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P17 ~ P19

1. 不满足集合元素的互异性

3. 一个集合与它的真子集不可能有相同的大小

假设二者大小相同，由于真子集中的元素均在集合中，二者应该相等，与真子集定义矛盾

8. 证明：$A \subset B \Leftrightarrow A \cap B = A$

$\Leftarrow$: $\because A \cap B = A$

$\therefore$ 对 $\forall a \in A$，均有 $a \in B$

$\therefore A \subset B$

$A \subset B \Rightarrow A \cap B = A$

$\because A \subset B$

$\therefore$ 对 $\forall a \in A$，$a \in B$

设 $A \cap B = C$，则 $A \subset C$

但 $C \subset A$，故 $A = C$

$\therefore A \cap B = A$

下证 $A \cap B = A \Leftrightarrow A \cup B = B$

$\Rightarrow$: $\because A \cap B = A$

$\therefore \forall a \in A, a \in B$

设 $A \cup B = C$，~~则 $C \subset B$~~

$\therefore$ ~~若 $\forall c \in C$ 且 $c \in A$，则 $c \in B$~~

~~若 $c \in C$，$c \in B$~~

则 $C \subset B$，同时 $B \subset C$

$\therefore B = C$，~~$A \cap B$~~ $A \cup B = B$

$\Leftarrow$: $\because A \cup B = B$

$\therefore \forall a \in A, a \in B$

$\therefore A \subset B$

$\therefore A \cap B = A$

9. 证明：

$\Rightarrow$: $\because A$、B 不相交

$\therefore \forall a \in A, a \notin B$

$\therefore A \setminus B = A$

$\Leftarrow$: $\because A \setminus B = A$

$\therefore \forall a \in A, a \notin B$

$\therefore A \cap B = \emptyset$

$\therefore A$、B 不相交

下证 $A \setminus B = A \Leftrightarrow A \cup B = A \Delta B$

$\Rightarrow$: $\because A \setminus B = A$

$\therefore \forall a \in A, a \notin B$

设 $\exists b \in B$, s.t. $b \in A$

则与上述矛盾，

故 $\forall b \in B, b \notin A$

即 $B \setminus A = B$，$A \cup B = (A \setminus B) \cup (B \setminus A) = A \Delta B$

$\Leftarrow$: $A \cup B = A \Delta B = (A \cup B) \setminus (A \cap B)$

则 $A \cap B = \emptyset$

A，B 不相交

$A \setminus B = A$

12. 对 $\forall x \in (A \cup B) \setminus (C \cup D)$

$x \in (A \cup B)$ 且 $x \notin (C \cup D)$

即 $x \in (A \cup B)$，$x \notin C$ 且 $x \notin D$

不妨设 $x \in A$

$\because x \in A$，$x \notin C$

$\therefore x \in (A \setminus C)$

$x \in (A \setminus C) \cup (B \setminus D)$

15. 证明：$A \subset B \Rightarrow I_A \leq I_B$：

① 对 $\forall x \in A$，$A \subset B$ 则 $x \in B$

有 $I_A(x) = I_B(x) = 1$

② 对 $\forall x \in B$ 但 $x \notin A$，

有 $I_A(x) = 0 < I_B(x) = 1$

③ 对 $\forall x \notin B$，则 $x \notin A$

有 $I_A(x) = I_B(x) = 0$

综上，$I_A \leq I_B$

$A \subset B \Leftarrow I_A \leq I_B$：

对 ~~∀x满足~~ $\forall x \in A$ 有 $I_A(x) = 1$，由于 $I_B \leq 1$

故 $I_B(x) = 1$，$x \in B$

$\therefore$ 对 $\forall x \in A$，$x \in B$

$A \subset B$

$A \cap B = \varnothing \Rightarrow I_A I_B = 0$：

① $x \in A$ 时，$A \cap B = \varnothing$ 则 $x \notin B$

$I_A(x) = 1$，$I_B(x) = 0$

$I_A I_B = 0$

② $x \in B$ 时，$A \cap B = \varnothing$ 则 $x \notin A$

$I_A(x) = 0$，$I_B(x) = 1$

$I_A I_B = 0$

③ $x \notin A$ 且 $x \notin B$ 时，

$I_A(x) = I_B(x) = 0$，$I_A I_B = 0$

综上，$I_A I_B = 0$

$A \cap B = \varnothing \Leftarrow I_A I_B = 0$

对 $\forall x \in A$，$I_A(x) = 1$，$I_A(x) I_B(x) = 0$

则 $I_B(x) = 0$，$x \notin B$

故 $A \cap B = \varnothing$

18. $\because a \vee b = \dfrac{a+b+|a-b|}{2}$，$a \wedge b = \dfrac{a+b-|a-b|}{2}$

$\therefore (a \vee b) + (a \wedge b) = a + b$

证 1.4.8：$I_{A \cup B} + I_{A \cap B} = I_A \wedge I_B + I_A \vee I_B = I_A + I_B$

19. $A\setminus B = A\setminus(A\cap B) = A - A\cap B$

$I_{A\setminus B} = I_{A-A\cap B} = I_A - I_{A\cap B} = I_A - I_A I_B$

$I_{A-B} = I_A - I_B$

20. $1 - I_{A\cup B\cup C} = I_{(A\cup B\cup C)^c}$

$= I_{A^c\cap B^c\cap C^c}$

$= I_{A^c} I_{B^c} I_{C^c}$

$= (1-I_A)(1-I_B)(1-I_C)$

$\therefore I_{A\cup B\cup C} = 1-(1-I_A)(1-I_B)(1-I_C)$

$= I_A I_B I_C - I_A I_B - I_B I_C - I_C I_A + I_A + I_B + I_C$

21. 由20可知，$I_A I_B I_C = I_{A\cup B\cup C} + I_A I_B + I_B I_C + I_C I_A - I_A - I_B - I_C$

$\because I_{ABC} = I_A I_B I_C$

$\therefore I_{ABC} = I_{A\cup B\cup C} + (I_{A\cap B} - \frac{I_A+I_B}{2}) + (I_{B\cap C} - \frac{I_B+I_C}{2}) + (I_{C\cap A} - \frac{I_C+I_A}{2})$

代入 $I_{A\cap B} = I_A + I_B - I_{A\cup B}$，$I_{B\cap C} = I_B + I_C - I_{B\cup C}$，$I_{C\cap A} = I_C + I_A - I_{C\cup A}$

$\therefore I_{ABC} = I_{A\cup B\cup C} - I_{A\cup B} - I_{B\cup C} - I_{C\cup A} + I_A + I_B + I_C$

P 42 ~ 43

1. 证明：(1) $\because a, b \geq 0$，$\forall A\subset\Omega$，$P(A)$、$Q(A) \geq 0$

$\therefore \forall A\subset\Omega$，$aP(A) + bQ(A) \geq 0$

(2) $\because P(\Omega) = Q(\Omega) = 1$，$a+b=1$

$\therefore aP(\Omega) + bP(\Omega) = a+b = 1$

(3) $\because$ 对 $\forall A_n\subset\Omega$，(A_n两两不相交)

$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$，$Q(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} Q(A_n)$

$\therefore aP(\bigcup_{n=1}^{\infty} A_n) + bQ(\bigcup_{n=1}^{\infty} A_n) = a\sum_{n=1}^{\infty} P(A_n) + b\sum_{n=1}^{\infty} Q(A_n)$

$= \sum_{n=1}^{\infty} aP(A_n) + bQ(A_n)$

比如 $\Omega = \{w_1, w_2, \cdots\}$

$P(\{w_i\}) = 2^{-i}$

$Q(\{w_i\}) = 2\cdot 3^{-i}$，$i = 1, 2, \cdots$

~~$a=b=\frac{1}{2}$~~ $a=1$，$b=0$时

结论仍然成立

7. 证明：(i) $\because P(A)\geq 0$，对$\forall A\subset\Omega$

$\therefore \frac{P(A)}{2}\geq 0$

(ii) $\because P(A+B)=P(A)+P(B)$，若$A\cap B=\varnothing$

$\therefore \frac{1}{2}P(A+B)=\frac{1}{2}P(A)+P(B)$

(iii) $\because P(\Omega)=1$

$\therefore \frac{1}{2}P(\Omega)=\frac{1}{2}\neq 1$

(i) ~~$\because P(A)\geq 0$，对$\forall A$~~ $\subset\Omega$

~~$\therefore$~~ $P^2(A)\geq 0$ 显然成立

(ii) ~~$\because P(A+B)=$~~ 假设 $P(A)=P(B)=\frac{1}{2}$

$P^2(A+B)=(P(A)+P(B))^2=1\neq P^2(A)+P^2(B)=\frac{1}{2}$

(iii) $\because P(\Omega)=1$

$\therefore P^2(\Omega)=1$

反例：Ω仅有一个元素如$\Omega=\{1\}$，(ii) 满足

8. (1) 证明：$\because A\cap B\cap C\subset A$

$\therefore P(A\cap B\cap C)\leq P(A)$

同理 $P(A\cap B\cap C)\leq P(B)$，$P(A\cap B\cap C)\leq P(C)$

$\therefore P(A\cap B\cap C)\leq P(A)\wedge P(B)\wedge P(C)$

(2) 证明：$\because A\subset A\cup B\cup C$

$\therefore P(A)\leq P(A\cup B\cup C)$

同理 $P(B)\leq P(A\cup B\cup C)$，$P(C)\leq P(A\cup B\cup C)$

$\therefore P(A\cup B\cup C)\geq P(A)\vee P(B)\vee P(C)$

9. $0\leq P(A\cup B)=P(A)+P(B)-P(AB)\leq 1$

$\therefore P(AB)\geq P(A)+P(B)-1$

取 $B=\varnothing$，$P(AB)=0\geq P(A)-1$ 成立

10. $P(A)=0$ 或 $P(A)=1$ 时有 $P(A)\cdot P(A)=1$

在 $[0,1]$ 内所有实数中任取一点，取到0的概率为 $P(A)$

则 $P(A)=0$ 但 $A=\{0\}$