科目：微积分	章节：3.1	做题人：钱雨杰

1、证明过程与定积分类似

2、略

3. $$\iint\limits_{[0,1]\times[0,1]} xy\,dxdy = \lim_{n\to+\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i}{n}\cdot\frac{j}{n}\cdot\left(\frac{1}{n^2}\right) = \lim_{n\to+\infty}\left(\sum_{i=1}^{n}\frac{i}{n^2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

4. $$\iint\limits_{[-a,a]\times[-a,a]} \sin(x+y)\,dxdy.$$

用 $x=\frac{i}{n}\cdot a$，$y=\frac{j}{n}\cdot a\ (-n+1\le i,j\le n)$ 将积分域分成 $(2n)^2$ 个小正方形.

在x轴上方的小正方形，选其右上顶点的函数值；x轴下方的小正方形，选其左下顶点的函数值.

$$\therefore \iint\limits_{[-a,a]^2} \sin(x+y)\,dxdy = \lim_{n\to+\infty}\left(\sum_{i=1}^{n}\sum_{j=-n+1}^{n}\sin\left(\frac{i+j}{n}\right)\left(\frac{1}{n^2}\right) + \sum_{i=1}^{n}\sum_{j=-n+1}^{n}\sin\left(\frac{-i-j}{n}\right)\left(\frac{1}{n^2}\right)\right) = 0$$

5、$\because f(x,y)\in R(I)$　$\therefore f$ 在 I 上的间断点集合是零面积集.

$\because J\subset I$.　$\therefore J$ 的间断点集合是 I 的间断点集合的子集，亦为零面积集

$\therefore f\in R(J)$.

6. 令分割 T_x：$0=x_0<x_1<\cdots<x_n=1$．T_y：$0=y_0<y_1<\cdots<y_m=1$.

$$\iint\limits_{[0,1]\times[0,1]} f(x)g(y)\,dxdy = \lim_{|T|\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i)g(y_j)\Delta x_i\Delta y_j = \lim_{|T|\to 0}\left(\sum_{i=1}^{n} f(x_i)\Delta x_i\right)\left(\sum_{j=1}^{m} g(y_j)\Delta y_j\right)$$

$$= \int_0^1 f(x)\,dx\cdot\int_0^1 g(y)\,dy.$$

$$\iint\limits_{[0,1]\times[0,1]} e^{-(x+y)}\,dxdy = \int_0^1 e^{-x}dx\int_0^1 e^{-y}dy = (1-e^{-1})^2$$

7. $f(x,y)$ 与 $|f(x,y)|$ 在 I 上的间断点集合是相同的.

故若 $f(x,y)$ 在 I 上可积，$|f(x,y)|$ 在 I 上可积.

令 $I_1=\{(x,y)\,|\,(x,y)\in I$ 且 $f(x,y)\ge 0\}$．$I_2=\{(x,y)\,|\,(x,y)\in I$ 且 $f(x,y)<0\}$.

$$\iint\limits_{I} f(x,y)\,dxdy = \iint\limits_{I_1} f(x,y)\,dxdy + \iint\limits_{I_2} f(x,y)\,dxdy$$

$$\iint\limits_{I} |f(x,y)|\,dxdy = \iint\limits_{I_1} f(x,y)\,dxdy - \iint\limits_{I_2} f(x,y)\,dxdy$$

$\because \iint\limits_{I_1} f(x,y)\,dxdy>0$．$\iint\limits_{I_2} f(x,y)\,dxdy<0$．由绝对值不等式 $\left|\iint\limits_{I} f(x,y)\,dxdy\right| \le \iint\limits_{I}|f(x,y)|\,dxdy$

8、略．参阅《数学分析》

科目：微积分	章节：3.1	做题人：钱雨杰

9. 记点列 $\{X_n\}$ 收敛于 A. 即 $\forall \varepsilon>0$. $\exists N$. $\forall n>N$. $\|X_n-A\|<\varepsilon$.

记 $A=(x_a, y_a)$. 取矩形 $I=[x_a-\varepsilon, x_a+\varepsilon]\times[y_a-\varepsilon, y_a+\varepsilon]$. $\sigma(I)=4\varepsilon^2$.

当 $n>N$ 时. X_n 落在矩形 I 内. 当 $n\le N$ 时. 有限个点，构成零面积集.

由 $\varepsilon>0$ 的任意性可知，$\{X_n\}$ 是零面积集.

10. (1) 考虑 $f(x,y)$ 的间断点. 当 $(1-x^2)^2+(1-y^2)^2=0$ 时. $f(x,y)$ 间断.

即有4个间断点 $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$.

$\therefore f(x,y)$ 在 $[-2,2]\times[-2,2]$ 上可积.

(2) 考虑 $f(x,y)$ 的间断点. 当 $y=x^2$ 时. $f(x,y)$ 间断.

由 $y=x^2$ 是二维平面上一条光滑曲线，是零面积集.

$\therefore f(x,y)$ 在 $[0,1]\times[0,1]$ 上可积.

11. 12、易证.

13. $\because f.\ g\in R(I)$ $\therefore f.\ g$ 的间断点集合均为零面积集.

$\because fg$ 的间断点集合 $\subset$ f 的间断点集合 $\cup$ g 的间断点集合 $\therefore$ 也是零面积集

$\therefore fg\in R(I)$. 同理. 若 $g\neq 0$. $\frac{f}{g}\in R(I)$.

科目：微积分	章节：3.2	做题人：姚炫容

1. 按定义，将二重化为Rieman和的形式，则显然。

2. $\frac{1}{100} > \frac{1}{100+\cos^2x+\cos^2y} > \frac{1}{102}$

$$\therefore 1.96 < \iint\limits_{|x|+|y|\le 10} \frac{dxdy}{100+\cos^2x+\cos^2y} < 2.$$

3.(1) 在D上，有 $x+y\ge 1$. $\therefore (x+y)^3 \ge (x+y)^2$ 等号只在一点取到.

$\cancel{\therefore \iint\limits_D (x+y)^2 dxdy < \iint\limits_D (x+y)^3}$ $\therefore \iint\limits_D (x+y)^2 dxdy < \iint\limits_D (x+y)^3 dxdy$

(2) 在D上. $x+y\le 1$ $\therefore \ln(x+y)\le 0$. $xy\ge 0$.

$\therefore xy \ge \ln(x+y)$ ~~且∃S⊂D S不为零面积集~~.

等号取到的集合为零面积集.

$\therefore \iint\limits_D \ln(x+y)dxdy < \iint\limits_D xy\,dxdy$.

4. 反证法. 若∃$(x_0,y_0)\in D$. s.t. $f(x_0,y_0)\ne 0$. 则由f非负.

得 $f(x_0,y_0)>0$. 又∵ $f\in C(D)$.

$\therefore \exists B\subset D$, s.t. $(x_0,y_0)\in B$. 且B不为零面积集.

$\forall (x,y)\in B$. 有 $f(x,y)>0$.

$\therefore \iint\limits_B f(x,y)dxdy > 0$. 又∵ $\iint\limits_{D-B} f(x,y)dxdy \ge 0$

$\therefore \iint\limits_D f(x,y)dxdy > 0$ 与题设矛盾.

△得证.

科目：微积分	章节：3.2	做题人：姚烁容

5. $\exists\, \xi, \eta$. s.t. $\xi^2+\eta^2 \le r^2$ 且

$$\frac{1}{r^2}\iint\limits_{x^2+y^2\le r^2} f(x,y)\,dxdy = \frac{1}{r^2}\cdot f(\xi,\eta)\cdot\iint\limits_{x^2+y^2\le r^2} dxdy$$

$$= f(\xi,\eta)\cdot\pi$$

当 $r\to 0^+$ 时 $f(\xi,\eta)\to f(0,0)$.

$\therefore$ 原式 $= \pi\cdot f(0,0)$.

科目：微积分	章节：§ 3.3	做题人：姜天健

1.

(1) 几何意义是半球的体积

$\therefore \iint\limits_{D} \sqrt{R^2-x^2-y^2}\,dxdy$, $D=\{(x,y)\,|\,x^2+y^2\le R^2\}$

$= \frac{2}{3}\pi R^3$

(2) 几何意义是从圆柱中挖去圆锥

$\iint\limits_{D} \sqrt{x^2+y^2}\,dxdy$. $D=\{(x,y)\,|\,x^2+y^2\le R^2\}$

$= \pi R^3 - \frac{1}{3}\pi R^3 = \frac{2}{3}\pi R^3$

(3)

$\iint\limits_{D} 1\,dxdy$, $D=\{(x,y)\,|\,|x|+|y|+|z|\le 1\}$

注：本题的 z 的取值未给出，这里默认为绝对值小于等于 1 的常数

几何意义为正方形的体积

$= 2(1-|z|)^2$

科目：微积分	章节：§3.3	做题人：姜天键

2.

(1)

$$\iint_I \frac{x^2}{1+y^2}dxdy,\ I=[0,1]^2$$

$$=\int_0^1 dx\int_0^1\frac{x^2}{1+y^2}dy$$

$$=\frac{1}{12}\pi$$

(2)

$$\iint_I x\cos(xy)dxdy,\ I=[0,\tfrac{\pi}{2}]\times[0,1]$$

$$=\int_0^{\frac{\pi}{2}}dx\int_0^1 x\cos(xy)dy$$

$$=1$$

(3)

$$\iint_I \sin(x+y)dxdy,\ I=[0,\pi]^2$$

$$=\int_0^{\pi}dx\cdot\int_0^{\pi}\sin(x+y)dy$$

$$=0$$

3.

∵ 函数 $f(x,y)$ 在 $I=[a,b]\times[c,d]$

上有连续的二阶偏导数

~~∴ $f(x,y)$ 在 I 上处处可微~~

~~∴ $f(x,y)$ 在 I 上处处连续~~

~~又∵ I 为矩形域~~

~~∴ $f(x,y)$ 在 I 上可积~~

∵ ~~$\forall x\in$~~ $[a,b]$ 有 $f(x,y)$ 在闭区

~~间上连续~~

∴ ~~关于~~ y 在 $[c,d]$ 上可积

∴ $\forall x$，由 y 在闭区间上连续

∴ $\frac{\partial^2 f}{\partial x\partial y}$ 可积

$$\therefore \iint_I f(x,y)d\sigma=\int_a^b dx\int_c^d\frac{\partial f(x,y)}{\partial x\partial y}dy$$

$$=\int_a^b dx\left(\frac{\partial f(x,d)}{\partial x}-\frac{\partial f(x,c)}{\partial x}\right)$$

$$=f(b,d)-f(b,c)+f(a,c)-f(a,d)$$

科目：微积分	章节：§3.3	做题人：[illegible]

4.

(1)

$D=\{(x,y)\mid x+y\le 1,\ y-x\le 1,\ y\ge 0\}$

$\therefore D=\{(x,y)\mid y-1\le x\le 1-y,\ 0\le y\le 1\}$

$\therefore \iint\limits_D f(x,y)\,d\sigma=\int_0^1 dy\int_{y-1}^{1-y} f(x,y)\,dx$

(2)

$D=\{(x,y)\mid y\ge x-2,\ x\ge y^2\}=D_1\cup D_2$

其中 $D_1=\{(x,y)\mid x\ge y^2,\ 0\le x\le 1\}$

$D_2=\{(x,y)\mid y^2\le x\le \cancel{y}+2,\ 1\le x\le 4\}$

$\iint\limits_D f(x,y)\,d\sigma=\iint\limits_{D_1} f(x,y)\,d\sigma+\iint\limits_{D_2} f(x,y)\,d\sigma$

$=\int_0^1 dx\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy+\int_1^4 dx\int_{x-2}^{\sqrt{x}} f(x,y)\,dy$

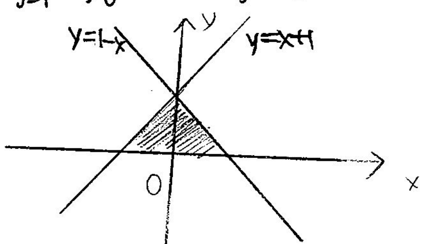
(3)

$D=\{(x,y)\mid \frac{2}{x}\le y\le 2x,\ 1\le x\le 2\}$

$\iint\limits_D f(x,y)\,d\sigma=\int_1^2 dx\int_{\frac{2}{x}}^{2x} f(x,y)\,dy$

5.

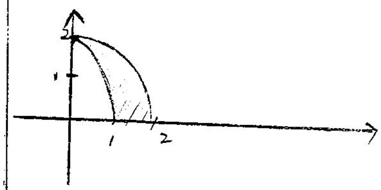
(1) $\int_{-1}^0 dx\int_0^{1+x} f(x,y)\,dy+\int_0^1 dx\int_0^{1-x} f(x,y)\,dy$



交换：$\int_0^1 dy\int_{y-1}^{1-y} f(x,y)\,dx$

(2)

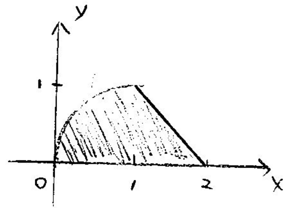
$\int_0^1 dx\int_{2\sqrt{1-x}}^{\sqrt{4-x^2}} f(x,y)\,dy+\int_1^2 dx\int_0^{\sqrt{4-x^2}} f(x,y)\,dy$



$\Rightarrow \int_0^2 dy\int_{1-\frac{y^2}{4}}^{\sqrt{4-y^2}} f(x,y)\,dx$

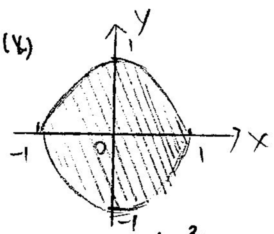
(3)

$\int_0^1 dx\int_0^{\sqrt{2x-x^2}} f(x,y)\,dy+\int_1^2 dx\int_0^{2-x} f(x,y)\,dy$



$\Rightarrow \int_0^1 dy\int_{1-\sqrt{1-y^2}}^{2-y} f(x,y)\,dx$

(4)



$\int_{-1}^1 dx\int_{x^2-1}^{1-x^2} f(x,y)\,dy$

$\Rightarrow \int_{-1}^0 dy\int_{-\sqrt{1+y}}^{\sqrt{1+y}} f(x,y)\,dx$

$+\int_0^1 dy\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,dx.$

科目:微积分	章节: § 3.3	做题人:姜天健

(3)

$$\iint_D |xy|\,dxdy \quad D=\{(x,y)\mid x^2+y^2\le R^2\}$$

$$= 4\int_0^R dx\int_0^{\sqrt{R^2-x^2}} xy\,dy$$

$$= 4\int_0^R \frac{(R^2-x^2)x}{2}dx$$

$$= \frac{R^4}{2}$$

(4)

$$\iint_D x\cos(xy)\,dxdy \quad D=\{(x,y)\mid x^2+y^2\le R^2\}$$

$\because$ $x\cos(xy)$ 关于 x 为奇函数

D 关于 y 轴对称

$\therefore$ $\iint_D x\cos(xy)\,dxdy=0$

(5) $\iint_D (x^2+y^2)\,dxdy \quad \partial D=(y=x)\cup(y=x+1)\cup(y=1)\cup(y=4)$

$$= \int_1^4 dy\int_{y-1}^{y}(x^2+y^2)dy$$

$$= \frac{71}{2}$$

(6) $\iint_D e^{x+y}dxdy,\ D=\{(x,y)\mid |x|+|y|\le 1\}$

$$= \int_{-1}^0 dx\int_{-1-x}^{x+1} e^{x+y}dy+\int_0^1 dx\int_{x-1}^{1-x} e^{x+y}dy$$

$$= e-e^{-1}$$

(7)

$$\iint_D \cos(x+y)\,dxdy,\ D=\{(x,y)\mid 0\le x,y\le\pi\}$$

$$= \int_0^{\pi}dx\int_0^{\pi}\cos(x+y)dy$$

$$= -4$$

(8)

$$\iint_D |\cos(x+y)|\,dxdy \quad D=[0,1]^2$$

$$= \int_0^{\frac{\pi}{2}-1}dx\int_0^1\cos(x+y)dy+\int_{\frac{\pi}{2}-1}^1 dx\int_0^{\frac{\pi}{2}-x}\cos(x+y)dy$$

$$-\int_{\frac{\pi}{2}-1}^1 dx\int_{\frac{\pi}{2}-x}^1\cos(x+y)dy$$

$$= 3-\pi+2\cos 1+\cos 2$$

(9) $\iint_D y^2dxdy$, D由 $\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}, t\in[0,2\pi]$

及 x 轴围成

由曲线的参数方程知其为摆线

记 $y=f(x)$

$$\therefore \text{原式}=\int_0^{2\pi a}dx\int_0^{f(x)}y^2dy$$

$dx=a(1-\cos t)dt$

$$\therefore \text{原式}=\int_0^{2\pi}y\cdot dt\cdot\int_0^y y^2dy$$

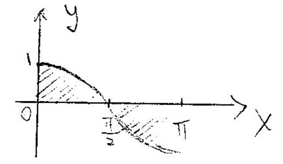
$$= \frac{a^4}{3}\int_0^{2\pi}(1-\cos t)^4dt$$

$$= \frac{35}{12}\pi a^4$$

科目：微积分	章节：§3.3	做题人：[illegible]

(5)

$$\int_0^{\pi} dx \int_0^{\cos x} f(x,y)\,dy$$



$$\Rightarrow \int_{-1}^{0} dy \int_{\pi}^{\arccos y} f(x,y)\,dx + \int_0^1 dy \int_0^{\arccos y} f(x,y)\,dx$$

6.

(1) $\iint\limits_D x y^2 dx dy$, $D=\{(x,y)\mid 4x\geq y^2, \cancel{x\geq 1}\ x\leq 1\}$

$\therefore \int_0^1 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} f(x,y)\,dy$

$= \int_0^1 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} x y^2 dy$

$= \frac{32}{21}$

(2) $\iint\limits_D \frac{1}{\sqrt{2a-x}} dx dy$, $D=\{(x,y)\mid (x-a)^2+(y-a)^2\leq 1,\ 0\leq x,y\leq a\}$

① 若 $|a|\leq \frac{\sqrt{2}}{2}$

$$\iint\limits_D \frac{1}{\sqrt{2a-x}} dx dy = 2a^{\frac{3}{2}}(\sqrt{2}-1)$$

② 若 $\frac{\sqrt{2}}{2}\leq a\leq 1$

$D = D_1 \cup D_2$

其中 $D_1 = \{(x,y)\mid 0\leq y\leq a,\ (x-a)^2+(y-a)^2\leq 1,\ 0\leq x\leq a-\sqrt{1-a^2}\}$

$D_2 = \{(x,y)\mid 0\leq y\leq a,\ a-\sqrt{1-a^2}\leq x\leq a\}$

$\therefore \iint\limits_D f(x,y)\,d\sigma$

$$= \int_0^{a-\sqrt{1-a^2}} \frac{\sqrt{1-(x-a)^2}}{\sqrt{2a-x}} dx + 2a\left((a+\sqrt{1-a^2})^{\frac{1}{2}} - \sqrt{a}\right)$$

③ 若 $a\geq 1$

$\iint\limits_D f(x,y)\,d\sigma$

$$= \int_{a-1}^{a} \frac{\sqrt{1-(x-a)^2}}{\sqrt{2a-x}} dx$$

注：由于水平所限，② ③中的定积分暂无法求解，求积出解的同学予以补上。

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(10)

$$\iint_D (x+y)dxdy \quad D=[0,1]^2$$

$$=\int_0^1 dx\int_0^1 (x+y)dy$$

$$=\int_0^1 dx\left(\int_0^{1-x}(x+y)dy+\int_{1-x}^1 (x+y)dy\right)$$

$$=\frac{1}{2}$$

7.

记区域D在$y\le 0$的部分为D_2

$\because D_1\cap D_2=\{y=0\}$

构成0面积集

$$\therefore \iint_D f(x,y)\,d\sigma=\iint_{D_1} f(x,y)\,d\sigma+\iint_{D_2} f(x,y)\,d\sigma.$$

~~①若 $f(x,-y)=f(x,y)$~~

对$\iint_{D_2} f(x,y)d\sigma$中，记$y=-y$后

$$\iint_{D_2} f(x,y)\,d\sigma=\iint_{D_1} f(x,-y)\,d\sigma$$

①若 $f(x,-y)=f(x,y)$

而 $\iint_D f(x,y)d\sigma=2\iint_{D_1} f(x,y)d\sigma$

②若 $f(x,-y)=-f(x,y)$

而 $\iint_D f(x,y)\,d\sigma=0$

当平面关于y轴对称时

$$\iint_D f(x,y)\,dxdy=\begin{cases}2\iint_{D_1} f(x,y)dxdy & f(-x,y)=f(x,y)\\ 0 & f(-x,y)=-f(x,y)\end{cases}$$

其中D_1为D在$x\ge 0$的部分

8. $D=\{(x,y)\mid x^2+y^2\le R^2\}$

$\because x^2y^3$为关于y的奇函数

D关于x轴对称

$$\therefore \iint_D x^2y^3dxdy=0$$

同理

$$\iint_D \sqrt{R^2-x^2}\,\sin y\,dxdy=0$$

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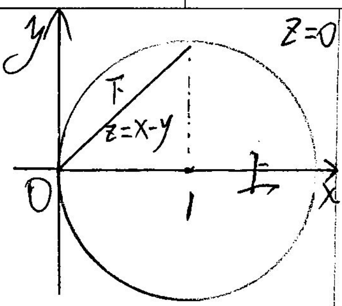
9.



$$\vec{V} = \iint_{D_{xy}} z\,dxdy = \iint_{D_{r\theta}} (1+r\cos\theta - r\sin\theta)\, r\,drd\theta$$

$$\begin{cases} x = 1 + r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\vec{V} = \int_0^1 dr \int_0^{2\pi} r[1+r\cos\theta - r\sin\theta]\, d\theta$$

$$= \pi$$

$$\vec{V_{下}} = \iint_{D_{xy}} z\,dxdy = \int_0^1 dx \int_{x}^{\sqrt{1-x^2}} dy\,(x-y)$$

$$= \int_0^1 \left(x\sqrt{1-x^2} - \frac{1}{2}\right) dx = -\frac{1}{6}$$

$$\vec{V_{上}} = \vec{V} - \vec{V_{下}} = \pi + \frac{1}{6}$$

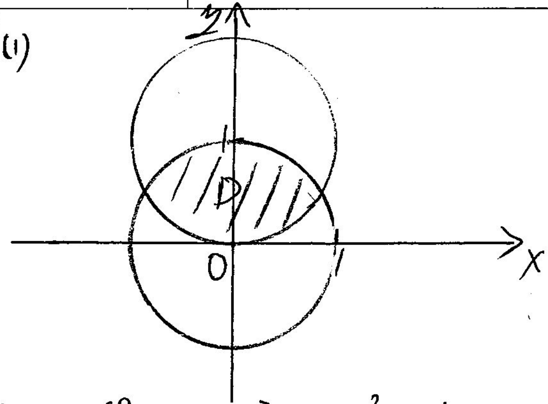
$$V_{上} = |\vec{V_{上}}| = \pi + \frac{1}{6} \quad V_{下} = |\vec{V_{下}}| = \frac{1}{6}$$

10.

$$V = \iint_{D_{xy}} z\,dxdy = \int_{-1}^{1} dx \int_{x^2}^{1} (x^2+y^2)\, dy$$

$$= \int_{-1}^{1} \left(x^2 + \frac{1}{3} - x^4 - \frac{1}{3}x^6\right) dx = \frac{88}{105}$$

11 (1)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r^2 \le 1,\ r^2 - 2r\sin\theta \le 0$$

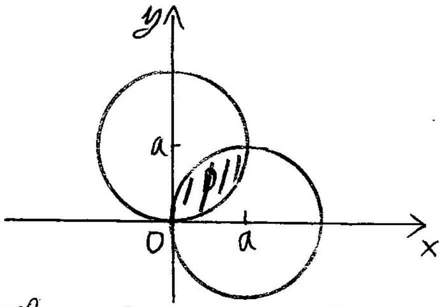
$$\Downarrow$$

$$0 \le r \le 1,\quad \sin\theta \ge \frac{r}{2}$$

$$\iint_D f(x,y)\,dxdy = \iint_{D_{r\theta}} f(x,y)\, r\,drd\theta =$$

$$\int_0^1 dr \int_{\arcsin(\frac{r}{2})}^{\pi - \arcsin(\frac{r}{2})} f(x,y)\, r\, d\theta$$

(2)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{matrix} r^2 - 2ar\sin\theta \le 0,\ r^2 - 2ar\cos\theta \le 0 \\ r \le 2a\sin\theta,\ r \le 2a\cos\theta \end{matrix}$$

$$\iint_D f(x,y)\,dxdy = \iint_{D_{r\theta}} f(x,y)\, r\,drd\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\min(2a\sin\theta,\ 2a\cos\theta)} f(x,y)\, r\,dr$$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^{2a\sin\theta} f(x,y)\, r\,dr \ +$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} f(x,y)\, r\,dr$$

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12(1) 变量代换:

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \quad 2\cos\theta \le r \le 4\cos\theta$$

$$\iint_D (x^2+y^2)dxdy = \iint_{Dr\theta} r^2 \cdot r\, drd\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{2\cos\theta}^{4\cos\theta} r^3 dr$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cdot 60\cos^4\theta$$

$$= \frac{45}{2}\pi$$

(2)

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \quad r^2\le 1 \ \& \ r^2 \le 2r\cos\theta$$

即 $0\le r\le 1 \ \& \ r\le 2\cos\theta$

$$\iint_D \sqrt{(x^2+y^2)^3}\, dxdy = \iint_{Dr\theta} r^3\cdot r\, drd\theta$$

$$= 2\int_0^{\frac{\pi}{3}} d\theta \int_0^1 r^4 dr + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^4 dr$$

$$= \frac{2}{15}\pi + \frac{64}{5}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^5\theta\, d\theta$$

$$= \frac{2}{15}\pi + \frac{512}{75} - \frac{98\sqrt{3}}{25}$$

(3)

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \quad x^2+y^2=x+y,$$

即 $r=\cos\theta+\sin\theta$

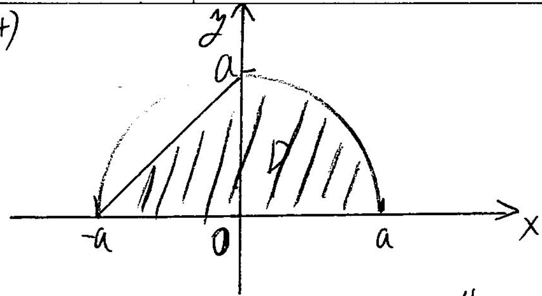
$$\iint_D (x+y)dxdy = \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \int_0^{\sin\theta+\cos\theta} r^2(\sin\theta+\cos\theta)dr$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \cdot \frac{1}{3}(\sin\theta+\cos\theta)^4$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{1}{3}\left(1+2\sin 2\theta + \frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{\pi}{2}$$

(4)



$$\iint_{D_左} (y-x)^2 dxdy = \int_{-a}^{0} dx \int_0^{x+a} dy \cdot (y-x)^2 = \frac{a^4}{4}$$

~~$= \int_{-a}^{0} dx \cdot \frac{1}{3}x^3 = $~~

$$\iint_{D_右} (y-x)^2 dxdy = \int_0^a dr \int_0^{\frac{\pi}{2}} d\theta \cdot (r\cos\theta - r\sin\theta)^2 \cdot r$$

$$= \int_0^a dr \cdot \int_0^{\frac{\pi}{2}} r^3(1-\sin 2\theta)d\theta$$

$$= \int_0^a r^3\left(\frac{\pi}{2}-1\right)dr$$

$$= \frac{1}{8}(\pi-2)a^4$$

$$\iint_D (y-x)^2 dxdy = \frac{1}{8}\pi a^4$$

(5)

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$$\iint_D \arctan\frac{y}{x}\, dxdy$$

$$= \int_0^1 dr \int_{\pi}^{\frac{3}{2}\pi} r\cdot \arctan(\tan\theta)d\theta$$

$$= \int_0^1 dr \int_{\pi}^{\frac{3}{2}\pi} r\cdot(\theta-\pi)d\theta$$

$$= \int_0^1 \frac{1}{8}\pi^2 r\, dr$$

$$= \frac{\pi^2}{16}$$

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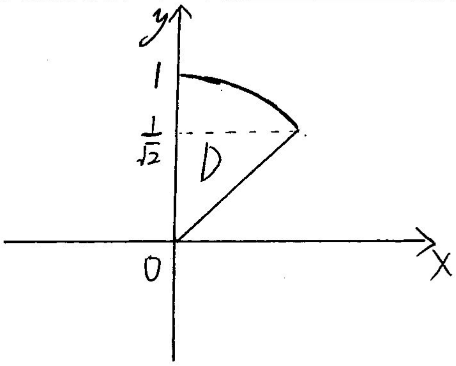
12(6)



原式 $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^1 dr \cdot e^{-r^2} \cdot r$

$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(e^{-1}-1)\, d\theta$

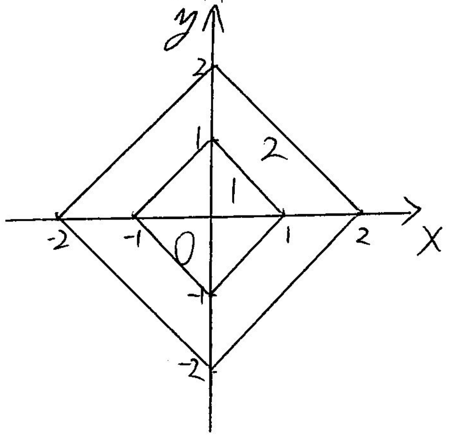
$= -\frac{\pi}{8}\left(\frac{1}{e}-1\right)$

$= \frac{\pi}{8}\left(1-\frac{1}{e}\right)$

(7)

$\iint\limits_D f(x,y)\,dxdy = 1\times(\sqrt{2})^2 + 2\times\left[(\sqrt{8})^2-(\sqrt{2})^2\right]$

$= 14$



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13. (1) 解：$D_{\rho\theta}=\{(\rho,\theta)\mid a\le\rho\le a\sqrt{2\cos 2\theta},\ \theta\in[-\frac{\pi}{6},\frac{\pi}{6}]\cup[\frac{5\pi}{6},\frac{7}{6}\pi]\}$.

$\therefore$ ~~面积~~ $\iint_D d\sigma=\iint_{D_{\rho\theta}}\rho\, d\theta d\rho=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta\int_a^{a\sqrt{2\cos2\theta}}\rho d\rho+\int_{\frac{5}{6}\pi}^{\frac{7}{6}\pi}d\theta\int_a^{a\sqrt{2\cos2\theta}}\rho d\rho$

$=(\sqrt{3}-\frac{\pi}{3})a^2$.

(2) 解：$D_{\rho\theta}=\{(\rho,\theta)\mid 0\le\rho\le\sqrt{3}a\sin\theta,\ \rho\le a(1+\cos\theta),\ 0\le\theta\le\pi\}$.

$=\{(\rho,\theta)\mid 0\le\rho\le\sqrt{3}a\sin\theta,\ \theta\in[0,\frac{\pi}{3}]\ \cancel{\cup[\pi,\frac{7\pi}{6}]}\}\cup$

$\{(\rho,\theta)\mid 0\le\rho\le a(1+\cos\theta),\ \theta\in[\frac{\pi}{3},\pi]\}$.

$\therefore$ 面积 $\iint_D d\sigma=\int_0^{\frac{\pi}{3}}d\theta\int_0^{\sqrt{3}a\sin\theta}\rho d\rho+\int_{\frac{\pi}{3}}^{\pi}d\theta\int_0^{a(1+\cos\theta)}\rho d\rho$

$=\frac{3}{2}a^2\int_0^{\frac{\pi}{3}}\sin^2\theta\, d\theta+\frac{1}{2}a^2\int_{\frac{\pi}{3}}^{\pi}(1+\cos\theta)^2 d\theta$

$=\frac{3}{2}a^2\int_0^{\frac{\pi}{3}}\frac{1-\cos2\theta}{2}d\theta+\frac{1}{2}a^2\int_{\frac{\pi}{3}}^{\pi}(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos2\theta)d\theta$

$=\frac{3\pi-3\sqrt{3}}{4}a^2$.

14. (1) 解：$D=\{(x,y)\mid 2\le xy\le4,\ x\le y\le 3x,\ x\ge0,\ y\ge0\}$.

作代换 $\begin{cases}u=xy\\ v=\frac{y}{x}\end{cases}$ 则 $D^*=\{(u,v)\mid 2\le u\le4,\ 1\le v\le3\}$.

$\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}y & x\\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix}=\frac{2y}{x}=2v\ne0$. $\therefore\frac{D(x,y)}{D(u,v)}=\frac{1}{2v}$.

$\therefore\iint_D x^2y^2dxdy=\int_2^4du\int_1^3\frac{u^2}{2v}dv=\frac{\ln3}{2}\int_2^4u^2du=\frac{28}{3}\ln3$.

(2) 解：$D=\{(x,y)\mid 1\le xy\le2,\ 1\le x^2-y^2\le2,\ x,y\ge0\}$.

作代换 $\begin{cases}u=xy\\ v=x^2-y^2\end{cases}$ $\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}y & x\\ 2x & -2y\end{vmatrix}=-2(x^2+y^2)\ne0$. $\frac{D(x,y)}{D(u,v)}=\frac{-1}{2(x^2+y^2)}$

$\therefore\iint_D(x^2+y^2)dxdy=\int_1^2du\int_1^2(x^2+y^2)\frac{+1}{2(x^2+y^2)}dv=\frac{1}{2}$.

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(3) $\iint\limits_D (x^2+y^2)dxdy$, $D=\{(x,y)\mid |x|+|y|\le 1\}$;

解: $D=\{(x,y)\mid -1\le x+y\le 1,\ -1\le x-y\le 1\}$. 作 $\begin{cases}u=x+y\\ v=x-y\end{cases}$ $\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}1&1\\1&-1\end{vmatrix}=-2$

$\therefore \frac{D(x,y)}{D(u,v)}=-\frac{1}{2}$. $\therefore \iint\limits_D (x^2+y^2)dxdy=\int_{-1}^{1}du\int_{-1}^{1}\frac{u^2+v^2}{2}\cdot\frac{1}{2}dv=\frac{2}{3}$.

(4) 解. $D=\{(x,y)\mid \frac{1}{3}\le y\le 2,\ 2-2y\le y^2-x\le 1+y\}$. 作 $\begin{cases}u=y^2-x\\ v=y\end{cases}$

$\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}-1&2y\\0&1\end{vmatrix}=-1$. $\therefore \frac{D(x,y)}{D(u,v)}=-1$.

$\iint\limits_D (x-y^2)dxdy=\iint\limits_{D^*}(-u)\cdot 1\,dudv=\int_{\frac{1}{3}}^{2}dv\int_{2-2v}^{1+v}(-u)du=-\frac{175}{54}$.

15. 求面积.

(1) 解: $\begin{cases}u=a_1x+b_1y+c_1\\ v=a_2x+b_2y+c_2.\end{cases}$ $\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}=a_1b_2-a_2b_1\neq 0$.

$\therefore \frac{D(x,y)}{D(u,v)}=\frac{1}{a_1b_2-a_2b_1}$ $\therefore$ 面积 $\iint\limits_D d\sigma=\iint\limits_{u^2+v^2\le 1}\frac{1}{|a_1b_2-a_2b_1|}dudv=\frac{\pi}{|a_1b_2-a_2b_1|}$.

(2) 解: $\begin{cases}u=\sqrt{x}\\ v=\sqrt{y}\end{cases}$. $\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}\frac{1}{2\sqrt{x}}&0\\0&\frac{1}{2\sqrt{y}}\end{vmatrix}=\frac{1}{4\sqrt{xy}}=\frac{1}{4uv}\neq 0$. $\frac{D(x,y)}{D(u,v)}=4uv$.

$\therefore \iint\limits_D d\sigma=\iint\limits_{\substack{u+v\le\sqrt{a}\\ u,v\ge 0}}4uv\,dudv=\frac{1}{6}a^2$.

16. 证明: (1) 作代换 $\begin{cases}u=x+y\\ v=x-y\end{cases}$ (注意到 $D=\{(x,y)\mid -1\le x+y\le 1,\ -1\le x-y\le 1\}$).

$\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}1&1\\1&-1\end{vmatrix}=-2$ $\therefore \frac{D(x,y)}{D(u,v)}=-\frac{1}{2}$ $\therefore \iint\limits_{|x|+|y|\le 1}f(x+y)dxdy=\int_{-1}^{1}du\int_{-1}^{1}\frac{1}{2}f(u)dv$

$=\int_{-1}^{1}f(u)du=\int_{-1}^{1}f(t)dt$. 得证.

(2) $D=\{(x,y)\mid 1\le xy\le 2,\ 1\le\frac{y}{x}\le 4,\ x>0,y>0\}$. 作 $\begin{cases}u=xy\\ v=\frac{y}{x}.\end{cases}$

$\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}y&x\\-\frac{y}{x^2}&\frac{1}{x}\end{vmatrix}=\frac{2y}{x}=2v\neq 0$. $\therefore \frac{D(x,y)}{D(u,v)}=\frac{1}{2v}$.

$\therefore \iint\limits_D f(xy)dxdy=\iint\limits_{\substack{1\le u\le 2\\ 1\le v\le 4}}f(u)\frac{1}{2v}dudv=\int_1^2 f(u)du\int_1^4\frac{1}{2v}dv=\ln 2\int_1^2 f(u)du$

$=\ln 2\int_1^2 f(t)dt$ 得证.

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有点意思… ←

17. 解：由对称性，$\iint\limits_{x^2+y^2\le R^2} \frac{f(x)}{f(x)+f(y)}dxdy = \iint\limits_{x^2+y^2\le R^2} \frac{f(y)}{f(x)+f(y)}dxdy = A$.

$\therefore 2A = \iint\limits_{x^2+y^2\le R^2} \frac{f(x)+f(y)}{f(x)+f(y)}dxdy = \iint\limits_{x^2+y^2\le R^2} dxdy = \pi R^2$，$A = \frac{1}{2}\pi R^2$.

$\therefore$ 原式 $= aA + bA = \frac{a+b}{2}\pi R^2$.

18. 解：$F(x) = \int_0^x dt \int_{t^2}^{x^2} f(t,s)ds$. 记 $g(x,t) = \int_{t^2}^{x^2} f(t,s)ds$. 则 $F(x) = \int_0^x g(x,t)dt$.

$F(x+\Delta x) - F(x) = \int_0^{x+\Delta x} g(x+\Delta x, t)dt - \int_0^x g(x,t)dt$

$= \int_0^x [g(x+\Delta x,t) - g(x,t)]dt + \int_x^{x+\Delta x} g(x+\Delta x, t)dt$.

$\therefore F'(x) = \lim\limits_{\Delta x\to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x} \overset{\lim\limits_{\Delta x\to 0}}{=} \left[\int_0^x \frac{g(x+\Delta x,t)-g(x,t)}{\Delta x}dt + \frac{\int_x^{x+\Delta x} g(x+\Delta x,t)dt}{\Delta x}\right]$

$\overset{\lim\limits_{\Delta x\to 0}}{=} \left[\int_0^x g_x'(x,t)dt + g(x+\Delta x, x)\right] = \lim\limits_{\Delta x\to 0}\left[\int_0^x 2xf(t,x^2)dt + g(x+\Delta x, x)\right]$

$= \int_0^x 2xf(t,x^2)dt$.