2009-2010 秋季学期概率论与数理统计期末考试参考答案和评分细则

一、选择题（10 分，每空 2 分，将选项对应的大写英文字母直接写在横线上）

题号	1	2	3	4	5
A 卷	A	A	B	B	C
B 卷	B	C	A	A	**B**

二、填空题（15 分，每空 3 分，将计算结果直接写在横线上）

题号	6	7	8	9	10
A 卷	0.3	0.25	p	$2(n-1)/n$	2
B 卷	p	$2(n-1)/n$	2	0.3	0.25

注：**A9B7** 也写成以下形式之一：$2-\dfrac{2}{n}$, $2\left(1-\dfrac{1}{n}\right)$, $\dfrac{\lambda(n-1)}{n}$, $\lambda-\dfrac{\lambda}{n}$, $\lambda\left(1-\dfrac{1}{n}\right)$

三、（15 分）

(1) 记 A_k 表示事件“第 k 次试验成功”。则

$$P\left(A_{k+1}\middle|A_k\right)=\frac{2}{3},\quad P\left(A_{k+1}\middle|\overline{A_k}\right)=\frac{1}{6}$$

于是

$$P\left(A_{k+1}\right)=P\left(A_k\right)P\left(A_{k+1}\middle|A_k\right)+P\left(\overline{A_k}\right)P\left(A_{k+1}\middle|\overline{A_k}\right)=P\left(A_k\right)\times\frac{2}{3}+\left(1-P\left(A_k\right)\right)\times\frac{1}{6}$$
$$=\frac{1}{6}+P\left(A_k\right)\times\frac{1}{2}$$

因此

$$P\left(A_2\right)=\frac{1}{6}+\frac{1}{3}\times\frac{1}{2}=\frac{1}{3}$$

利用上述递推关系及数学归纳法可以证明对任意正整数 k， $P\left(A_k\right)=\dfrac{1}{3}$

(2)

$$P\left(A_2\middle|A_3\right)=\frac{P(A_2A_3)}{P(A_3)}$$
$$=\frac{P(A_2)P\left(A_3\middle|A_2\right)}{P(A_3)}$$
$$=P\left(A_3\middle|A_2\right)=\frac{2}{3}$$

(3) $P\left(X=1\right)=P(A_1)=\dfrac{1}{3}$ ；

当 $k>1$ 时，

$$
\begin{aligned}
P\left(X=k\right) &= P\left(\overline{A_1}\,\overline{A_2}\cdots\overline{A_{k-1}}A_k\right) \\
&= P\left(\overline{A_1}\right)P\left(\overline{A_2}\middle|\overline{A_1}\right)\cdots P\left(\overline{A_{k-1}}\middle|\overline{A_1}\,\overline{A_2}\cdots\overline{A_{k-2}}\right)P\left(A_k\middle|\overline{A_1}\,\overline{A_2}\cdots\overline{A_{k-1}}\right) \\
&= P\left(\overline{A_1}\right)P\left(\overline{A_2}\middle|\overline{A_1}\right)\cdots P\left(\overline{A_{k-1}}\middle|\overline{A_{k-2}}\right)P\left(A_k\middle|\overline{A_{k-1}}\right) \\
&= \frac{2}{3}\times\left(\frac{5}{6}\right)^{k-2}\times\frac{1}{6}=\frac{1}{9}\times\left(\frac{5}{6}\right)^{k}
\end{aligned}
$$

A 四(B 五)、(20 分)

(1)

$$
\begin{aligned}
F_X(t) &= P\left(X\le t\right) \\
&= 1-P\left(X>t\right) \\
&= 1-P\left(X_1>t, X_2>t\right) \\
&= 1-P\left(X_1>t\right)P\left(X_2>t\right) \\
&= 1-e^{-(\mu_1+\mu_2)t},\ \ （t\ge 0）
\end{aligned}
$$

即 X 服从指数分布 $Exp(\mu_1+\mu_2)$ 。

(2)

解法 1：

$$
\begin{aligned}
P\left(X_1<X_2\right) &= \iint\limits_{x<y} f_{X,Y}(x,y)dxdy \\
&= \int_0^{+\infty}\int_x^{+\infty}\mu_1 e^{-\mu_1 x}\mu_2 e^{-\mu_2 y}dydx \\
&= \frac{\mu_1}{\mu_1+\mu_2}\int_0^{+\infty}(\mu_1+\mu_2)e^{-(\mu_1+\mu_2)x}\int_x^{+\infty}\mu_2 e^{-\mu_2(y-x)}dydx \\
&= \frac{\mu_1}{\mu_1+\mu_2}
\end{aligned}
$$

解法 2：

令 $\begin{cases} U=X_1-X_2 \\ V=X_2 \end{cases}$ 。故

$$
p_{U,V}(u,v)=\mu_1 e^{-\mu_1(u+v)}1_{u+v>0}\cdot\mu_2 e^{-\mu_2 v}1_{v>0}
$$

进而，

$$
p_U(u)=\int_{\max(0,-u)}^{+\infty}\mu_1\mu_2 e^{-\mu_1 u}e^{-(\mu_1+\mu_2)v}dv=\begin{cases}\dfrac{\mu_1\mu_2}{\mu_1+\mu_2}e^{-\mu_1 u}, & u>0; \\ \dfrac{\mu_1\mu_2}{\mu_1+\mu_2}e^{\mu_2 u}, & u<0.\end{cases}
$$

因此

$$P\left(X_1 < X_2\right) = P(U<0) = \int_{-\infty}^{0} \frac{\mu_1\mu_2}{\mu_1+\mu_2} e^{\mu_2 u} dv$$

$$= \frac{\mu_1}{\mu_1+\mu_2}$$

解法 3：

令$\begin{cases} W = X_1 / X_2 \\ V = X_2 \end{cases}$。故

$$p_{W,V}(w,v) = \mu_1\mu_2 e^{-\mu_1 wv - \mu_2 v} 1_{w>0,v>0}$$

进而，

$$p_W(w) = \frac{\mu_1\mu_2}{\left(\mu_1 w + \mu_2\right)^2} 1_{w>0}$$

因此

$$P\left(X_1 < X_2\right) = P(W<1) = \int_0^1 \frac{\mu_1\mu_2}{\left(\mu_1 w + \mu_2\right)^2} dw$$

$$= \frac{\mu_1}{\mu_1+\mu_2}$$

(3)

解法 1：

$$\begin{aligned} F_{X_1|X_1<X_2}(t) &= P\left(X_1 \le t \middle| X_1 < X_2\right) \\ &= 1 - P\left(X_1 > t \middle| X_1 < X_2\right) \\ &= 1 - \frac{P(t < X_1 < X_2)}{P(X_1 < X_2)} \\ &= 1 - \frac{\mu_1+\mu_2}{\mu_1} \int_{\max(t,0)}^{+\infty} \int_x^{+\infty} \mu_1 e^{-\mu_1 x} \mu_2 e^{-\mu_2 y} dydx \\ &= 1 - \int_{\max(t,0)}^{+\infty} (\mu_1+\mu_2) e^{-(\mu_1+\mu_2)x} \int_x^{+\infty} \mu_2 e^{-\mu_2(y-x)} dydx \\ &= 1 - e^{-(\mu_1+\mu_2)\max(t,0)} = \begin{cases} 1 - e^{-(\mu_1+\mu_2)t}, & t \ge 0; \\ 0, & t < 0. \end{cases} \end{aligned}$$

即在已知 $X_1 < X_2$ 的条件下， X_1 服从指数分布 $Exp(\mu_1+\mu_2)$，从而

$$E\left(X_1 \middle| X_1 < X_2\right) = \frac{1}{\mu_1+\mu_2}$$

解法 2：

$$
\begin{aligned}
p_{X_1|X_1<X_2}(t) &= \frac{P\left(X_1 \in dt, X_1 < X_2\right)}{P(X_1 < X_2)} \\
&= \frac{\int_t^{+\infty} \mu_1\mu_2 e^{-\mu_1 t - \mu_2 y} dy}{\frac{\mu_1}{\mu_1+\mu_2}} \\
&= (\mu_1+\mu_2)e^{-(\mu_1+\mu_2)t}
\end{aligned}
$$

所以，

$$
\begin{aligned}
E\left(X_1 \middle| X_1 < X_2\right) &= \int_0^{+\infty} t p_{X_1|X_1<X_2}(t)dt \\
&= \frac{1}{\mu_1+\mu_2}
\end{aligned}
$$

<u>解法 3</u>：

$$
\begin{aligned}
E\left(X_1 \middle| X_1 < X_2\right) &= \frac{E\left(X_1 1_{X_1<X_2}\right)}{P(X_1 < X_2)} \\
&= \frac{\int_0^{+\infty}\left(\int_t^{+\infty} t \cdot \mu_1 e^{-\mu_1 t} \cdot \mu_2 e^{-\mu_2 y} dy\right)dt}{\frac{\mu_1}{\mu_1+\mu_2}} \\
&= \frac{1}{\mu_1+\mu_2}
\end{aligned}
$$

(4)

<u>证法 1</u>：

$$
\begin{aligned}
P\left(X_1 \le t, X_1 < X_2\right) &= P(X_1 < X_2)P\left(X_1 \le t \middle| X_1 < X_2\right) \\
&= \frac{\mu_1}{\mu_1+\mu_2}\left(1 - e^{-(\mu_1+\mu_2)t}\right) \\
&= P\left(X_1 < X_2\right)P\left(X \le t\right)
\end{aligned}
$$

<u>证法 2</u>：

$$
\begin{aligned}
p_{I,X}\left(1,t\right) &= p_{I,X_1}\left(1,t\right) \\
&= P\left(X_1 < X_2 \middle| X_1 = t\right) p_{X_1}\left(t\right) \\
&= P\left(t < X_2 \middle| X_1 = t\right) p_{X_1}\left(t\right) \\
&= P\left(t < X_2\right) p_{X_1}\left(t\right) \\
&= e^{-\mu_2 t} \cdot \mu_1 e^{-\mu_1 t} = \frac{\mu_1}{\mu_1+\mu_2} \cdot (\mu_1+\mu_2)e^{-(\mu_1+\mu_2)t} \\
&= P(I=1)p_X(t)
\end{aligned}
$$

A 五(B 四)、(15 分)

(1)

$$F_Z(t)=P\left(Z\le t\right)=P\left(-\ln X\le t\right)=P(X\ge e^{-t})=\begin{cases}1-e^{-t}, & t\ge 0;\\ 0, & t<0.\end{cases}$$

故 $Z\sim Exp(1)$

(2)

解法 1：卷积公式

$$\begin{aligned}f_{X+Y}(t)&=\int_{-\infty}^{+\infty}f_X(x)f_Y(t-x)dx\\&=\int_0^1 e^{-(t-x)}1_{t-x>0}dx\\&=1_{t>0}e^{-t}\int_0^{\min(1,t)}e^x dx\\&=1_{t>0}e^{-t}\left(e^{\min(1,t)}-1\right)\\&=\begin{cases}e^{-t}\left(e-1\right), & t>1;\\ 1-e^{-t}, & 0<t<1;\\ 0, & t<0.\end{cases}\end{aligned}$$

解法 2：先求概率分布函数

$$\begin{aligned}F_{X+Y}(t)&=P(X+Y\le t)\\&=\begin{cases}\int_0^1\int_0^{t-x}f_X(x)f_Y(y)dydx, & t\ge 1;\\ \int_0^t\int_0^{t-x}f_X(x)f_Y(y)dydx, & 0\le t<1;\\ 0 & t<0.\end{cases}\\&=\begin{cases}\int_0^1 1-e^{x-t}dx=1-e^{-t}(e-1) & t\ge 1;\\ \int_0^t 1-e^{x-t}dx=t+e^{-t}-1 & 0\le t<1;\\ 0 & t<0.\end{cases}\end{aligned}$$

再求导

得概率密度函数（见左边）。

(3)

$$P\left(Y\le 1 \middle| X\le e^{\frac{-(Y-1)^2}{2}}\right)=\frac{P\left(Y\le 1, X\le e^{\frac{-(Y-1)^2}{2}}\right)}{P\left(X\le e^{\frac{-(Y-1)^2}{2}}\right)}$$

$$=\frac{\iint\limits_{y\le 1, x\le e^{\frac{-(y-1)^2}{2}}} 1_{0<x<1}\cdot e^{-y}1_{y>0}\,dxdy}{\iint\limits_{x\le e^{\frac{-(y-1)^2}{2}}} 1_{0<x<1}\cdot e^{-y}1_{y>0}\,dxdy}=\frac{\int_0^1 e^{-y}\left(\int_0^{e^{\frac{-(y-1)^2}{2}}}dx\right)dy}{\int_0^{+\infty} e^{-y}\left(\int_0^{e^{\frac{-(y-1)^2}{2}}}dx\right)dy}$$

$$=\frac{\int_0^1 e^{-y}e^{\frac{-(y-1)^2}{2}}dy}{\int_0^{+\infty} e^{-y}e^{\frac{-(y-1)^2}{2}}dy}=\frac{e^{-\frac{1}{2}}\sqrt{2\pi}\int_0^1\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy}{e^{-\frac{1}{2}}\sqrt{2\pi}\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy}\int_0^1\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy$$

$$=\frac{\Phi(1)-\Phi(0)}{1-\Phi(0)}=2\Phi(1)-1.$$

六、(13 分)

(1) 由

$$EX=\int_1^{+\infty}x\cdot\frac{1}{\lambda}e^{-\frac{(x-1)}{\lambda}}dx$$
$$=1+\lambda$$

得

$$\lambda=EX-1$$

用样本均值 $\overline{X}$ **代换**上式中的总体均值 EX ，得到 λ 的矩估计量：

$$\hat{\lambda}_M=\overline{X}-1$$

由

$$E\hat{\lambda}_M=E\left(\overline{X}-1\right)$$
$$=EX-1$$
$$=\lambda,\qquad\forall\lambda>0$$

知 $\hat{\lambda}_M=\overline{X}-1$ 是 λ 的无偏估计。

(2) 由

$$Var(X)=\lambda^2$$

知

$$E(S^2)=Var(X)=\lambda^2$$

由中心极限定理，当 n 充分大时，$\overline{X}$ 近似服从如下正态分布

$$\overline{X}\overset{\cdot}{\sim}N\left(\lambda+1,\frac{\lambda^2}{n}\right)$$

七、(12 分)

(1) 由 $\sqrt{\frac{9}{S^2}}\left(\overline{X}-\mu\right)\sim t(8)$，$P\left(\sqrt{\frac{9}{S^2}}\left(\overline{X}-\mu\right)\le t_{0.95}(8)\right)=0.95$，得 μ 的 95%置信度的置信下界为

$$\overline{x}-t_{0.95}(8)\sqrt{\frac{s^2}{9}}=7.68-t_{0.95}(8)\times\sqrt{\frac{0.64}{9}}=7.68-\frac{0.8}{3}\times1.860=-7.184$$

(2) 因

$$P_{\mu\ge8}\left(\sqrt{\frac{9}{S^2}}\left(\overline{X}-8\right)\le t_{0.05}(8)\right)\le P_{\mu\ge8}\left(\sqrt{\frac{9}{S^2}}\left(\overline{X}-\mu\right)\le t_{0.05}(8)\right)=0.05$$

故拒绝域为 $\sqrt{\frac{9}{S^2}}\left(\overline{X}-8\right)\le t_{0.05}(8)=-t_{0.95}(8)=-1.860$。代入样本值

$$\sqrt{\frac{9}{0.64}}\left(7.68-8\right)=-0.32\times\frac{3}{0.8}=-1.2>-1.860$$

这表明样本尚未落入拒绝域，故不能拒绝原假设 $\mu\ge8$。