1. 左-右: $\lim\limits_{c\to 0+}\left[\int_0^1 f(x)\,dx-\int_c^1 f(x)\,dx\right]$

$= \lim\limits_{c\to 0+}\int_0^c f(x)\,dx$

$\le \lim\limits_{c\to 0+} c\cdot \sup|f(x)| = 0$

$f(x)$ 可积.

2. (1) $\underbrace{\int_2^{+\infty}\frac{|\sin x|}{x\sqrt{x^2+1}}dx}_{\text{收敛}} \le \int_2^{+\infty}\frac{1}{x\sqrt{x^2+1}}dx \le \int_2^{+\infty}\frac{dx}{x^2}$ 收敛

有界函数. $\infty$ 为瑕点.
只需要考虑 $\infty$ 处的函数极限.

则 $\int_2^{+\infty}\frac{\sin x}{x\sqrt{x^2+1}}dx$ 收敛.

(2). $\int_0^1\frac{1}{\sqrt{x-x^3}}dx$. 瑕点 $x=0$ / $x=1$

$=\int_0^1\frac{1}{\sqrt{x(1-x)(1+x)}}dx$

$=\int_0^{1/2}\frac{1}{\sqrt{x(1-x)(1+x)}}dx+\int_{1/2}^1\frac{1}{\sqrt{x(1-x)(1+x)}}dx \le \int_0^{1/2}\frac{2}{\sqrt{x}}dx+\underbrace{\int_{1/2}^1\frac{2}{\sqrt{1-x}}dx}_{\int_0^{1/2}\frac{2}{\sqrt{t}}dt}$ 收敛.

(3) $\int_0^{\pi/2}\frac{1-\cos x}{x^n}dx$. 瑕点 $x=0$.

$x=0+$ $\frac{1-\cos x}{x^n}\sim\frac{\frac12 x^2}{x^n}=\frac12 x^{2-n}$  $2-n$ ~~>-1~~ $n<3$ 时收敛, $n\ge 3$ 时发散.

(4) $\int_0^1\frac{\ln x}{1-x}dx$. 瑕点 $x=0$ / $x=1$

$x\to 0+$ $\frac{\ln x}{1-x}\sim\ln x$  $\int_0^{1/2}\ln x\,dx=(x\ln x-x)\Big|_0^{1/2}$ 收敛

$x\to 1-$ $\frac{\ln x}{1-x}\sim\underbrace{-\frac{1}{x}}_{(\text{略})}$  $\int_{1/2}^1-\frac1x\,dx$ 收敛.

$=\int_0^1\frac{\ln(1-t)}{t}dt$

$=\int_0^1\frac{-t-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{4}\cdots}{t}dt$

$=\int_0^1\left(-1-\frac{t}{2}-\frac{t^2}{3}-\cdots\right)dt$

$=\sum_{n=1}^{\infty}-\frac{1}{n^2}=-\frac{\pi^2}{6}$

(5). $\int_0^\infty\frac{x^p\arctan x}{1+x^q}dx$  瑕点(可能): $\infty$ $(p,q>0)$

$x\to\infty$ $\frac{x^p\arctan x}{1+x^q}\sim x^{p-q}$  $p-q<-1$ 即 $p<q-1$ 收敛.

(6) $\int_1^\infty\frac{dx}{x^p\sqrt{\ln x}}$. 瑕点 ~~■~~. $x=1$ $(p>0)$.

$x\to 1+$ $\frac{1}{\sqrt{\ln x}}\sim\frac{1}{\sqrt{x-1}}$  则 $\int_1^2\frac{1}{x^p\ln x}dx\sim\int_1^2\frac{dx}{\sqrt{\ln x}}\sim\int_1^2\frac{dx}{\sqrt{x-1}}$ 收敛.

$x\to\infty$ $\frac{1}{x^{p+\varepsilon}}\le\frac{1}{x^p\sqrt{\ln x}}\le\frac{1}{x^p}$ 其中 $\varepsilon$ 为 $\forall$ 正数.

易知 $p<1$ 收敛, $p>1$ 发散. 若 $p=1$. $\int_1^\infty\frac{1}{x\ln x}dx=\int_0^\infty\frac{dt}{t}$. 发散.

综上 $p<1$ 收敛. $p\ge 1$ 发散.

(7) $\int_0^1 x^{p-1}(1-x)^{q-1}\ln x\,dx$ 瑕点：$x=0$ / $x=1$ $(p,q>0)$

$= \int_0^{1/2} x^{p-1}(1-x)^{q-1}\ln x\,dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}\ln x\,dx$

$x\to 0+ \quad \sim \int_0^{1/2} x^{p-1}\ln x\,dx \qquad x\to 1- \quad \sim \int_{1/2}^1 (1-x)^{q-1}\ln x\,dx$

(8) $\int_0^\infty \frac{\sin^2 x}{x^p}dx$ 瑕点 $x=0$，$x\to\infty$.

$x\to 0 \sim \int_0^{1/2}\frac{x^2}{x^p}dx$ 当 $p-2<1$，$p<3$ 收敛.

$x\to\infty \sim \int_1^\infty \frac{1}{x^p}dx$ $p>1$ 收敛. $\Rightarrow$ $1<p<3$ 收敛. $p\geq 3$ 发散.

现考虑 $p\leq 1$ 情况 $x=0$ 处收敛. $x\to\infty$ $\int_{\frac{\pi}{4}}^\infty \frac{\sin^2 x}{x^p}dx \leq \int_{\frac{\pi}{4}}^\infty \frac{\sin^2 x}{x}dx$

$\int_{\frac{\pi}{4}}^\infty \frac{\sin^2 x}{x}dx = \int_{\frac{\pi}{4}}^\infty \frac{1-\cos 2x}{x}dx \geq \int_{\frac{\pi}{4}}^\infty \frac{1}{x}dx$ 发散.

(证明 $\int_{\frac{\pi}{4}}^\infty \frac{1-\cos 2x}{x}dx \geq \int_{\frac{\pi}{4}}^\infty \frac{1}{x}dx$.

阴影面积逐渐↓. 故 $\int_{\frac{\pi}{4}}^\infty \frac{\cos 2x}{x}dx \leq 0$

$\Rightarrow \int_{\frac{\pi}{4}}^\infty \frac{1}{x}dx - \int_{\frac{\pi}{4}}^\infty \frac{\cos 2x}{x}dx > \int_{\frac{\pi}{4}}^\infty \frac{1}{x}dx$ 发散)

综上：$1<p<3$ 收敛. $p\geq 3$、$p\leq 1$ 发散.

3. 先判断是否可积. 瑕点：$x=\pi/2$. 做变量代换.

$y=\frac{\pi}{2}-x$. $\underline{\int_0^{\pi/2}\ln(\cos x)dx = \int_0^{\pi/2}\ln(\sin y)dy}$. 瑕点 $y=0$.

$y=0 \sim \int_0^\varepsilon \ln y\,dy \ (\varepsilon>0) = y\ln y - y\Big|_0^\varepsilon \ (\varepsilon>0)$ 有限. 则积分存在.

$I = \int_0^{\pi/2}\ln(\cos x)dx = \int_0^{\pi/2}\ln(\sin x)dx$

$2I = \int_0^{\pi/2}\ln(\cos x\sin x)dx = \int_0^{\pi/2}\ln(\sin 2x)dx \underset{-\int_0^{\pi/2}\ln 2dx}{\wedge} = \int_0^\pi \frac{\ln\sin y}{2}dy \underset{-\int_0^{\pi/2}\ln 2dx}{\wedge} = \int_0^{\pi/2}\ln\sin x\,dx - \int_0^{\pi/2}\ln 2\,dx$

$\Rightarrow I = -\ln 2\cdot\frac{\pi}{2}$.

4. (1) 显然 $x=0$ 不是瑕点.

$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}dx \to$

| | |
|---|---|
| $\frac{1}{x}$ | $-\frac{1}{x^2}$ |
| $\sin x$ | $-\cos x$ |

$\Rightarrow \lim_{A\to\infty}\left(\underbrace{-\frac{\cos x}{x}\Big|_0^A}_{\text{无意义}} - \int_0^A \frac{\cos x}{x^2}dx\right)$

↘

| | |
|---|---|
| $\frac{1}{x}$ | $-\frac{1}{x^2}$ |
| $\sin x$ | $\underline{\underline{1-\cos x}}$ ☆ |

$\Rightarrow \lim_{A\to\infty}\left(\underbrace{\frac{1-\cos x}{x}\Big|_0^A}_{\text{有限}} - \underbrace{\int_0^A \frac{1-\cos x}{x^2}dx}_{\lim_{A\to\infty}\int_0^A\frac{dx}{x^2}\text{ 收敛}}\right) \Rightarrow$ 收敛.

(2) $I(\lambda)=\int_0^\infty \frac{\sin x}{x}e^{-\lambda x}\,dx$

① $\left|\frac{\sin x}{x}\right|\le 1$ 且 $\int_0^\infty e^{-\lambda x}\,dx$ 收敛，则 $I(\lambda)$ 收敛

② $e^{-\lambda x}\le 1$ $\int_0^\infty \frac{\sin x}{x}\,dx$ 收敛，则 $I(\lambda)$ 收敛.

(3) $\int_{2k\pi}^{2(k+1)\pi}\left|\frac{\sin x}{x}\right|dx = g(k)$

$g(k)\ge \int_{2k\pi}^{2(k+1)\pi}\frac{|\sin x|}{2(k+1)\pi}\,dx = \frac{2}{(k+1)\pi}$

$\int_0^\infty \left|\frac{\sin x}{x}\right|dx = \sum_{k=0}^\infty g(k) \ge \sum_{k=0}^\infty \frac{2}{(k+1)\pi}$ 发散.

5. (1) $D_N(x)=\sum_{n=-N}^{N} e^{inx} = \underbrace{e^{-iNx}+e^{-i(N-1)x}+\cdots}_{积分为0}+1+\underbrace{\cdots+e^{i(N-1)x}+e^{iNx}}_{积分为0.}$

$\int_{-\pi}^{\pi} D_N(x)\,dx=\int_{-\pi}^{\pi}dx = 2\pi.$

(2) 因 $\lim_{x\to0-}\frac{x-2\sin\frac x2}{x\sin\frac x2} = \lim_{x\to0-}\frac{-2\cdot\frac{x^3}{8\cdot3!}}{x\cdot x/2}=0=\lim_{x\to0+}\frac{x-2\sin\frac x2}{x\sin\frac x2}.$ 可去.

(3) $\lim_{N\to\infty}\int_{-\pi}^{\pi} f(x)\left(\sin(N+\tfrac12)x\right)dx$

$=\lim_{N\to\infty}\left[\int_{-\pi}^{\pi} \underline{f(x)\cos\tfrac x2}\cdot\sin Nx\,dx + \int_{-\pi}^{\pi} \underline{f(x)\sin\tfrac x2}\cos Nx\,dx\right] = 0.$

(4) $\underbrace{\lim_{N\to\infty}\int_{-\pi}^{\pi} \frac{\sin(N-\frac12)x}{\sin\frac x2}\,dx}_{2\pi} = \lim_{N\to\infty}\left[\int_{-\pi}^{\pi}\frac{\sin(N+\frac12)x}{x/2}\,dx + \underbrace{\int_{-\pi}^{\pi}\left(\frac{1}{\sin\frac x2}-\frac2x\right)\sin(N+\tfrac12)x\,dx}_{0}\right]$

$=\lim_{N\to\infty}\int_{-\pi}^{\pi}\frac{\sin(N+\frac12)x}{x/2}\,dx$

$=\int_{-\infty}^{\infty} 2\frac{\sin y}{y}\,dy$ // $y=(N+\frac12)x$.

$\Rightarrow \int_0^\infty \frac{\sin y}{y}\,dy = \pi/2.$