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11. $P(A|C)=\dfrac{P(AC)}{P(C)}$, $P(B|C)=\dfrac{P(BC)}{P(C)}$

$P(A|C)\geq P(B|C) \Rightarrow \dfrac{P(AC)}{P(C)}\geq\dfrac{P(BC)}{P(C)}$

$\because P(C)>0$

$\therefore P(AC)\geq P(BC)$ (1)

同理. $P(AC^c)\geq P(BC^c)$ (2)

(1)+(2), $P(AC)+P(AC^c)\geq P(BC)+P(BC^c)$

$P(AC)+P(AC^c)=P(AC\cup AC^c)=P(A\cap\Omega)=P(A)$

$P(BC)+P(BC^c)=P(B)$

故 $P(A)\geq P(B)$

16. (1) 只需证 $P(((A_1\cup A_2)A_3)\cap(A_4^c\cup A_5^c))=P((A_1\cup A_2)A_3)P(A_4^c\cup A_5^c)$ 即可

左边 $=P((A_1\cup A_2)\cap(A_3(A_4A_5)^c))=P(A_1A_3(A_4A_5)^c\cup A_2A_3(A_4A_5)^c)$

$=P(A_1A_3(A_4A_5)^c)+P(A_2A_3(A_4A_5)^c)-P(A_1A_2A_3(A_4A_5)^c)$

$=P(A_1A_3A_4^c)+P(A_1A_3A_5^c)-P(A_1A_3A_4^cA_5^c)+P(A_2A_3A_4^c)+P(A_2A_3A_5^c)-P(A_2A_3A_4^cA_5^c)$
$-P(A_1A_2A_3A_4^c)-P(A_1A_2A_3A_5^c)+P(A_1A_2A_3A_4^cA_5^c)$

$=p_1p_3(1-p_4)+p_1p_3(1-p_5)-p_1p_3(1-p_4)(1-p_5)+p_2p_3(1-p_4)+p_2p_3(1-p_5)\cdots$

$=(p_1+p_2-p_1p_2)p_3(1-p_4p_5)$

右边 $=[P(A_1A_3)+P(A_2A_3)-P(A_1A_2A_3)][1-P(A_4A_5)]$

$=(p_1+p_2-p_1p_2)p_3(1-p_4p_5)=$ 左边

故 $(A_1\cup A_2)A_3$ 与 $A_4^c\cup A_5^c$ 独立

(2) $P((A_1\cup A_2)\cap(A_3\cap A_4))=P(A_1A_3A_4\cup A_2A_3A_4)=P(A_1A_3A_4)+P(A_2A_3A_4)-P(A_1A_2A_3A_4)$

$=(p_1+p_2-p_1p_2)p_3p_4$

$=P(A_1\cup A_2)P(A_3A_4)$

$P((A_3\cap A_4)\cap A_5^c)=P(A_3A_4-A_3A_4A_5)=P(A_3A_4)-P(A_3A_4A_5)=P(A_3A_4)[1-P(A_5)]=P(A_3A_4)P(A_5^c)$

$P(A_5^c\cap(A_1\cup A_2))=P(A_1A_5^c)+P(A_2A_5^c)-P(A_1A_2A_5^c)$

$=[P(A_1)+P(A_2)-P(A_1)P(A_2)]P(A_5^c)$

$=P(A_1\cup A_2)P(A_5^c)$

$P((A_1\cup A_2)\cap A_3A_4\cap A_5^c)=P(A_1A_3A_4A_5^c)+P(A_2A_3A_4A_5^c)-P(A_1A_2A_3A_4A_5^c)$

$=[P(A_1)+P(A_2)-P(A_1A_2)]P(A_3A_4)P(A_5^c)$

$=P(A_1\cup A_2)P(A_3A_4)P(A_5^c)$

故 $A_1\cup A_2$、$A_3\cap A_4$、A_5^c 相互独立

17. 设 A_i={该老虎机是第i个老虎机}，$P(A_1)=P(A_2)=P(A_3)$

B={4次中获得2次回报}

$\cancel{P(A_1)=\frac{P(BA_1)}{P(B|A_1)}}$

$$P(B)=P(A_1)\cdot C_4^2(\tfrac{1}{3})^2(\tfrac{2}{3})^2+P(A_2)\cdot C_4^2(\tfrac{1}{2})^2(\tfrac{1}{2})^2+P(A_3)(\tfrac{2}{3})^2(\tfrac{1}{3})^2$$
$$=\frac{16}{81}+\frac{1}{8}=\frac{209}{648}$$

$\cancel{P(A_1|B)=\frac{P(A_1)\cdot C_4^2(\frac{1}{3})^2(\frac{2}{3})^2}{P(B)}}$

$$P(A_1|B)=\frac{P(A_1B)}{P(B)}=\frac{P(A_1)\cdot C_4^2(\frac{1}{3})^2(\frac{2}{3})^2}{P(B)}=\frac{64}{209}$$

$$P(A_2|B)=\frac{P(A_2B)}{P(B)}=\frac{P(A_2)\cdot C_4^2(\frac{1}{2})^2(\frac{1}{2})^2}{P(B)}=\frac{\frac{1}{8}}{\frac{209}{648}}=\frac{81}{209}$$

$$P(A_3|B)=\frac{P(A_3B)}{P(B)}=\frac{P(A_3)\cdot C_4^2(\frac{2}{3})^2(\frac{1}{2})^2}{P(B)}=\frac{64}{209}$$

记 C={下次还可以获得回报}

$$P(C|B)=P(A_1|B)P((C|B)|(A_1|B))+P(A_2|B)P((C|B)|(A_2|B))+P(A_3|B)P((C|B)|(A_3|B))$$

显然 $P((C|B)|(A_1|B))=P(C|A_1)=\frac{1}{3}$，$P((C|B)|(A_2|B))=P(C|A_2)=\frac{1}{2}$，$P((C|B)|(A_3|B))=P(C|A_3)=\frac{2}{3}$

$$\therefore P(C|B)=\frac{1}{3}\cdot\frac{64}{209}+\frac{1}{2}\cdot\frac{81}{209}+\frac{2}{3}\cdot\frac{64}{209}$$
$$=\frac{1}{2}$$

18. 记 A_i={他通过第i个考试~~的概率~~}，B={获得最终资格}

$P(A_1)=p$，$P(A_1^c)=1-p$

$P(A_2|A_1)=p$，$P(A_2|A_1^c)=\frac{p}{2}$，$P(A_2^c|A_1)=1-p$，$P(A_2^c|A_1^c)=1-\frac{p}{2}$

$$P(A_2)=P(A_1)P(A_2|A_1)+P(A_1^c)P(A_2|A_1^c)=p\cdot p+(1-p)\cdot\frac{p}{2}=\frac{p+p^2}{2}$$

$$P(A_2^c)=1-P(A_2)=\frac{2-p-p^2}{2}$$

$P(A_3|A_2)=p$，$P(A_3|A_2^c)=\frac{p}{2}$

$$P(A_3)=P(A_2)P(A_3|A_2)+P(A_2^c)P(A_3|A_2^c)=\frac{p+p^2}{2}\cdot p+\frac{2-p-p^2}{2}\cdot\frac{p}{2}=\frac{2p+p^2+p^3}{4}$$

$$P(A_3^c)=\frac{4-2p-p^2-p^3}{4}$$

$$P(A_4)=P(A_3)P(A_4|A_3)+P(A_3^c)P(A_4|A_3^c)=\frac{2p+p^2+p^3}{4}\cdot p+\frac{4-2p-p^2-p^3}{4}\cdot\frac{p}{2}=\frac{4p+2p^2+p^3+p^4}{8}$$

$P(A_4^c)=\frac{8-4p-2p^2-p^3-p^4}{8}$，但是上面这些并没有用

$$P(B)=P(A_1A_2A_3A_4)+P(A_1A_2A_3A_4^c)+P(A_1A_2A_3^cA_4)+P(A_1A_2^cA_3A_4)+P(A_1^cA_2A_3A_4)$$

$P(A_1A_2A_3A_4) = \cancel{P(A)}\, P(A_1)\,P(A_2|A_1)\,P(A_3|A_1A_2)\,P(A_4|A_1A_2A_3)$

$= P(A_1)\,P(A_2|A_1)\,P(A_3|A_2)\,P(A_4|A_3)$

$= p\cdot p\cdot p\cdot p = p^4$

$P(A_1A_2A_3A_4^c) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_2)\,P(A_4^c|A_3) = p\cdot p\cdot p\cdot (1-p) = (1-p)p^3$

$P(A_1A_2A_3^cA_4) = P(A_1)\,P(A_2|A_1)\,P(A_3^c|A_2)\,P(A_4|A_3^c) = p\cdot p\cdot (1-p)\cdot \frac{p}{2} = (1-p)\frac{p^3}{2}$

$P(A_1A_2^cA_3A_4) = P(A_1)\,P(A_2^c|A_1)\,P(A_3|A_2^c)\,P(A_4|A_3) = p\cdot (1-p)\cdot \frac{p}{2}\cdot p = (1-p)\frac{p^3}{2}$

$P(A_1^cA_2A_3A_4) = P(A_1^c)\,P(A_2|A_1^c)\,P(A_3|A_2)\,P(A_4|A_3) = (1-p)\cdot \frac{p}{2}\cdot p\cdot p = (1-p)\frac{p^3}{2}$

$P(B) = p^4 + p^3 - p^4 + \frac{3}{2}(p^3 - p^4) = \frac{5p^3}{2} - \frac{3p^4}{2} = \frac{5p^3 - 3p^4}{2}$

23. (1) ~~$P(Y_1\ge 0) = P(Y_1 = 1) = \frac{1}{2}$~~ 记 $A = \{对\ \forall n: 1\le n\le 4,\ Y_n \ge 0\}$

~~$P(Y_2\ge 0) = P(Y_2=0) + P(Y_2=2)$~~ $P(A) = P(Y_1\ge 0)\,P(Y_2\ge 0|Y_1\ge 0)\,P(Y_3\ge 0|Y_1,Y_2\ge 0)\,P(Y_4\ge 0|Y_1,Y_2,Y_3\ge 0)$

~~$P\ Y_n \ge 0 =$~~

$P(Y_1\ge 0) = \frac{1}{2}$

$Y_1\ge 0$ 即 $Y_1 = 1$ 的情况下，无论如何走 $Y_2 = 0$ 或 2，$Y_2 \ge 0$

$P(Y_2\ge 0|Y_1\ge 0) = 1$

$Y_2\ge 0$ 的情况下，$P(Y_2=0) = P(Y_2=2) = \frac{1}{2}$，仅当 $Y_2=0$ 且向左走时，$Y_3<0$

故 $P(Y_3\ge 0|Y_1,Y_2\ge 0) = \frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2} = \frac{3}{4}$

$Y_3\ge 0$ 情况下，$Y_3 = 1$ 或 3，无论如何走 $Y_4\ge 0$ 恒成立

故 $P(Y_4\ge 0|Y_1,Y_2,Y_3\ge 0) = 1$

$\therefore P(A) = \frac{1}{2}\cdot 1\cdot \frac{3}{4}\cdot 1 = \frac{3}{8}$

(2) 记 $B = \{对\ \forall n: 1\le n\le 4,\ |Y_n|\le 2\}$

$P(B) = P(|Y_1|\le 2)\,P(|Y_2|\le 2\,|\,|Y_1|\le 2)\,P(|Y_3|\le 2\,|\,|Y_1|,|Y_2|\le 2)\,P(|Y_4|\le 2\,|\,|Y_1|,|Y_2|,|Y_3|\le 2)$

$P(|Y_1|\le 2)$ 与 $P(|Y_2|\le 2\,|\,|Y_1|\le 2)$ 显然为 1

$P(Y_2=2) = P(Y_2=-2) = \frac{1}{4}$，$P(Y_2=0) = \frac{1}{2}$

$P(|Y_3|\le 2\,|\,|Y_1|\le 2, |Y_2|\le 2) = P(Y_2=0) + \frac{1}{2}P(Y_2=2) + \frac{1}{2}P(Y_2=-2) = \frac{3}{4}$

在 $|Y_1|\le 2$、$|Y_2|\le 2$ 前提下，$|Y_3|\le 2$ 当且仅当 $Y_3 = -1$ 或 0 或 1

故 $|Y_4|\le 2$ 恒成立，$P(|Y_4|\le 2\,|\,|Y_1|,|Y_2|,|Y_3|\le 2) = 1$

$P(B) = \frac{3}{4}$

(3) ~~记 $C=\{x \mid \forall n: 1\le n\le 4,\ Y_n\ge 0\}$~~

$$P(A\mid Y_4=0)=\frac{P(A\cap(Y_4=0))}{P(Y_4=0)}=\frac{P(A)\,P(Y_4=0\mid A)}{P(Y_4=0)}$$

$$P(Y_4=0)=C_4^2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^2=\frac{3}{8}$$

~~P~~ $P(Y_3=1\mid Y_1,Y_2\ge 0)=\frac{2}{3}$，$P(Y_3=3\mid Y_1,Y_2\ge 0)=\frac{1}{3}$

故 $P(Y_4=0\mid A)=\frac{1}{2}P(Y_3=1\mid Y_1,Y_2\ge 0)=\frac{1}{3}$

$$P(A\mid Y_4=0)=\frac{\frac{1}{3}\cdot\frac{3}{8}}{\frac{3}{8}}=\frac{1}{3}$$

25. (1) 记得到正面的次数为X，$X=\{0,1,2,3,4,5,6$

$$P(X=k)=\frac{1}{6}\binom{k}{k}\frac{1}{2^k}+\frac{1}{6}\binom{k+1}{k}\frac{1}{2^{k+1}}+\cdots+\frac{1}{6}\binom{6}{k}\frac{1}{2^6}$$

$$=\frac{1}{6}\sum_{n=k}^{6}\binom{n}{k}\frac{1}{2^n}$$

(2) 显然 $n=3,4,5,6$，记Y为得到的点数

$$P(Y=n\mid X=3)=\frac{P(Y=n,X=3)}{P(X=3)}=\frac{P(X=3\mid Y=n)\,P(Y=n)}{P(X=3)}$$

$$P(X=3)=\frac{1}{6}\sum_{n=3}^{6}\binom{n}{3}\frac{1}{2^n}=\frac{1}{6}\left(\frac{1}{2^3}+3\cdot\frac{1}{2^4}+10\cdot\frac{1}{2^5}+20\cdot\frac{1}{2^6}\right)=\frac{5}{32}$$

$P(Y=n)=\frac{1}{6}$　　$P(X=3\mid Y=n)=\binom{n}{3}\frac{1}{2^3}\cdot\frac{1}{2^{n-3}}=\frac{\binom{n}{3}}{2^n}$

故 $$P(Y=n\mid X=3)=\frac{\frac{\binom{n}{3}}{2^n}\cdot\frac{1}{6}}{\frac{5}{32}}=\frac{16}{15}\cdot\frac{\binom{n}{3}}{2^n}$$

~~28. $P(X<Y)=P(Y=1)P(X<Y\mid Y=1)+P(Y=2)$~~|

28. 将 Ω 分为有可数个集合 $\{Y=1\},\{Y=2\},\cdots,\{Y=n\},\cdots$

$$P(X\le Y)=P(Y=1)P(X\le Y|Y=1)+P(Y=2)P(X\le Y|Y=2)+\cdots+P(Y=n)P(X\le Y|Y=n)+\cdots$$
$$=\sum_{k=1}^{\infty}P(Y=k)\,P(X\le k)$$
$$=\sum_{k=1}^{\infty}\left[P(Y=k)\sum_{i=1}^{k}P(X=i)\right]$$
$$=\sum_{k=1}^{\infty}\left(p_k\sum_{i=1}^{k}p_i\right)$$

$$P(X=Y)=P(Y=1)P(X=Y|Y=1)+P(Y=2)P(X=Y|Y=2)+\cdots+P(Y=n)P(X=Y|Y=n)+\cdots$$
$$=\sum_{k=1}^{\infty}P(Y=k)\,P(X=k)$$
$$=\sum_{k=1}^{\infty}p_k^2$$

41. 先证 $\varphi(s+t)=\varphi(s)\varphi(t) \iff \varphi(t)=e^{-\lambda t}$ $(s,t\ge 0)$

$\Rightarrow$: $s=t=0$. $\varphi(0)=\varphi^2(0)$, $\varphi(0)=1$ ($\varphi(0)=0$舍去)

记 $\alpha=\varphi(1)$, $\varphi(1)=\varphi(\frac{n-1}{n})\varphi(\frac{1}{n})=\varphi(\frac{n-2}{n})\varphi^2(\frac{1}{n})=\cdots=\varphi^n(\frac{1}{n})$, $n\in\mathbb{N}^+$

故 $\varphi(\frac{1}{n})=\alpha^{\frac{1}{n}}$; $\varphi(\frac{m}{n})=\varphi(\frac{m-1}{n})\varphi(\frac{1}{n})=\varphi(\frac{m-2}{n})\varphi^2(\frac{1}{n})=\cdots=\varphi^m(\frac{1}{n})=\alpha^{\frac{m}{n}}$, $m\in\mathbb{N}^+$

故 $\varphi(t)=\alpha^t$ 在正有理数上成立

对 $\forall r\in\mathbb{R}^+\setminus\mathbb{Q}^+$, 取 $\{a_n\}$、$\{b_n\}$ 满足 $a_n\le b_n$ 且 $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=r$, $a_n, b_n\in\mathbb{Q}^+$

~~则~~ $\varphi(t)$ 单调不增 则 $\varphi(b_n)\le\varphi(r)\le\varphi(a_n)$, 令 $n\to\infty$ 并由夹逼原理

$\varphi(r)=\lim_{n\to\infty}\varphi(a_n)=\lim_{n\to\infty}\alpha^{a_n}=\alpha^r$, r是正无理数

故 $\varphi(t)=\alpha^t=\varphi^t(1)=e^{-\lambda t}$, 其中 $\lambda=\ln\frac{1}{\varphi(1)}\ge 0$

$\Leftarrow$: $\varphi(t)=e^{-\lambda t}$, $\varphi(s+t)=e^{-\lambda(s+t)}=e^{-\lambda s}\cdot e^{-\lambda t}=\varphi(s)\varphi(t)$

下证满足 $P(T>s+t|T>s)=P(T>t)$, $(s,t\ge 0)$ $\iff$ T满足指数分布

$\Rightarrow$: $P(T>s+t|T>s)=\frac{P(T>s+t)}{P(T>s)}=P(T>t)$, $P(T>t)$ 关于 t 显然单调非增

故 $P(T>s+t)=P(T>s)\cdot P(T>t)$, 可令 $\varphi(t)=P(T>t)$,

则 $P(T>t)=e^{-\lambda t}$ $t>0$

$F_T(t)=1-P(T>t)=1-e^{-\lambda t}$ $(t>0)$, T满足指数分布

$\Leftarrow$: T满足 $F_T(t)=1-e^{-\lambda t}$, 故 $P(T>t)=1-F_T(t)=e^{-\lambda t}$

$P(T>s+t)=e^{-\lambda(s+t)}=e^{-\lambda s}\cdot e^{-\lambda t}=P(T>s)P(T>t)$

$\therefore P(T>s+t|T>s)=P(T>s+t)/P(T>s)=P(T>t)$

证毕