Key points:

- **Find the initial voltage of the capacitor in phase 1**
- **Write the equations according to KCL/KVL**
- **Solve the differential equation**

Step forcing function

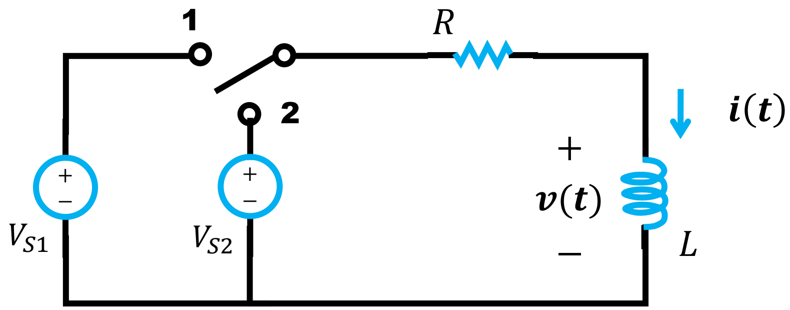


$$v(t) = V_0 u(t - t_0)$$

where $u(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$

Example 5: charging of an inductor

QUESTION: Assume there is no charge on the inductor L before the switch is turned on. Find the response after the switch is switching between node 1 and 2.



- When the switch turned to node 1, according to KVL

$$V_{S1} = i(t)R + L\frac{di(t)}{dt} \quad \rightarrow \quad \frac{di(t)}{dt} + i(t)\frac{R}{L} = \frac{V_{S1}}{L}$$

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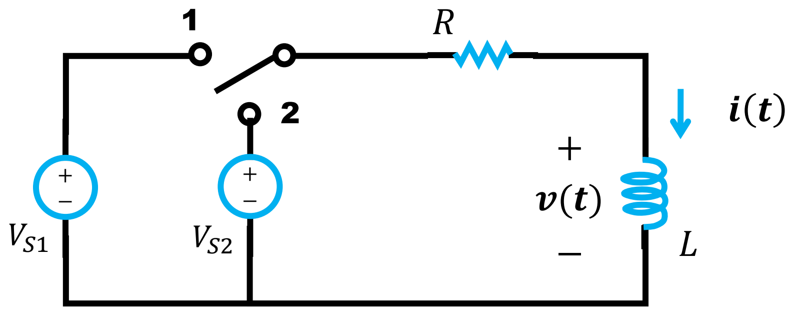

$$\frac{di(t)}{dt} + i(t)\frac{R}{L} = \frac{V_{S1}}{L}$$

- Step 1a: Assume the particular integral solution $i_p(t) = K_1$
- Step 1b: Substitute $i_p(t)$ to the equation → $K_1 = \frac{V_{S1}}{R}$
- Step 2: Find the solution to the homogeneous equation

$$\frac{di(t)}{dt} + i(t)\frac{R}{L} = 0$$

Assume $i_c(t) = K_2 e^{-a_1 t}$ → $a_1 = \frac{R}{L}$

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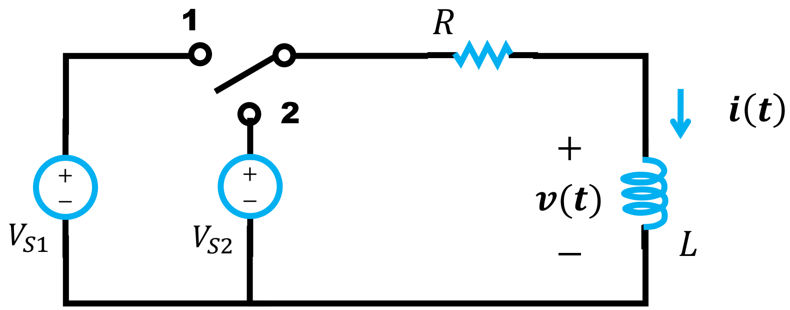

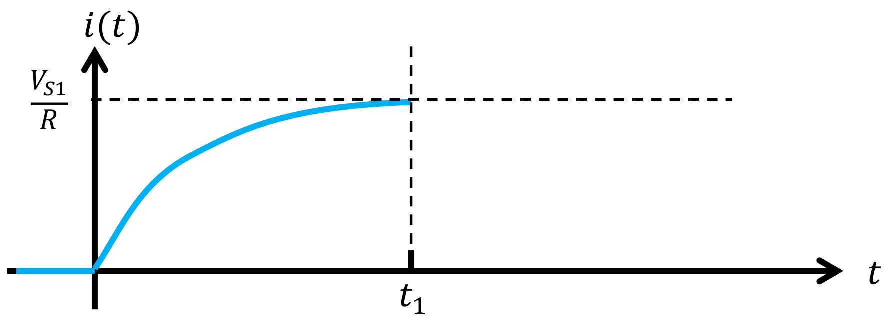

- The full solution $i(t) = i_p(t) + i_c(t) = \frac{V_{S1}}{R} + K_2 e^{-\frac{R}{L}t}$
- Step 3: find the value at one instant of time $i(0) = 0$ → $K_2 = -\frac{V_{S1}}{R}$
- The full solution for $t \in [0, t_1)$ $i(t) = \frac{V_{S1}}{R} - \frac{V_{S1}}{R} e^{-\frac{R}{L}t}$

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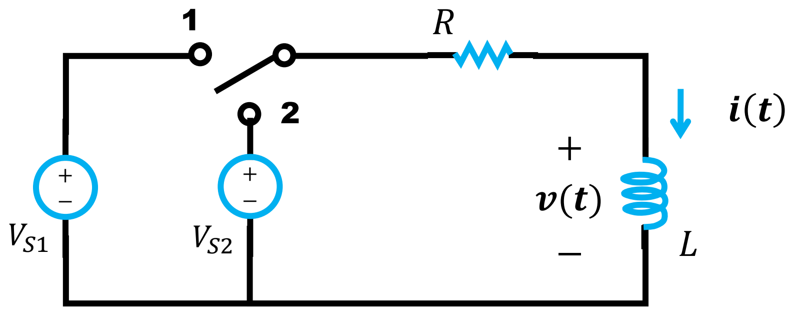
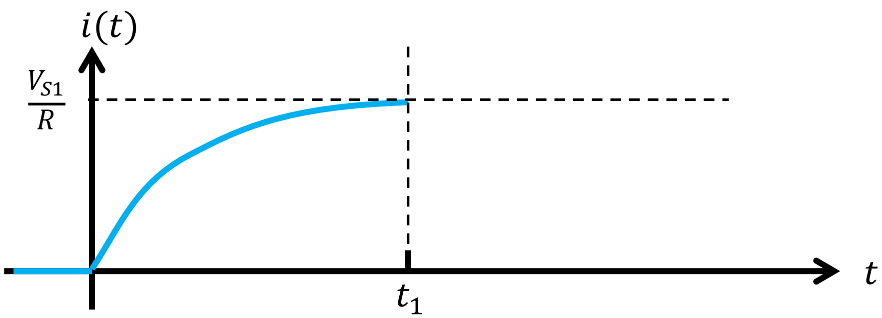




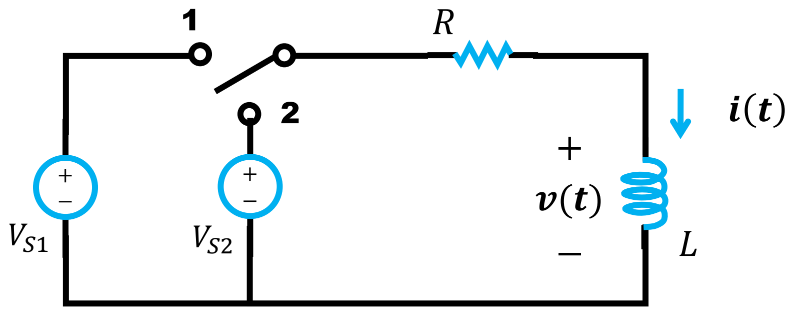
- Assume the switch turns to node 2 @$t = t_1$
- According to KVL

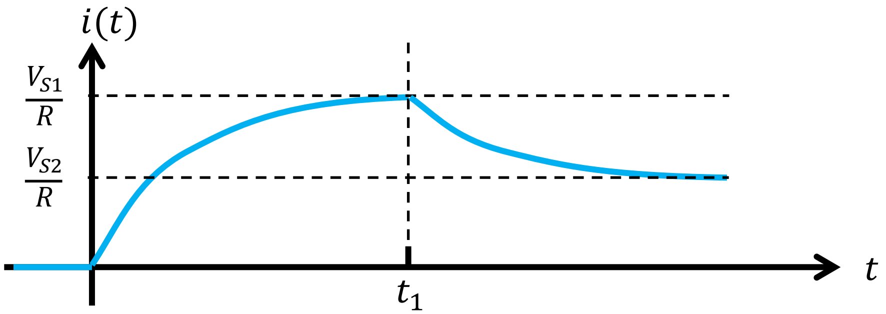
$$V_{S2} = i(t-t_1)R + L\frac{di(t-t_1)}{dt} \rightarrow \frac{di(t-t_1)}{dt} + i(t-t_1)\frac{R}{L} = \frac{V_{S2}}{L}$$

- The full solution must be $\quad i(t) = \frac{V_{S2}}{R} + K_2 e^{-\frac{R}{L}(t-t_1)}$

Example 5: charging of an inductor

QUESTION: Assume there is no charge on the inductor L before the switch is turned on. Find the response after the switch is switching between node 1 and 2.





- find the value at one instant of time $i(t_1) = \frac{V_{S1}}{R}$ $\rightarrow K_2 = \frac{V_{S1}}{R} - \frac{V_{S2}}{R}$
- The full solution for $t \geq t_1$

$$i(t) = \frac{V_{S2}}{R} + \left(\frac{V_{S1}}{R} - \frac{V_{S2}}{R}\right) e^{-\frac{R}{L}(t-t_1)}$$

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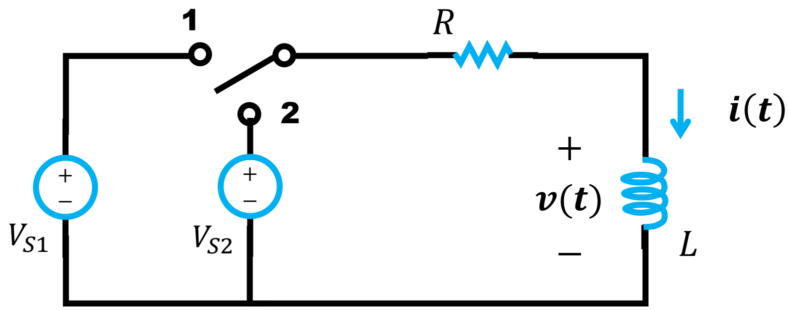
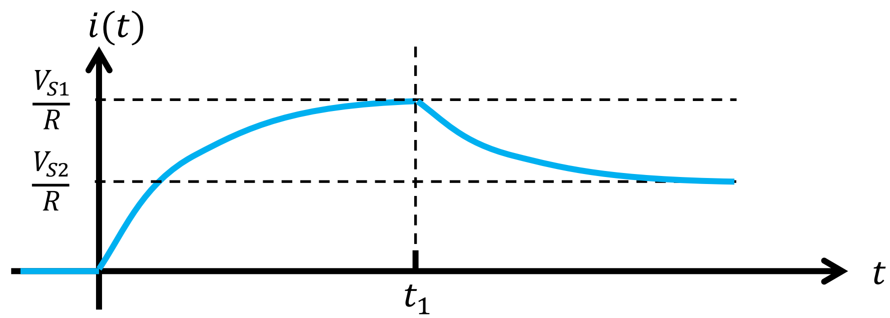




$$i(t) = \frac{V_{S2}}{R} + \left(\frac{V_{S1}}{R} - \frac{V_{S2}}{R}\right) e^{-\frac{R}{L}(t-t_1)}$$

$i(\infty)$ $i(t_1)$ $i(\infty)$

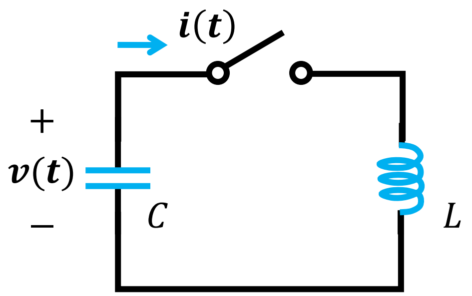
Steady state response (forced response) + **Transient response (natural response)** = **Complete Response**

Outlines

- 1st order circuit
  - Source free RC circuit
  - Source free RL circuit
  - Pulse response
- **2nd order circuits**

Example 6: Source free LC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change after the switch is turned on.



- According to KCL

$$-C\frac{dv(t)}{dt} = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

$$\rightarrow \quad \frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

Review: Calculus

QUESTION: Find the solution $x(t)$ of the following equation

$$\frac{d^2}{dt^2}x(t) + a_1\frac{d}{dt}x(t) + a_2x(t) = f(t)$$

2nd order differential equation

- Step 1: Find a solution $x(t) = x_c(t)$ to the homogeneous equation

$$\frac{d^2}{dt^2}x(t) + a_1\frac{d}{dt}x(t) + a_2x(t) = 0$$

Assume $x_c(t) = Ke^{st}$. Substitute it to the equation

$$s^2Ke^{st} + a_1sKe^{st} + a_2Ke^{st} = 0$$

Since $Ke^{st} \neq 0$

→ $s^2 + a_1s + a_2 = 0$

Define $2\zeta\omega_0 = a_1, \omega_o^2 = a_2$

→ $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

Review: Calculus

QUESTION: Find the solution $x(t)$ of the following equation

$$\frac{d^2}{dt^2}x(t) + a_1\frac{d}{dt}x(t) + a_2 x(t) = f(t) \qquad \text{Define} \quad \begin{cases} 2\zeta\omega_0 = a_1 \\ \omega_o^2 = a_2 \end{cases}$$

- Step 1: Find a solution $x(t) = x_c(t)$ to the homogeneous equation

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \qquad \text{where} \quad \begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$$

- Step 2: Find any solution to the original equation $x(t) = x_p(t)$
- Step 3: the solution to the original equation can be written as

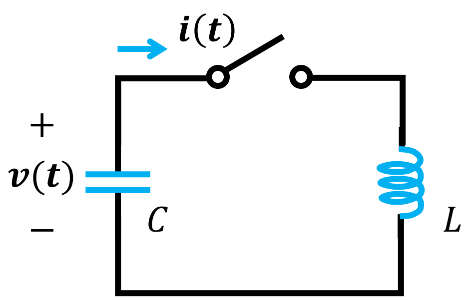
$$x(t) = x_p(t) + x_c(t)$$

$x_c(t)$ → Complementary solution

$x_p(t)$ → Particular integral solution

Example 6: Source free LC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change after the switch is turned on.



$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

- Step 1a: Find the **particular integral solution** $v_p(t)$

Assume $v_p(t) = \mathrm{A}$

- Step 1b: substitute $v_p(t)$ to the equation

$$\frac{1}{LC}A = 0 \quad \rightarrow \quad A = 0$$

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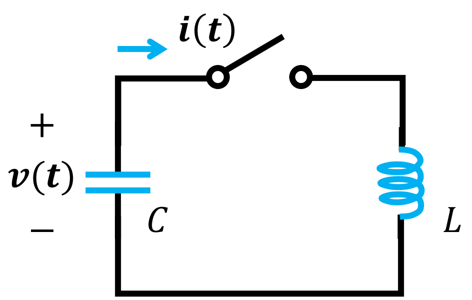

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

- Step 2a: find the **homogeneous equation**

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

- Step 2b: find the **complementary solution** $v_c(t)$ to the homogeneous equation

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$\begin{cases} S_1 = j\omega_0 \\ S_2 = -j\omega_0 \end{cases} \quad \text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

[Particular integral solution

$v_p(t) = 0$]

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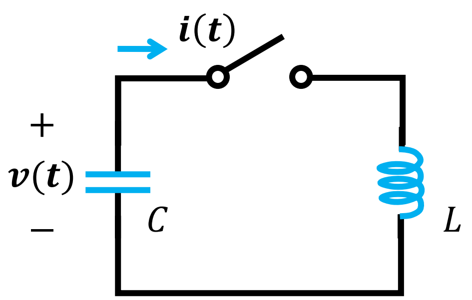

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

- Step 3a: find the value of $v(0)$ and $\left.\frac{dv(t)}{dt}\right|_{t=0}$

- Step 3b: substitute $v(0)$ and $\left.\frac{dv(t)}{dt}\right|_{t=0}$ to $v(t)$

$$\begin{cases} v(0) = K_1 + K_2 \\ \left.\frac{dv(t)}{dt}\right|_{t=0} = K_1 s_1 + K_2 s_2 = -\frac{i(0)}{C} \end{cases}$$

K_1 and K_2 can be solved

Particular integral solution

$$v_p(t) = 0$$

Complementary solution

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

where $S_1 = j\omega_0, S_2 = -j\omega_0$

Full solution $v(t) = v_p(t) + v_c(t)$

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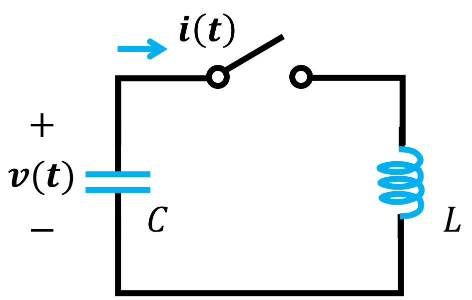

$$\begin{cases} v(t) = Acos(\omega_0 t + \theta) \\ i(t) = \sqrt{\frac{C}{L}} Asin(\omega_0 t + \theta) \end{cases}$$

where

$$\begin{cases} A = \sqrt{v^2(0) + \frac{L}{C} i^2(0)} \\ \omega_0 = \frac{1}{\sqrt{LC}} \\ \theta = tan^{-1}\left(\sqrt{\frac{L}{C}} \frac{i(0)}{v(0)}\right) \end{cases}$$

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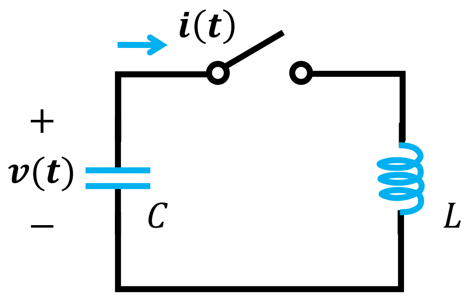

$$\begin{cases} v(t) = A\cos(\omega_0 t + \theta) \\ i(t) = \sqrt{\frac{C}{L}} A\sin(\omega_0 t + \theta) \end{cases}$$

Energy stored in the capacitor/inductor

$$\begin{cases} w_C(t) = \frac{1}{2}Cv^2(t) = \left(\frac{1}{2}Cv^2(0) + \frac{1}{2}Li^2(t)\right)cos^2(\omega_0 t + \theta) \\ w_L(t) = \frac{1}{2}Li^2(t) = \left(\frac{1}{2}Cv^2(0) + \frac{1}{2}Li^2(t)\right)sin^2(\omega_0 t + \theta) \end{cases}$$

The total energy doesn't change

$$\boldsymbol{w(t) = w_C(t) + w_L(t) = \frac{1}{2}Cv^2(0) + \frac{1}{2}Li^2(t)}$$

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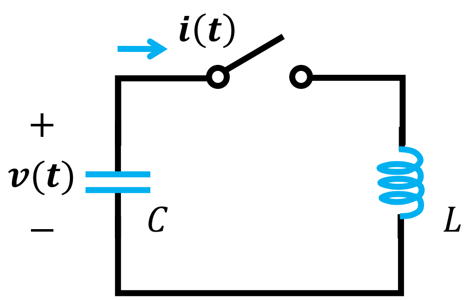
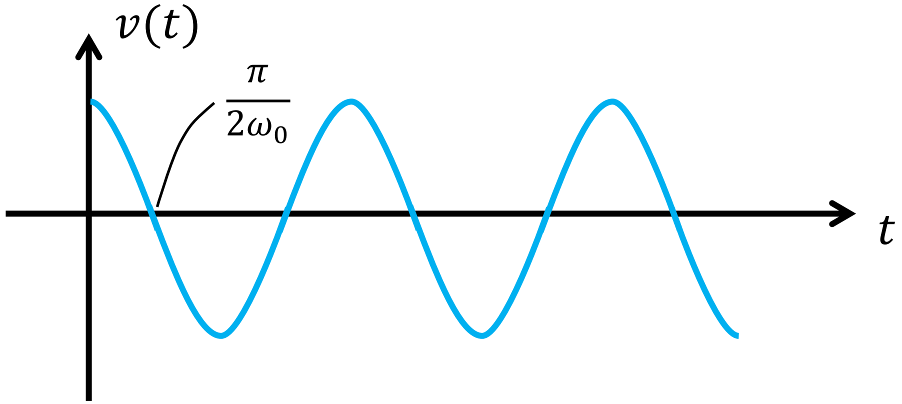
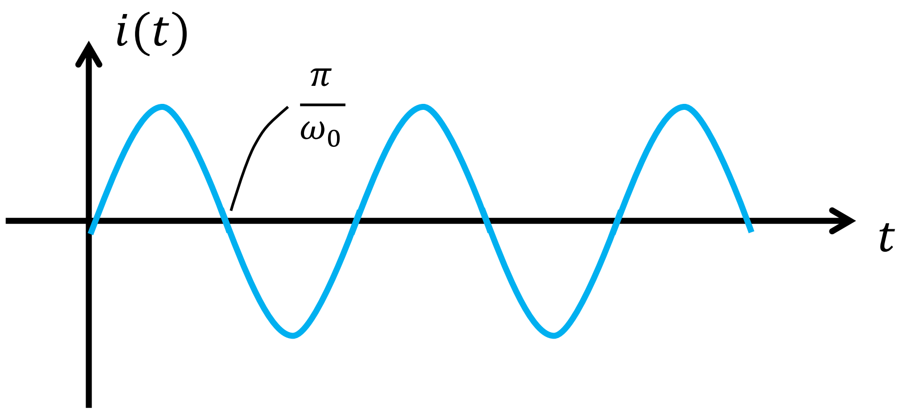




$$\begin{cases} v(t) = A\cos(\omega_0 t + \theta) \\ i(t) = \sqrt{\frac{C}{L}} A\sin(\omega_0 t + \theta) \end{cases}$$

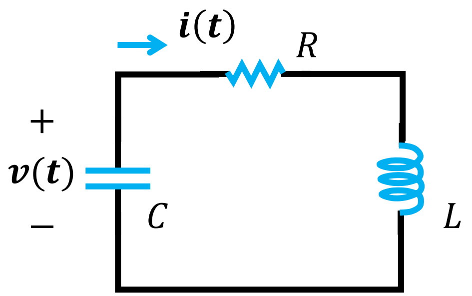


Outlines

- 1st order circuit
  - Source free RC circuit
  - Source free RL circuit
  - Pulse response
- 2nd order circuits
  - Source free LC circuit
  - **Source free RLC circuit**

Example 7: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change.



- According to KCL

$$i(t) = -C\frac{dv(t)}{dt}$$

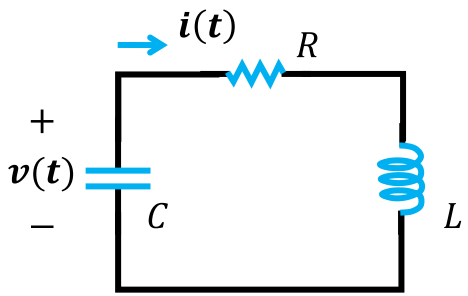
- According to KVL

$$-v(t) + i(t)R + L\frac{di(t)}{dt} = 0$$

$$\rightarrow \quad \frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

Example 7: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change.



$$\frac{d^2v(t)}{dt^2}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

- Step 1a: Find the **particular integral solution** $v_p(t)$

Assume $v_p(t) = \mathrm{A}$

- Step 1b: substitute $v_p(t)$ to the equation

$$\frac{1}{LC}A=0 \quad \rightarrow \quad A=0$$

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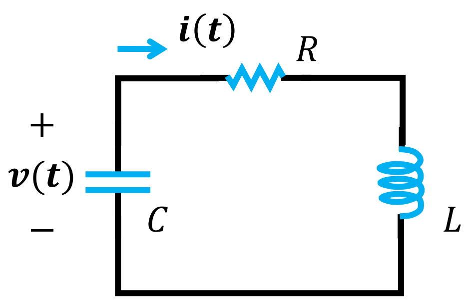

$$\frac{d^2v(t)}{dt^2}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

- Step 2a: find the **homogeneous equation**

$$\frac{d^2v(t)}{dt^2}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

Particular integral solution

$$v_p(t)=0$$

- Step 2b: find the **complementary solution** $v_c(t)$ to the homogeneous equation

$$v_c(t)=K_1e^{s_1t}+K_2e^{s_2t}$$

$$\rightarrow \begin{cases} S_1=-\zeta\omega_0+\omega_0\sqrt{\zeta^2-1} \\ S_2=-\zeta\omega_0-\omega_0\sqrt{\zeta^2-1} \end{cases} \quad \text{where} \quad \begin{cases} 2\zeta\omega_0=\frac{R}{L} \\ \omega_o^2=\frac{1}{LC} \end{cases}$$

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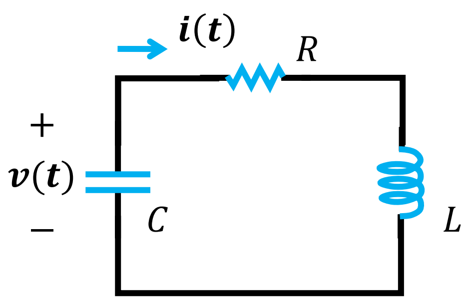
QUESTION: Find how the voltage on the capacitor and the current through the inductor change.



$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

- Step 3a: find the value of $v(0)$ and $\left.\frac{dv(t)}{dt}\right|_{t=0}$
- Step 3b: substitute $v(0)$ and $\left.\frac{dv(t)}{dt}\right|_{t=0}$ to $v(t)$

$$\begin{cases} v(0) = K_1 + K_2 \\ \left.\frac{dv(t)}{dt}\right|_{t=0} = K_1 s_1 + K_2 s_2 = -\frac{i(0)}{C} \end{cases}$$

K_1 and K_2 can be solved

Particular integral solution

$$v_p(t) = 0$$

Complementary solution

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$

Full solution $v(t) = v_p(t) + v_c(t)$

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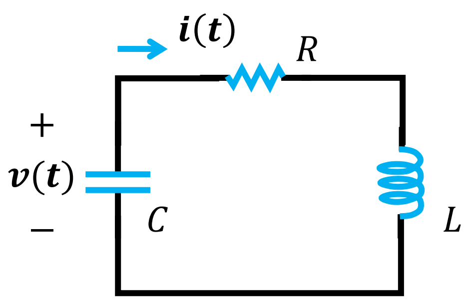


Particular integral solution

$$v_p(t) = 0$$

Complementary solution

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$

Full solution $v(t) = v_p(t) + v_c(t)$

A DISCUSSION ON ζ

- If $\zeta = 1$ → **critically damped**

→ $S_1 = S_2 = -\zeta\omega_0$

$$v_c(t) = K_1 e^{-\zeta\omega_0 t} + K_2 t e^{-\zeta\omega_0 t}$$

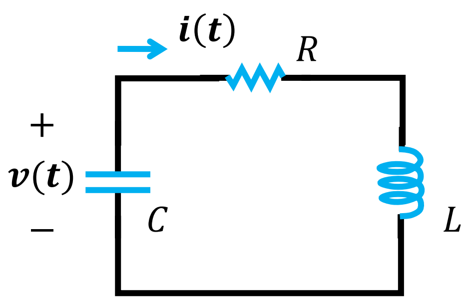
- If $\zeta > 1$ → **overdamped**

$$v_c(t) = K_1 e^{\left(-\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}\right)t} + K_2 e^{\left(-\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}\right)t}$$

The response is the sum of two decaying exponentials

Example 7: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change.



Particular integral solution

$$v_p(t) = 0$$

Complementary solution

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$

Full solution $v(t) = v_p(t) + v_c(t)$

A DISCUSSION ON ζ

- If $\zeta < 1$ → **underdamped**

$$\begin{cases} S_1 = -\zeta\omega_0 + j\omega_0\sqrt{1 - \zeta^2} \\ S_2 = -\zeta\omega_0 - j\omega_0\sqrt{1 - \zeta^2} \end{cases}$$

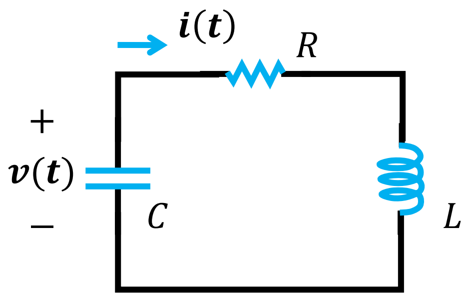
Define $\omega_d = \omega_0\sqrt{1 - \zeta^2}$

$$v_c(t) = e^{-\zeta\omega_0 t}[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)]$$

The response is an exponentially damped oscillatory response

Example 7: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change.



Particular integral solution

$$v_p(t) = 0$$

Complementary solution

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

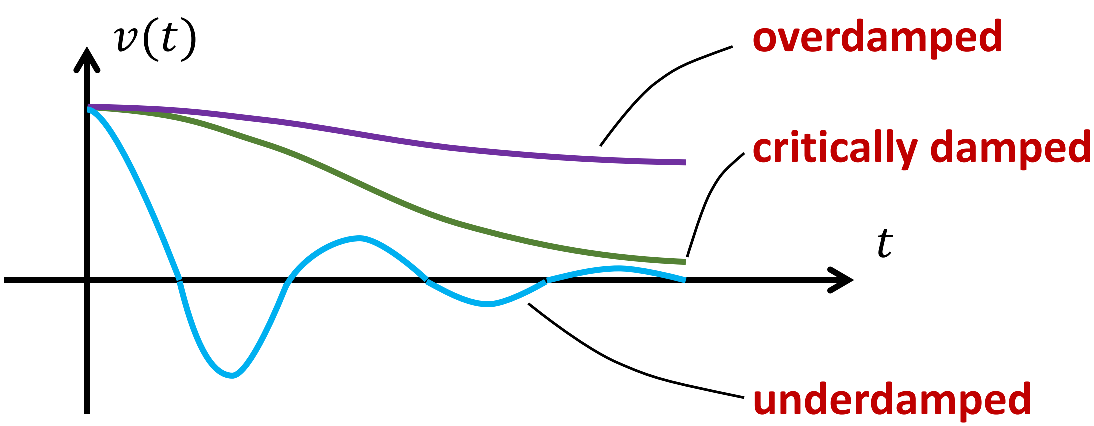
Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$

Full solution $v(t) = v_p(t) + v_c(t)$

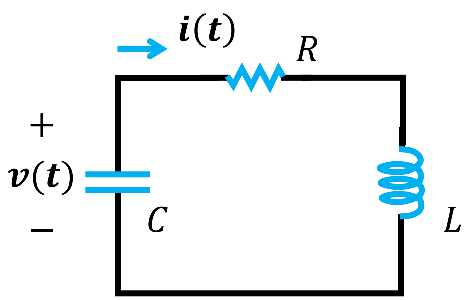
A DISCUSSION ON ζ

- If $\zeta < 1$ → **underdamped**
- If $\zeta = 1$ → **critically damped**
- If $\zeta > 1$ → **overdamped**



Example 7: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change.



$$v_c(t) = e^{-\zeta\omega_0 t}[A_1\cos(\omega_d t) + A_2\sin(\omega_d t)]$$

Define **DAMPING FACTOR** $\alpha = \frac{R}{2L}$

Define **RESONATE FREQUENCY** $\omega_0 = \frac{1}{\sqrt{LC}}$

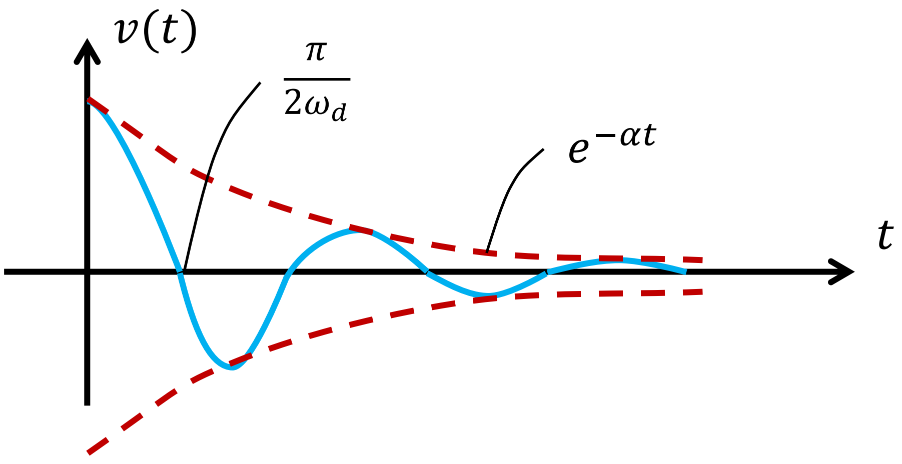
Define **DAMPING RATIO** $\zeta = \frac{\alpha}{\omega_0} = \frac{R}{2\omega_0 L}$

$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

Solution: $v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$

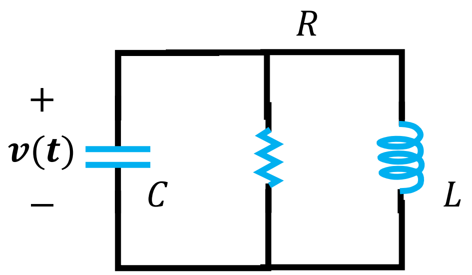


Outlines

- 1st order circuit
  - Source free RC circuit
  - Source free RL circuit
  - Pulse response
- 2nd order circuits
  - Source free LC circuit
  - Source free series RLC circuit
  - **Source free series parallel circuit**

Example 8: parallel connected RLC circuit

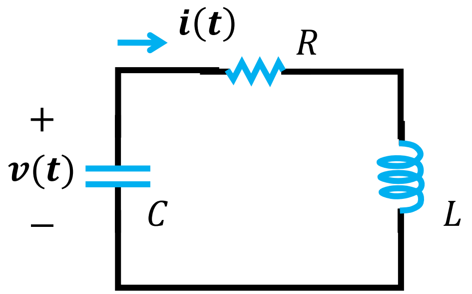
QUESTION: Find how the voltage on the capacitor and the current through the inductor change.



- According to KCL

$$C\frac{dv(t)}{dt}+\frac{v(t)}{R}+\frac{1}{L}\int_{-\infty}^{t}v(t)dt=0$$

$$\rightarrow \quad \frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

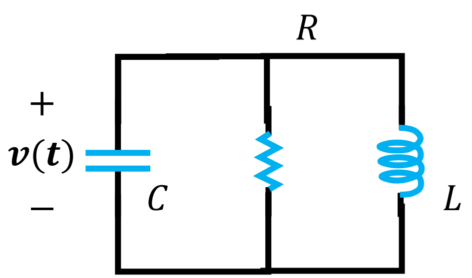


- Series connected RLC circuit

$$\frac{d^2v(t)}{dt^2}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

Example 8: parallel connected RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change.

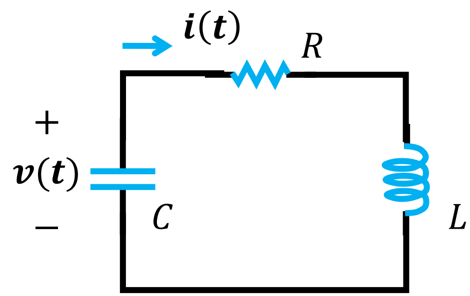


$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

Solution: $v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{1}{2\omega_0 RC}$



$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

Solution: $v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

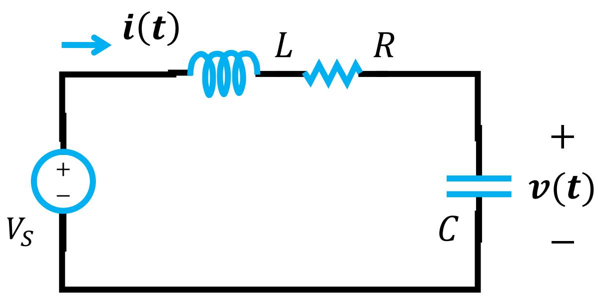
and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$

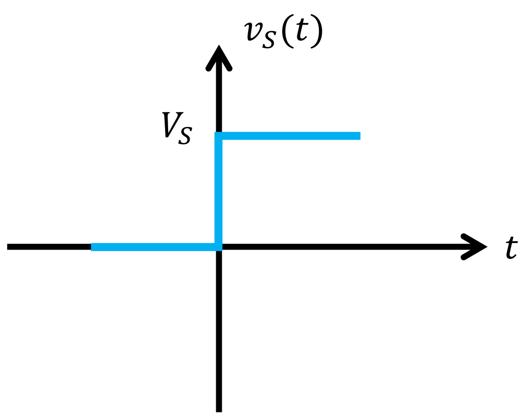
Outlines

- 1st order circuit
  - Source free RC circuit
  - Source free RL circuit
  - Pulse response
- 2nd order circuits
  - Source free LC circuit
  - Source free series RLC circuit
  - Source free series parallel circuit
  - Response of RLC circuit

Example 9: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change with an input voltage of step function.





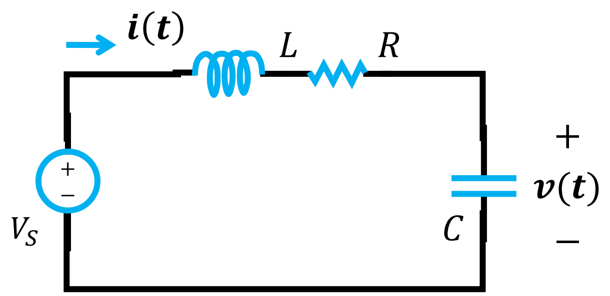
- According to KVL

$$-v(t) + i(t)R + L\frac{di(t)}{dt} = V_S$$

$$\rightarrow \frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{V_S}{LC}$$

Example 9: source free RLC circuit

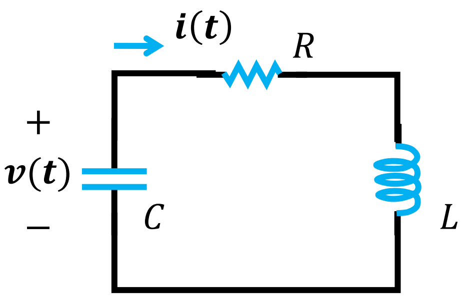
QUESTION: Find how the voltage on the capacitor and the current through the inductor change with an input voltage of step function.



- According to KVL

$$-v(t) + i(t)R + L\frac{di(t)}{dt} = V_S$$

$$\rightarrow \quad \frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{V_S}{LC}$$



$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

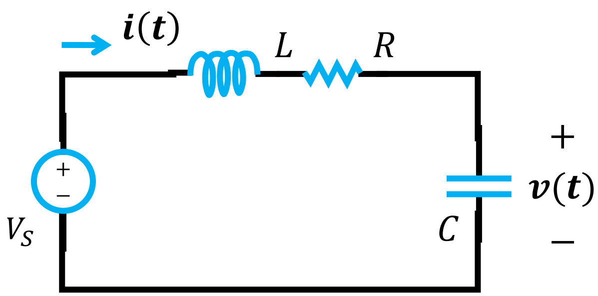
Solution: $v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$

Example 9: source free RLC circuit

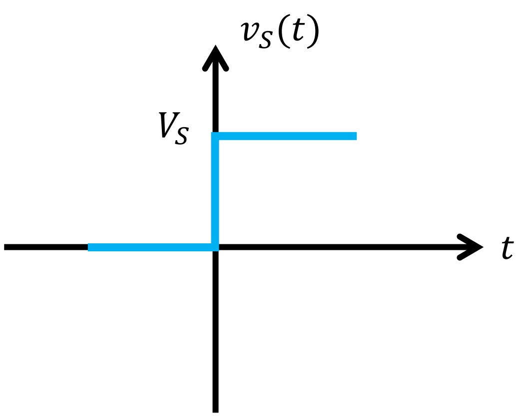
QUESTION: Find how the voltage on the capacitor and the current through the inductor change with an input voltage of step function.



$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{V_S}{LC}$$

- Step 1a: Find the **particular integral solution** $v_p(t)$

Assume $v_p(t) = \mathrm{A}$

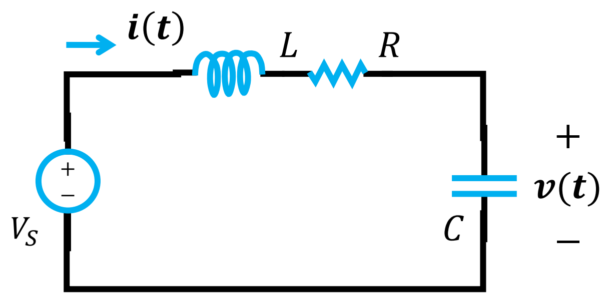


- Step 1b: substitute $v_p(t)$ to the equation

$$\frac{1}{LC}A = \frac{V_S}{LC} \quad \rightarrow \quad A = V_S$$

Example 9: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change with an input voltage of step function.



Particular integral solution

$v_p(t) = V_S$

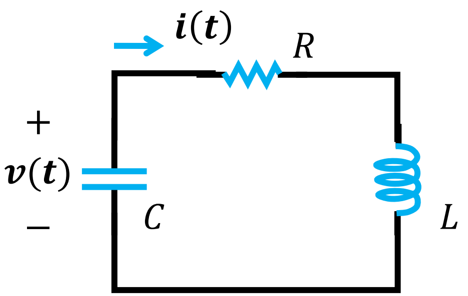
Complementary solution

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

and $\omega_0 = \frac{1}{\sqrt{LC}}, \zeta = \frac{R}{2\omega_0 L}$

Full solution $v(t) = v_p(t) + v_c(t)$



Particular integral solution

$v_p(t) = 0$

Complementary solution

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

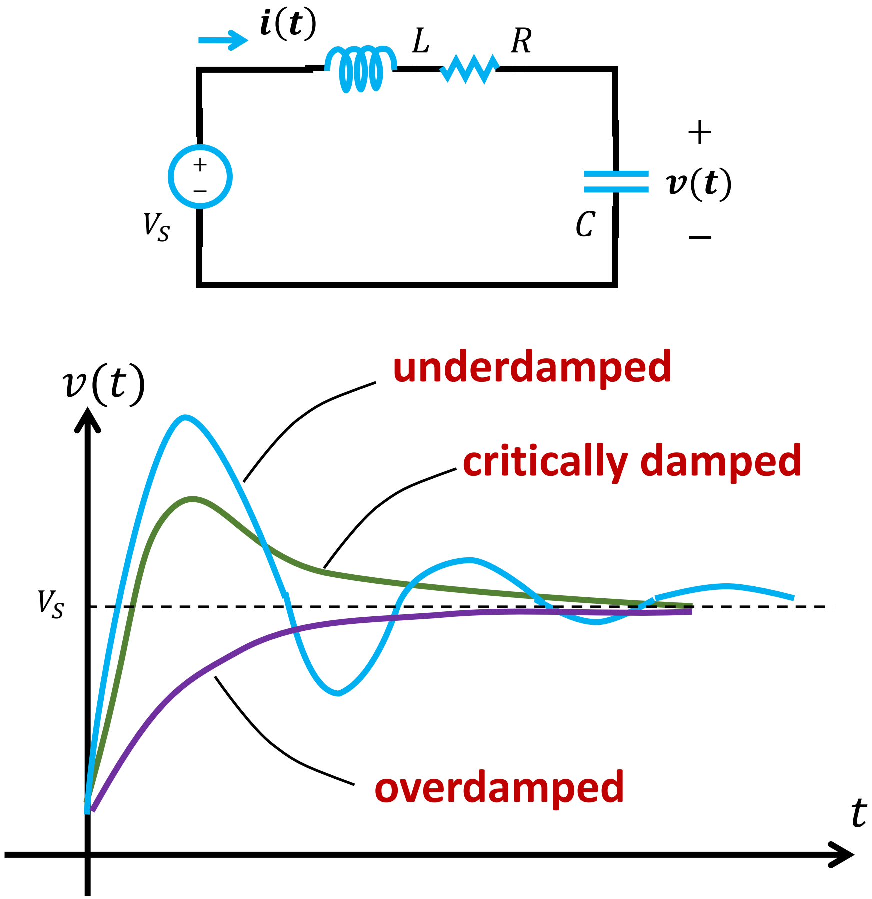
Where $\begin{cases} S_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} \\ S_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1} \end{cases}$

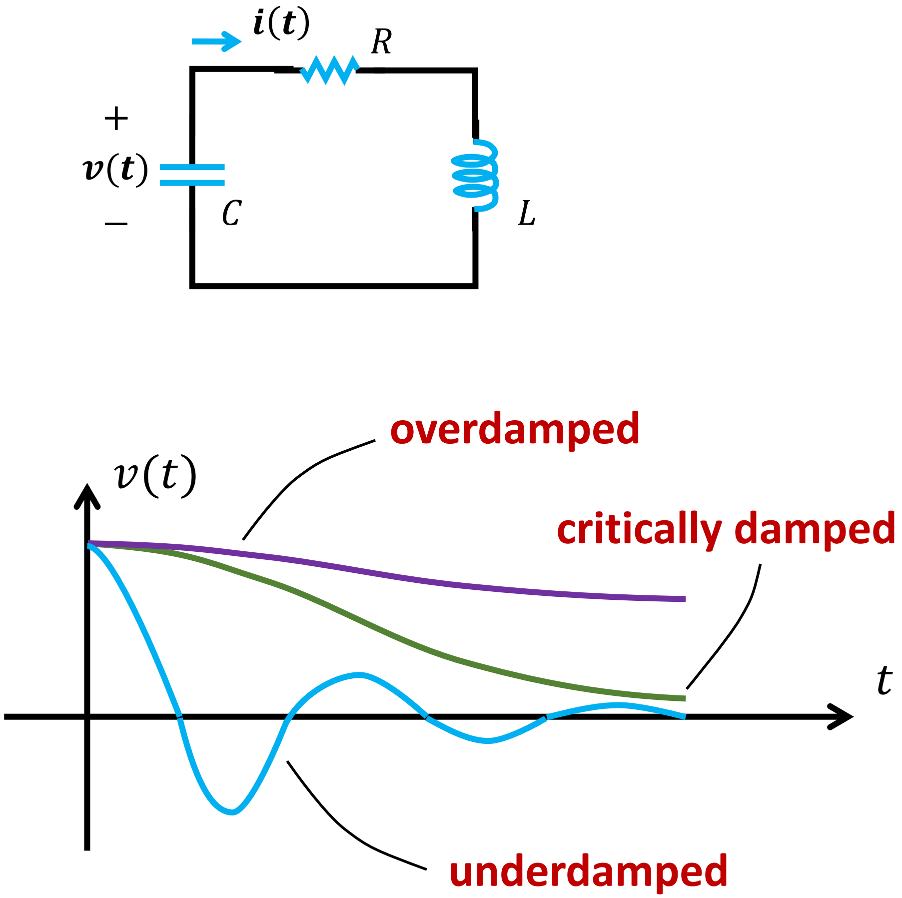
and $\omega_0 = \frac{1}{\sqrt{LC}}, \zeta = \frac{R}{2\omega_0 L}$

Full solution $v(t) = v_p(t) + v_c(t)$

Example 9: source free RLC circuit

QUESTION: Find how the voltage on the capacitor and the current through the inductor change with an input voltage of step function.





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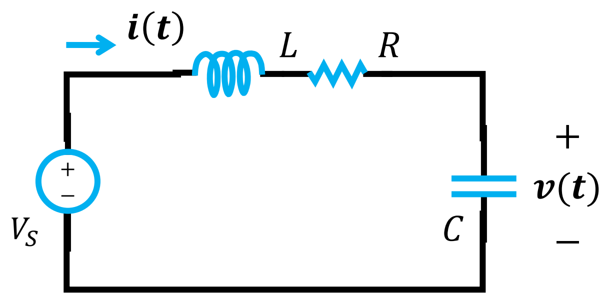
QUESTION: Find how the voltage on the capacitor and the current through the inductor change with an input voltage of step function.

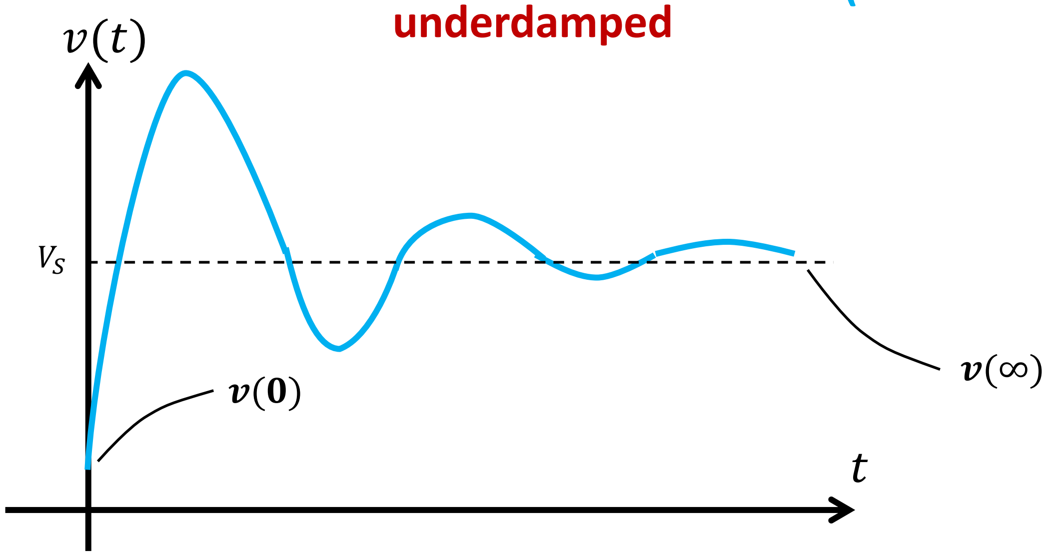


$$v(t) = V_S + e^{-\zeta\omega_0 t}[A_1\cos(\omega_d t) + A_2\sin(\omega_d t)]$$

Steady state response (forced response) + Transient response (natural response)

= Complete Response

underdamped



Outlines

- How to analyze 1st/2nd order circuit in time domain?
  - Write the circuit equation according to KVL/KCL
  - Solve the differential equation
    - Step 1a: Find the **particular integral solution** $v_p(t)$
    - Step 1b: substitute $v_p(t)$ to the equation to solve the unknown
    - Step 2a: find the **homogeneous equation**
    - Step 2b: find the **complementary solution** $v_c(t)$ to the homogeneous equation
    - Step 3a: find the initial voltage/current values
    - Step 3b: substitute the initials to the full solution to solve the unknown

Reading tasks & learning goals

- Reading tasks
  - Basic Engineering Circuit Analysis, 10th edition
    - Chapter 7
- Learning goals
  - Be able to calculate initial values for inductor currents & capacitor voltages in transient circuits
  - Be able to calculate V/I in 1st order transient circuit
  - Be able to calculate V/I in 2nd order transient circuit
  - Know what is time constant, steady state response, transient response, complete response