1. (略)

2. 解：⟨1⟩ $0 \le Z \le Y \le \theta$　　令 $F(y,z) = P(Y \le y,\ Z \le z)$

① 若 $\min\{y,z\} \le 0$　　$F(y,z) = 0$

② 若 $0 \le y \le z \le \theta$　　$F(y,z) = P(Y \le y) = \left(\frac{y}{\theta}\right)^n$

若 $0 \le z \le y \le \theta$　　$F(y,z) = P(Y \le y) - P(Y \le y,\ Z > z)$

$$= \left(\frac{y}{\theta}\right)^n - \left(\frac{y-z}{\theta}\right)^n$$

③ 若 $0 \le y \le \theta \le z$　　$F(y,z) = P(Y \le y) = \left(\frac{y}{\theta}\right)^n$

若 $0 \le z \le \theta \le y$　　$F(y,z) = P(Z \le z) = 1 - \left(\frac{\theta - z}{\theta}\right)^n$

④ 若 $\min\{y,z\} \ge \theta$　　$F(y,z) = 1$

⟨2⟩ $F_Y(y) = \begin{cases} \left(\frac{y}{\theta}\right)^n & y \in (0,\theta] \\ 1 & y \in [\theta, +\infty) \end{cases}$　　$f_Y(y) = \begin{cases} \frac{n y^{n-1}}{\theta^n} & y \in (0,\theta) \\ 0 & y \notin (0,\theta) \end{cases}$

$$EY = \int_0^\theta y \frac{n y^{n-1}}{\theta^n} dy = \frac{n\theta}{n+1}$$

$EY^2 = \frac{n\theta^2}{n+2}$　　$Var(Y) = \frac{n\theta^2}{(n+1)^2(n+2)}$

记 $W = \theta - Z$　　则 $F_W(w) = P(\theta - Z \le w) = P(Z \ge \theta - w)$

W 与 Y 同分布.　　$= \begin{cases} 0 & w \le 0 \\ \left(\frac{w}{\theta}\right)^n & w \in (0,\theta) \\ 1 & w \ge \theta \end{cases}$

则 $E(Z) = E(\theta - W) = \frac{\theta}{n+1}$

$Var(Z) = Var(W) = \frac{n\theta^2}{(n+1)^2(n+2)}$

3解：〈1〉 当 $x \in [-1, 1]$ $\quad P_X(x) = \int_0^1 P(x, y)\,dy = \int_{-1}^{1} \frac{1+xy}{4}\,dy = \frac{1}{2}$

故 $P_X(x) = \frac{1}{2} \cdot I_{\{-1 \le x \le 1\}} \quad P_Y(y) = \frac{1}{2} \cdot I_{\{1 \le y \le 1\}}$.

$P_X(x) \cdot P_Y(y) = \frac{1}{4} I_{\{(x,y) \in [-1,1]^2\}} \neq P(x, y)$ 故 X、Y不独立

〈2〉 令 $V = X^2 \quad W = Y^2$

$F_{V,W} = P(X^2 \le V, Y^2 \le W) = P(-\sqrt{V} \le X \le \sqrt{V}, -\sqrt{W} \le Y \le \sqrt{W}) = \int_{-\sqrt{W}}^{\sqrt{W}} \int_{-\sqrt{V}}^{\sqrt{V}} \frac{1+xy}{4}\,dx\,dy = \sqrt{W \cdot V}$

$F_V(v) = P(X^2 \le v) = \sqrt{v} \quad F_W(w) = \sqrt{w}$

易知 $F_{V,W} = F_V(v) \cdot F_W(w)$ 故 X^2, Y^2 独立.

4. 解：〈1〉 同题2有 $F_U(u) = \begin{cases} 0 & u \le 0 \\ -u^2+2u & u \in (0,1) \\ 1 & u \ge 1 \end{cases} \quad F_V(v) = \begin{cases} 0 & v \le 0 \\ v^2 & v \in (0,1) \\ 1 & v \ge 1 \end{cases}$

〈2〉 $F_{U,V}(u,v) = \begin{cases} 0 & \min(u,v) \le 0 \\ v^2 & 0 \le v \le u \le 1 \\ v^2 - (v-u)^2 & 0 \le u \le v \le 1 \\ v^2 & 0 \le v \le 1 \le u \\ 1-(1-u)^2 & 0 \le u \le 1 \le v \\ 1 & \min(u,v) \ge 1 \end{cases}$

仅在 $\{(u,v) \mid 0 < u \le v < 1\}$ 区域 $f_{U,V} = 2$

其余为0，故 U, V 服从区域

$\{(u,v) \mid 0 < u \le v < 1\}$ 上的均匀分布.

〈3〉 $P_{W,V}(w,v) = P_{U,V}(wv, v) \left| \frac{\partial(wv, v)}{\partial(w, v)} \right| = 2 \cdot I_{\{0 \le w \cdot v \le v \le 1\}} \cdot |v|$

$= 2v \cdot I_{\{0 < v < 1\}} \cdot I_{\{0 < w < 1\}}$

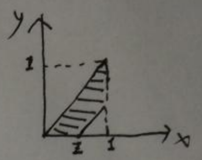
$P_W(w) = \int_R P_{W,V}(w,v)\,dv = 1\{0 < w < 1\}$

$P_V(v) = \int_R P_{W,V}(w,v)\,dw = 2v\, I_{\{0 < v < 1\}}$

易知 W与V 独立.

5、解：$F_Z(z)=P(Z\le z)$

$$=\iint_{\{(x,y)\mid 0<x<1,\ 0<y<x<1,\ x-y\le z\}} 3x\,dx\,dy \quad \text{(如右图阴影区所示)}$$



$$=\frac{3}{2}z-\frac{1}{2}z^3 \quad (0<z<1)$$

$$\Rightarrow p_Z(z)=\left(\frac{3}{2}-\frac{3}{2}z^2\right)\cdot 1_{\{0<z<1\}}$$

6、解：〈1〉记 $(X_1,Y_1)\sim N(0,0,1,1,\frac{1}{3})$

$(X_2,Y_2)\sim N(0,0,1,1,-\frac{1}{3})$.

$$P_X(x)=\int_{-\infty}^{+\infty}p(x,y)dy=\int_{-\infty}^{+\infty}\frac{1}{2}(\varphi_1(x,y)+\varphi_2(x,y))dy=\frac{1}{2}P_{X_1}(x)+\frac{1}{2}P_{X_2}(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

同理 $P_Y(y)=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{y^2}{2}}$

〈2〉
$$E(XY)=\iint_{R^2}xy\,p(x,y)dxdy=\iint_{R^2}xy\frac{1}{2}(\varphi_1(x,y)+\varphi_2(x,y))dxdy$$

$$=\frac{1}{2}E(X_1Y_1)+\frac{1}{2}E(X_2Y_2)=\frac{1}{2}(EX_1EY_1+\rho_1\sqrt{Var(X_1)Var(Y_1)})+\frac{1}{2}(E(X_2)E(Y_2)+\rho_2\sqrt{Var(X_2)Var(Y_2)})$$

$$=\frac{1}{2}(0\cdot 0+\frac{1}{3}\sqrt{1\cdot 1})+\frac{1}{2}((-\frac{1}{3})\cdot\sqrt{1\cdot 1})=0.$$

$$\Rightarrow \rho_{XY}=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}=\frac{E(XY)-EXEY}{\sqrt{Var(X)Var(Y)}}=0.$$

〈3〉
$$\varphi_1(x,y)=\frac{1}{2\pi\sqrt{1-\frac{1}{9}}}\exp\left\{-\frac{1}{2(1-\frac{1}{9})}(x^2-\frac{2}{3}xy+y^2)\right\}$$

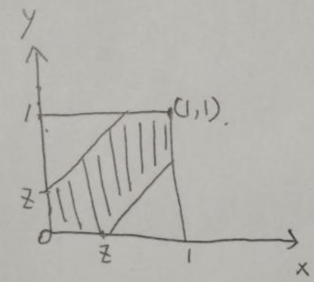
$$=\frac{3}{4\pi\sqrt{2}}\exp\left\{-\frac{9}{16}(x^2-\frac{2}{3}xy+y^2)\right\}.$$

同理，$\varphi_2(x,y)=\frac{3}{4\pi\sqrt{2}}\cdot\exp\left\{-\frac{9}{16}(x^2+\frac{2}{3}xy+y^2)\right\}$.

故 $P(x,y)\neq P_X(x)P_Y(y)$ $\Rightarrow$ X、Y不独立.

7.解：$Z=\max\{X_1,X_2\}-\min\{X_1,X_2\}=|X_1-X_2| \in (0,1)$.

则 $F_Z(z)=P(Z\le z)=\iint\limits_{|x_1-x_2|\le z}(2x)\cdot(2y)\,dxdy = 1-\iint\limits_{|x_1-x_2|>z}4xy\,dxdy$



$=1-2\int_z^1\int_0^{y-z}4xy\,dxdy$

$=1-2\int_z^1 2y(y-z)^2\,dy$

$=\frac{4}{3}z^3-4z+\frac{8}{3}$ $(z\in(0,1))$.

8.证明：独立 ⇒ 不相关 ✓

不相关 ⇒ 独立. 假定X. Y不相关

设X取值a, b. Y取值 c, d.

记 $X_1=\frac{x-a}{b-a}$ $X_2=\frac{y-c}{d-c}$ ⇒ X_1与X_2不相关

(X_1,X_2)分布为

$X_1 \backslash X_2$	0	1	
0	P_{11}	P_{12}	$1-\alpha$
1	P_{21}	P_{22}	α
	$1-\beta$	β	

$$\begin{cases} P_{11}+P_{12}=1-\alpha \\ P_{21}+\alpha\beta=\alpha \\ P_{11}+P_{21}=1-\beta \\ P_{12}+\alpha\beta=\beta \end{cases}$$

(其中 $P_{22}=\alpha\beta$ 由 $EX_1X_2=EX_1EX_2$ 得)

解之 $P_{11}=(1-\alpha)(1-\beta)$ $P_{12}=(1-\alpha)\beta$ $P_{21}=\alpha(1-\beta)$ $P_{22}=\alpha\beta$.

X_1与X_2独立 ⇒ X与Y独立.

9、证明：〈1〉 $F_Z(z) = P(XY \le z)$

$= P(Y=1)P(XY \le z \mid Y=1) + P(Y=-1)P(XY \le z \mid Y=-1)$

$= P(Y=1)P(X \le z) + P(Y=-1)P(X \ge -z)$

$= \Phi(z)$. $\Rightarrow Z \sim N(0,1)$.

〈2〉 易算得 $Cov(X, Z) = E(X^2Y) - E(X)\cdot E(XY) = 0$.

$\Rightarrow$ X与Z不相关.

为证明 X,Z 不独立，考查

$P(X \le 1, XY \ge 1) = P(Y=1)P(X \le 1, XY \ge 1 \mid Y=1) + P(Y=-1)P(X \le 1, XY \ge 1 \mid Y=-1)$

$= 0.5\, P(X \le -1)$

$= 0.5(1-\Phi(1))$

而 $P(X \le 1)\cdot P(XY \ge 1) = \Phi(1)(1-\Phi(1)) \ne 0.5(1-\Phi(1))$

知 X与Z不独立.

10、〈1〉 作变换 $\begin{cases} U = X+Y \\ V = \dfrac{X}{X+Y} \end{cases} \Rightarrow \begin{cases} X = UV \\ Y = U(1-V) \end{cases}$

则 $p_{U,V}(u,v) = p_X(uv)\, p_Y(u(1-v))\, |-U| = e^{-uv} e^{-u(1-v)} |-u| = u\cdot e^{-u}$.

〈2〉 $p_U(u) = \cancel{\int_{-\infty}^{+\infty}} \int_0^1 p_{u,v}(u,v)dv = \int_0^1 u e^{-u} dv = u\cdot e^{-u} \quad u>0$.

$p_V(v) = \int_{-\infty}^{+\infty} p_{u,v}(u,v)du = \int_{-\infty}^{+\infty} u e^{-u} du = 1 \quad v \in (0,1)$.

$\Rightarrow p_{U,V}(u,v) = p_U(u)\cdot p_V(v)$ 独立性得证。