# 微积分 T（3）
# 第一次作业答案

2023 年 10 月 7 日

1. (1) 证明. 由 Euler 公式可知

$$
\begin{aligned}
f(x) &\sim \hat{f}(0) + \sum_{n\geqslant 1}\Big(\hat{f}(n)\,e^{inx} + \hat{f}(-n)\,e^{-inx}\Big)\\
&\sim \hat{f}(0) + \sum_{n\geqslant 1}\Big(\hat{f}(n)\,(\cos nx + i\sin nx) + \hat{f}(-n)\,(\cos nx - i\sin nx)\Big)\\
&\sim \hat{f}(0) + \sum_{n\geqslant 1}\Big(\big(\hat{f}(n)+\hat{f}(-n)\big)\cos nx + \big(\hat{f}(n)-\hat{f}(-n)\big)\,i\sin nx\Big)
\end{aligned}
$$

因此，该结论成立。 □

(2) 证明. 如果 $f$ 是偶函数，设 $x=-y$ 得到

$\int_{\pi}^{-\pi} f(-y)\, e^{iny}\, \frac{d(-y)}{dy}\, dy.$

原本 $x=-\pi$，即 $-\pi\to\pi$. 所以 $y$ 就是 $\pi\to-\pi$.

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,\mathrm{d}x &= \int_{\pi}^{-\pi} f(-y)\,e^{iny}\,(-y)'\,\mathrm{d}y\\
&= \int_{-\pi}^{\pi} f(y)\,e^{iny}\,\mathrm{d}y
\end{aligned}
$$

即 $\int_{x_1}^{x_2} f(x)\,dx$. 令 $y=-x$.

$\int_{x_1}^{x_2} f(x)\,dx = \int_{-y_1}^{-y_2} f(-y)\,d(-y)$

偶 $= -\int_{-y_1}^{-y_2} f(y)\,dy$

$= \int_{-y_2}^{-y_1} f(y)\,dy$.

$x_2 = \pi = -y_2$

$x_1 = -\pi = -y_1$

除以 $2\pi$ 即得到 $\hat{f}(n)=\hat{f}(-n)$。由第 (1) 题的结论可知，$f$ 的 Fourier 级数是余弦级数。 □

(3) 证明. 如果 $f$ 是奇函数，设 $x=-y$ 得到

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,\mathrm{d}x &= \int_{\pi}^{-\pi} f(-y)\,e^{iny}\,(-y)'\,\mathrm{d}y\\
&= -\int_{-\pi}^{\pi} f(y)\,e^{iny}\,\mathrm{d}y
\end{aligned}
$$

除以 $2\pi$ 即得到 $\hat{f}(n)=-\hat{f}(-n)$。由第 (1) 题的结论可知，$f$ 的 Fourier 级数是正弦级数。 □



(2) 令 $x=\frac{\pi}{2}$, $f(x)=f(-x)$

$\Rightarrow \hat{f}(0) + \sum_{n\geq1}(\hat{f}(n)-\hat{f}(-n))\, i\sin nx = \hat{f}(0) - \sum_{n\geq1}(\hat{f}(n)-\hat{f}(-n))\, i\sin nx.$

$\Rightarrow \sum_{n\geq1}[\hat{f}(n)-\hat{f}(-n)](-1)^n i = -\sum_{n\geq1}(-1)^n i\,[\hat{f}(n)-\hat{f}(-n)].$

$\Rightarrow 2\sum_{n\geq1}(\hat{f}(n)-\hat{f}(-n))(-1)^n = 0.$

(4) **证明.** 设 $n$ 是奇数，则有

$$e^{-in\pi} = (-1)^n = -1$$

由于 $f$ 以 $2\pi$ 为周期，设 $x = y + \pi$ 则有

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, \mathrm{d}x &= \int_{0}^{2\pi} f(x)\, e^{-inx}\, \mathrm{d}x \\
&= \int_{-\pi}^{\pi} f(y+\pi)\, e^{-in(y+\pi)}\, (y+\pi)'\, \mathrm{d}y \\
&= e^{-in\pi} \int_{-\pi}^{\pi} f(y)\, e^{-iny}\, \mathrm{d}y \\
&= -\int_{-\pi}^{\pi} f(y)\, e^{-iny}\, \mathrm{d}y
\end{aligned}$$

除以 $2\pi$ 得到 $\hat{f}(n) = -\hat{f}(n)$，因此 $\hat{f}(n) = 0$。 □

(5) **证明.** 由于 $f$ 是实值函数，

$$\begin{aligned}
\overline{\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, \mathrm{d}x} &= \int_{-\pi}^{\pi} \overline{f(x)\, e^{-inx}}\, \mathrm{d}x \\
&= \int_{-\pi}^{\pi} f(x)\, \overline{e^{-inx}}\, \mathrm{d}x \\
&= \int_{-\pi}^{\pi} f(x)\, e^{inx}\, \mathrm{d}x
\end{aligned}$$

除以 $2\pi$ 即得到 $\overline{\hat{f}(n)} = \hat{f}(-n)$。 □

2. **证明.** 由于 $f''$ 在 $[-\pi, \pi]$ 上连续，可以取

$$C = \sup_{x \in [-\pi,\pi]} |f''(x)| < +\infty$$

$f$ 以 $2\pi$ 为周期，故 $f$ 在 $[-\pi, \pi]$ 上分部积分产生的边界项全部抵消。设 $n$ 是非零整数，分部积分两次得到

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left( \frac{e^{-inx}}{n^2} \right)'' \mathrm{d}x \\
&= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) \left( \frac{e^{-inx}}{n^2} \right)' \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f''(x)\, \frac{e^{-inx}}{n^2}\, \mathrm{d}x
\end{aligned}$$

$f(x)$ 连续. $f(x+2\pi) = f(x)$

$\Rightarrow$ $f'(x+2\pi) = f'(x)$

$\cancel{f(\pi) - f(-\pi)}$

$+ \frac{1}{2\pi} f g' \Big|_{-\pi}^{\pi} = 0.$

$- \frac{1}{2\pi} f g' \Big|_{-\pi}^{\pi} = 0$

$\int_{-\pi}^{\pi} f g''\, dx = fg' \Big|_{-\pi}^{\pi} - \int f' g'\, dx$

$= + \int f'' g\, dx - f' g \Big|_{-\pi}^{\pi}$

进而，其绝对值

$$
\begin{aligned}
\left|\hat{f}(n)\right| &= \frac{1}{2\pi n^2}\left|\int_{-\pi}^{\pi} f''(x)\, e^{-inx}\, \mathrm{d}x\right| \\
&\leqslant \frac{1}{2\pi n^2}\int_{-\pi}^{\pi} \left|f''(x)\, e^{-inx}\right| \mathrm{d}x \\
&= \frac{1}{2\pi n^2}\int_{-\pi}^{\pi} C\, \mathrm{d}x \\
&= \frac{C}{n^2}
\end{aligned}
$$

□

3. 由于 $f(x)$ 是奇函数，其 Fourier 级数由正弦级数构成。当 $n \neq 0$ 时，

（上行为求导
下行为积分）

$x(\pi-x)$ $\quad \pi-2x \quad -2 \quad 0$

$\sin nx \quad -\frac{\cos nx}{n} \quad -\frac{\sin nx}{n^2} \quad \frac{\cos nx}{n^3}$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, \mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\,(\pi-x)\sin nx\, \mathrm{d}x \\
&= -\frac{2}{\pi n}x\,(\pi-x)\cos nx\Big|_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}(\pi-2x)\cos nx\, \mathrm{d}x \\
&= -\frac{2}{\pi n}x\,(\pi-x)\cos nx\Big|_0^{\pi} + \frac{2}{\pi n^2}(\pi-2x)\sin nx\Big|_0^{\pi} + \frac{4}{\pi n^2}\int_0^{\pi}\sin nx\, \mathrm{d}x \\
&= -\frac{2}{\pi n}x\,(\pi-x)\cos nx\Big|_0^{\pi} + \frac{2}{\pi n^2}(\pi-2x)\sin nx\Big|_0^{\pi} - \frac{4}{\pi n^3}\cos nx\Big|_0^{\pi} \\
&= 0 + 0 - \frac{4}{\pi n^3}\left((-1)^n - 1\right) \\
&= \frac{4}{\pi n^3}\left(1-(-1)^n\right)
\end{aligned}
$$

$$b_n = \frac{2}{\pi}\left(-\frac{x(\pi-x)}{n}\cos nx + \frac{(\pi-2x)\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right)\Big|_0^{\pi}$$

$$= \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

因此，$f(x)$ 的 Fourier 级数为

$$f(x) \sim \sum_{n=1}^{\infty}\frac{4}{\pi n^3}\left(1-(-1)^n\right)\sin nx$$

4. 由于 $f(x)$ 是偶函数，其 Fourier 级数由余弦级数构成。

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\, \mathrm{d}x = \frac{1}{\pi}x^2\Big|_0^{\pi} = \pi$$

当 $n \neq 0$ 时，

$x \quad 1 \quad 0$

$+ \quad - \quad +$

$\cos nx \quad +\frac{\sin nx}{n} \quad -\frac{\cos nx}{n^2}$

$\frac{2}{\pi}\left[x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right]\Big|_0^{\pi}$

$=\frac{2}{\pi n^2}\left((-1)^n-1\right)$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi n}x\sin nx\Big|_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi}\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi n}x\sin nx\Big|_0^{\pi} + \frac{2}{\pi n^2}\cos nx\Big|_0^{\pi} \\
&= 0 + \frac{2}{\pi n^2}\left((-1)^n - 1\right) \\
&= \frac{2}{\pi n^2}\left((-1)^n - 1\right)
\end{aligned}
$$

因此，$f(x)$ 的 Fourier 级数为

$$
f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos nx
$$

5. (1) 证明. 注意到

$$
\left|\hat{f}(n)\,e^{inx}\right| = \left|\hat{f}(n)\right| \qquad \sum_{n=-\infty}^{+\infty}\left|\hat{f}(n)\right| < +\infty
$$

由 Weierstrass 强级数判别法可知，该函数项级数一致收敛：

$$
\sum_{n=-\infty}^{+\infty}\hat{f}(n)\,e^{inx} \qquad \square
$$

(2) 由于上述函数项级数一致收敛，极限函数 $g(x)$ 也是以 $2\pi$ 为周期

的连续函数，且其 Fourier 系数

$$
\begin{aligned}
\hat{g}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,\mathrm{e}^{-imx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{+\infty}\hat{f}(n)\,e^{inx}\right)e^{-imx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{+\infty}\hat{f}(n)\,\mathrm{e}^{i(n-m)x}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-\pi}^{\pi}\hat{f}(n)\,\mathrm{e}^{i(n-m)x}\,\mathrm{d}x \\
&= \sum_{n=-\infty}^{+\infty}\hat{f}(n)\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{i(n-m)x}\,\mathrm{d}x \\
&= \sum_{n=-\infty}^{+\infty}\hat{f}(n)\,\delta_{n,m} \\
&= \hat{f}(m)
\end{aligned}
$$

$$
\begin{cases} n=m & \int_{-\pi}^{\pi} dx = 2\pi \\ n\neq m & \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx \end{cases}
$$

$$
= \frac{e^{i(n-m)x}}{i(n-m)}\Big|_{-\pi}^{\pi} = 0.
$$

(3) 证明. 设周期为 $2\pi$ 的连续函数

$$
h(x) = g(x) - f(x)
$$

则其 Fourier 系数满足

$$
\hat{h}(n) = \hat{g}(n) - \hat{f}(n) = 0
$$

由参考书定理 2.1 的结论可知 $h(x) = 0$，因此 $g(x) = f(x)$。 □

(4) 证明. 由前述结论，在 $[-\pi, \pi]$ 上有一致收敛的函数项级数

$$
\frac{(\pi - x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}
$$

取 $x = 0$ 得到

$$
\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2} \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}
$$

□

1.

2. k阶可微 $\Rightarrow f^{(k)}(x) = f^{(k)}(x+2\pi)$.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx.$$

$$\begin{array}{ccccccc} f(x) & & f'(x) & & f''(x) & \cdots & f^{(k)}(x),\ (-1)^k \\ e^{-inx} & + & \dfrac{e^{-inx}}{-in} & - & \dfrac{e^{-inx}}{(-in)^2} & + \ \cdots & \dfrac{e^{-inx}}{(-in)^k} \end{array}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f^{(k)}(x)\,(-1)^k\, e^{-inx}}{(-in)^k}\,dx = \hat{f}(n)$$

$$|\hat{f}(n)| = \frac{1}{2\pi n^k}\left|\int_{-\pi}^{\pi} f^{(k)}(x)\,(-1)^k e^{-inx}\,dx\right|$$

$$\le \frac{1}{2\pi n^k}\int_{-\pi}^{\pi} \left| f^{(k)}(x)\,(-1)^k e^{-inx}\right| dx.$$

$f^{(k)}(x)$ 连续. 故 $x \in [-\pi,\pi]$ $f^{(k)}(x)$ 有上界.

则 $|\hat{f}(n)| \le \dfrac{C}{n^k}$ // $C = \max|f^{(k)}(x)|$ $x \in [-\pi,\pi]$.

3. (1). $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi n^2}((-1)^n-1)$.

偶函数，无 $b_n$. 即 $|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n-1)\cos nx$.

(2) 令 $x=0$. $|x| = 0 = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^n-1\right) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)^2}$

即 $\sum_{n\ge1,奇}\frac{1}{n^2} = \frac{\pi^2}{8}$. $\underbrace{\sum_{n\ge1}\frac{1}{n^2}}_{S_n} = \sum_{n\ge1奇}\frac{1}{n^2} + \underbrace{\sum_{n\ge1偶}\frac{1}{n^2}}_{\frac{S_n}{4}}$

$\Rightarrow \frac{3}{4}S_n = \frac{\pi^2}{8} \Rightarrow S_n = \frac{\pi^2}{6}$.

(3). 帕塞瓦尔等式：$\sum_{n=-\infty}^{\infty} |C_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx$

其中 $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$\boxed{\frac{a_n}{2}}$ 故.

(1)问中 $C_n = \frac{i}{\pi n^2}((-1)^n - 1) \quad n \in (-\infty, \infty)$.

(将 $\cos x$ 展开 $\frac{e^{ix}+e^{-ix}}{2}$).

故 $\sum_{n=-\infty \text{奇}}^{\infty} \frac{4}{n^4 \pi^2} + \frac{\pi^2}{4} \leftarrow c_0^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$

$\Rightarrow \sum_{n\geq 1 \text{奇}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$.

$S_n = \sum_{n\geq 1}^{\infty} \frac{1}{n^4} \qquad S_n = \frac{\pi^4}{96} + \frac{S_n}{16}$

$\Rightarrow S_n = \frac{\pi^4}{90}$.

4. (1) 奇函数. $f(0) = 0$. ~~$f(\pi) = 0$~~. $a(n) = 0$.

$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \, dx = \frac{4}{\pi n^3}(1-(-1)^n)$

~~故~~ (2) 由帕塞瓦尔等式：$\sum_{n=-\infty \text{奇}}^{\infty} \frac{16}{\pi^2 n^6} = \frac{1}{\pi}\int_{0}^{\pi} x^2(\pi - x)^2 dx = \frac{\pi^4}{30}$.

$\Rightarrow \sum_{n\geq 1 \text{奇}}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{960}$

$S_n = \sum_{n\geq 1}^{\infty} \frac{1}{n^6} \qquad S_n = \frac{S_n}{64} + \frac{\pi^6}{960} \Rightarrow S_n = \frac{\pi^6}{945}$.

5.

$$f(n) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi}{\sin\pi\alpha} e^{i\pi\alpha} e^{-i\pi x} e^{-inx} dx$$

$$= \frac{e^{i\pi\alpha}}{2\sin\pi\alpha}\int_0^{2\pi} e^{-i(\alpha+n)x} dx$$

$$= \frac{e^{i\pi\alpha}}{2\sin\pi\alpha(-i)} \cancel{\times}\, e^{-i(\alpha+n)x}\Big|_0^{2\pi}$$

$$= \frac{e^{i\pi\alpha}}{2\sin\pi\alpha(-i)(\alpha+n)}\left(\underbrace{e^{-i2\pi n}}_{1}\cdot e^{-i2\pi\alpha} - 1\right)$$

$$= \frac{e^{-i\pi\alpha} - e^{i\pi\alpha}}{2\sin\pi\alpha(-i)(\alpha+n)} = \frac{(-2i)\sin\pi\alpha}{(-i)2\sin\pi\alpha(\alpha+n)} = \frac{1}{n+\alpha}.$$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty}\frac{e^{inx}}{n+\alpha}.$$

由帕塞瓦尔等式: $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(n+\alpha)^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi}{\sin\pi\alpha}\right)^2 dx$

$$= \frac{\pi^2}{(\sin\pi\alpha)^2}.$$

6. Prove $\displaystyle\int_0^{\infty}\frac{\sin x}{x} = \frac{\pi}{2}$

f为R上区间I的可积函数. $\displaystyle\lim_{s\to+\infty}\int_I f(t)e^{-ist}dt = 0$. ← Riemann-Lebesgue 引理.

$\dfrac{1}{\sin\frac{x}{2}} - \dfrac{2}{x}$. $x=0$ 为可去间断点

上述极限为0. 即实部.虚部都是0. 故 $\displaystyle\lim_{N\to\infty}\int_0^{\pi}\left(\frac{1}{\sin\frac{x}{2}} - \frac{2}{x}\right)\sin(N+\tfrac{1}{2})x\,dx = 0.$

$$\int_0^{\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}dx = \pi. \quad // \sin\left(\frac{2N+1}{2}x\right) = \underline{\sin\tfrac{2N+1}{2}x - \sin\tfrac{2N-1}{2}x} + \underline{\sin\tfrac{2N-1}{2}x - \sin\tfrac{2N-3}{2}x + \sin\tfrac{2N-3}{2}x \cdots - \sin\tfrac{x}{2}} + \sin\tfrac{x}{2}$$

$$= \left(2\sum_{i=1}^{N}\cos ix + 1\right)\sin\frac{x}{2}.$$

$$\Rightarrow \lim_{N\to\infty}\int_0^{\pi}\frac{\sin(N+\frac{1}{2})x}{x}dx = \pi/2.$$

令 $(N+\frac{1}{2})x = t$ $\quad\Rightarrow \displaystyle\lim_{N\to\infty}\int_0^{(N+\frac{1}{2})\pi}\frac{\sin t}{t/(N+\frac{1}{2})}\frac{dt}{(N+\frac{1}{2})} = \pi/2$

$$\Rightarrow \int_0^{\infty}\frac{\sin x}{x}dx = \pi/2.$$

7. (a) $\int_0^T f(t)dt=0 \Rightarrow \hat{f}(0)=0$. $f(x)=\sum_{n\neq 0}\hat{f}(n)e^{in\frac{2\pi}{T}x}$.

$f\in C^1$，即 $f(x)$ 与 $f'(x)$ 有同样的周期.

故 $f'(x)=\sum_{n\neq 0}\hat{f'}(n)e^{in\frac{2\pi}{T}x} = \left(\sum_{n\neq 0}\hat{f}(n)e^{in\frac{2\pi}{T}x}\right)'$

$\Rightarrow \hat{f'}(n)=\frac{2\pi i n}{T}\hat{f}(n)$. 按定义代入.

$$\int_0^T|f(t)|^2dt = T\sum_{n\neq 0}|\hat{f}(n)|^2 = \frac{T^3}{4\pi^2}\sum_{n\neq 0}\frac{|\hat{f'}(n)|^2}{n^2}$$

$$\le \frac{T^3}{4\pi^2}\sum_{n\neq 0}|\hat{f'}(n)|^2 = \frac{T^2}{4\pi^2}\int_0^T|f'(t)|^2dt$$

(b) $$\left|\int_0^T \bar{f}(t)g(t)dt\right|^2 = \left|\int_0^T f(t)\Big(g(t)-\underbrace{\int_0^T g(x)dx}_{数}\Big)dt\right|^2$$

$\int_0^T f(t)dt=0$.

$$\le \int_0^T|f(t)|^2dt\int_0^T\left|g(t)-\int_0^T g(x)dx\right|^2dt$$

$$= \frac{T^2}{4\pi^2}\int_0^T|f(t)|^2dt\int_0^T\left|\frac{d}{dt}\Big(g(t)-\int_0^T g(x)dx\Big)\right|^2dt$$

(a)中结论.

$$= \frac{T^2}{4\pi^2}\int_0^T|f(t)|^2dt\int_0^T|g'(t)|^2dt.$$

(c) 要满足 $\int_0^T f(t)dt=0$. 故关于 $b$ 进行解析延拓. 使得 $\int_a^{2b-a}f(t)dt=0$.

且 $T=2(b-a)$. 代入 (a). 即证.

1. 反证法. 题干中要求的所有 $f(n)=0$.

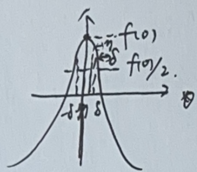
则对 $\forall p(\theta)$, $p(\theta)$ 为任意的三角函数多项式.

要求 $\int p_k(\theta)f(\theta)d\theta=0$ // $p_k(\theta)=p(\theta)$ 幂也成立.

现反证 $\lim_{k\to\infty}\int p_k(\theta)f(\theta)d\theta\to\infty$.

不失一般性 $f(\theta)$ 定义在 $[-\pi,\pi]$, 假设 $f(0)>0$. 现推矛盾

由于 $f(x)$ 连续.




总有 $0<\delta<\pi/2$. 可以有 $f(\theta)>0$, $|\theta|<\delta$.

现取 $p(\theta)=\varepsilon+\cos\theta$. $\varepsilon>0)$.

现有3个区间. $|\theta|<\eta$: $p(\theta)=\varepsilon+\cos\theta>1+\varepsilon/2$

$\eta\le|\theta|<\delta$: $p(\theta)=\varepsilon+\cos\theta<1-\varepsilon/2$

$\delta\le|\theta|\le\pi$: $p(\theta)=\varepsilon+\cos\theta<1-\varepsilon/2$.

现先承认上述条件. 再将 $\int_{-\pi}^{\pi}p_k(\theta)f(\theta)d\theta$ 分在上述3个区间计算.

$$\left|\int_{|\theta|>\delta}f(\theta)p_k(\theta)d\theta\right|\le 2\pi\cdot B(1-\varepsilon/2)^k\xrightarrow{\lim k\to\infty}0$$

$$\text{并}\int_{\eta\le|\theta|<\delta}f(\theta)p_k(\theta)d\theta\geq 0\xrightarrow{\lim_{k\to\infty}}\geq 0$$

$$\int_{|\theta|<\eta}f(\theta)p_k(\theta)d\theta\geq 2\eta\frac{f(0)}{2}(1+\varepsilon/2)^k\cdot\xrightarrow{\lim_{k\to\infty}}\infty.$$

与已知矛盾. 故 $f(0)=0$. 由于 0 的位置任意. $\Rightarrow f(\theta)=0$. 又连续.

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现上述3个区间是否存在: 首先 $\delta\le|\theta|\le\pi$, $\varepsilon+\cos\theta<1-\varepsilon/2\Rightarrow\varepsilon<\frac{2}{3}(1-\cos\theta)$ 即可

故区间 2. 3 是 OK 的. 现寻找 $\eta$. 当 $|\theta|<\eta$. $\varepsilon+\cos\theta\geq 1+\varepsilon/2$.

即 $\cos\theta>1-\varepsilon/2$. 又由于 $\varepsilon<\frac{2}{3}(1-\cos\theta)\le\frac{2}{3}(1-\cos\delta)$

$\Rightarrow\cos\theta\geq 1-\frac{1}{3}(1-\cos\delta)=\frac{2}{3}+\frac{\cos\delta}{3}=\frac{1}{3}(2+\cos\delta)$

即 $\Longrightarrow |\theta|<\arccos\left(\frac{2}{3}+\frac{\cos\delta}{3}\right)$ 即可, 即 $\eta=\arccos\left(\frac{2}{3}+\frac{\cos\delta}{3}\right)$

或 $p(\theta)$ 连续 $p(0)=1+\varepsilon$ 故一定有 $p(\eta)=1+\varepsilon/2$.