微积分A(2)期中复习

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0/ 学期初的建议

- 重视作业，一定认真完成作业，切实理解方法

标准. 不会做的题,听完讲解后,自己能够独立做出来.

- 建议多预习、自学，赶在大课进度前面

本学期所学内容：

- 多元函数微分学（比较容易）
- 含参数积分（难、抽象）

（以上两项：期中）

- 多重积分和曲线曲面积分（理论简单但难于计算）
- 常数项级数（中等），函数项级数（难），幂级数（容易）

（以上两项：期末）

1 / 多元连续函数、偏导数、全微分

1. 多元函数在一点处的极限

Def. $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m, x_0\in\mathbb{R}^n, \mathrm{A}\in\mathbb{R}^m$, f在x_0的某个去心邻域$B_0(x_0,r)$中有定义. 若$\forall\varepsilon>0, \exists\delta\in(0,r)$, s.t.

$$\|f(x)-\mathrm{A}\|<\varepsilon,\quad \forall x\in B_0(x_0,\delta),$$

则称$x\to x_0$时, $f(x)$以A为极限, 记作$\lim\limits_{x\to x_0} f(x)=\mathrm{A}$.

Remark. $\lim\limits_{x\to x_0} f(x)=\mathrm{A}$, 则:

不论动点x沿什么路径趋于定点x_0, 都有$f(x)\to\mathrm{A}$.

Question. 如何证明 $\lim\limits_{x\to x_0} f(x)$ 不存在？

例. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x+y}$ 是否存在？

解: $\lim\limits_{\substack{x\to 0\\ y=0}} \dfrac{x}{x+y} = \lim\limits_{x\to 0} \dfrac{x}{x} = \lim\limits_{x\to 0} 1 = 1$

$$\lim\limits_{\substack{y\to 0\\ x=0}} \frac{x}{x+y} = \lim\limits_{y\to 0} 0 = 0.$$

故 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x+y}$ 不存在. □

Question. 如何证明 $\lim_{x\to x_0} f(x)$不存在？

例. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$是否存在？

解：$\lim_{\substack{x\to 0\\ y=0}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=0}} \frac{0}{x^2} = 0$

$$\lim_{\substack{x\to 0\\ y=x}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=x}} \frac{x^2}{2x^2} = \lim_{\substack{x\to 0\\ y=x}} \frac{1}{2} = \frac{1}{2}$$

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$不存在！

Question. 如何证明 $\lim\limits_{x\to x_0} f(x)$不存在?

例. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x+y}$是否存在?

解: $\lim\limits_{\substack{x\to 0\\ y=0}} \dfrac{xy}{x+y} = 0,$

$$\lim_{\substack{x\to 0\\ y=x^2-x}} \frac{xy}{x+y} = \lim_{x\to 0} \frac{x^3-x^2}{x^2} = -1.$$

故 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x+y}$不存在. □

2. 多元函数极限的性质:四则运算、夹挤原理、复合极限定理

Thm. $f,g:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m, x_0\in\mathbb{R}^n$,若 $\lim\limits_{x\to x_0} f(x)$与 $\lim\limits_{x\to x_0} g(x)$ 都存在, 则

(1) $\lim\limits_{x\to x_0}\left(f(x)\pm g(x)\right)=\lim\limits_{x\to x_0} f(x)\pm\lim\limits_{x\to x_0} g(x)$;

(2) $m=1$时, $\lim\limits_{x\to x_0} f(x)g(x)=\lim\limits_{x\to x_0} f(x)\cdot\lim\limits_{x\to x_0} g(x)$;

(3) $m=1$且 $\lim\limits_{x\to x_0} g(x)\neq 0$时, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$.

1 / 多元连续函数、偏导数、全微分

2. 多元函数极限的性质:四则运算、夹挤原理、复合极限定理

Thm. (夹挤原理) $f,g,h:B_0(x_0,\delta)\subset\mathbb{R}^n\to\mathbb{R}$,若

$$f(x)\le g(x)\le h(x),\forall x\in B_0(x_0,\delta),$$

$$\lim_{x\to x_0} f(x)=\lim_{x\to x_0} h(x)=\text{A},$$

则

$$\lim_{x\to x_0} g(x)=\text{A}.$$

思路. 均值不等式是常用技巧:$|xy|\le\dfrac{x^2+y^2}{2}$

2. 多元函数极限的性质:四则运算、夹挤原理、复合极限定理

例(夹挤).

(1) $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^3+y^3}{x^2+y^2}=$__________;(2) $\lim\limits_{(x,y)\to(0,0)}x\sin\dfrac{1}{y}+y\cos\dfrac{1}{x}=$__________;

解.(1) $\dfrac{x^3+y^3}{x^2+y^2}=\dfrac{(x+y)(x^2-xy+y^2)}{x^2+y^2}$ $\because |x^2-xy+y^2|\le x^2+y^2+|xy|\le\dfrac{3}{2}(x^2+y^2)$

$\therefore \left|\dfrac{(x+y)(x^2-xy+y^2)}{x^2+y^2}\right|\le\dfrac{3}{2}|x+y|$ $\therefore \lim\limits_{(x,y)\to(0,0)}\dfrac{x^3+y^3}{x^2+y^2}=0$

(2) $\left|x\sin\dfrac{1}{y}+y\cos\dfrac{1}{x}\right|\le\left|x\sin\dfrac{1}{y}\right|+\left|y\cos\dfrac{1}{x}\right|\le|x|+|y|$

$\therefore \lim\limits_{(x,y)\to(0,0)}x\sin\dfrac{1}{y}+y\cos\dfrac{1}{x}=0$

2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

例(复合极限定理允许了结合一元函数的一些极限).

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{1+x^2+y^2}-1}{\sin(x^2+y^2)} = \underline{\qquad\qquad}$$

解. 视$x^2+y^2=r$

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{1+x^2+y^2}-1}{\sin(x^2+y^2)} = \lim_{r\to0+} \frac{\sqrt[3]{1+r}-1}{\sin r} = \lim_{r\to0+} \frac{\frac{1}{3}r}{\sin r} = \frac{1}{3}$$

2. 多元函数极限的性质:四则运算、夹挤原理、复合极限定理

练习 $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy-\sin(xy)}{xy-xy\cos(xy)}=$ __________

解. 视$xy=r$

$$\lim_{(x,y)\to(0,0)}\frac{xy-\sin(xy)}{xy-xy\cos(xy)}=\lim_{r\to0}\frac{r-\sin(r)}{r-r\cos(r)}=\lim_{r\to0}\frac{r-\sin r}{r(1-\cos r)}$$

$$=\lim_{r\to0}\frac{r-\sin r}{\frac{1}{2}r^3}=2\lim_{r\to0}\frac{r-\sin r}{r^3}=2\lim_{r\to0}\frac{1-\cos r}{3r^2}=1/3$$

3. 累次极限和二重极限

Def.(累次极限) $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)=\lim\limits_{y\to y_0}\left(\lim\limits_{x\to x_0} f(x,y)\right)$

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right)$$

Remark. 任意固定$y\neq y_0$, 若 $\lim\limits_{x\to x_0} f(x,y)$存在, 记为

$$g(y)=\lim_{x\to x_0} f(x,y).$$

若 $\lim\limits_{y\to y_0} g(y)=A$, 则 $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)=\lim\limits_{y\to y_0} g(y)=A.$

3. 累次极限和二重极限

Remark. 求算 $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$时候,先计算 $\lim\limits_{x\to x_0} f(x,y)$，此时把$y$看做常数,

显然这次极限计算后x被消掉,之后再令$y\to y_0$.

例. 求算 $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$时候,先计算 $\lim\limits_{x\to x_0} f(x,y)$，此时把$y$看做常数,

显然这次极限计算后x被消掉,之后再令$y\to y_0$.

1 / 多元连续函数、偏导数、全微分

例. (2020春) $D=\{(x,y)\,|\,x+y\neq 0\}$, $f(x,y)=\dfrac{x-y}{x+y}$

问: $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$, $\lim_{(x,y)\to(0,0)} f(x,y)$是否存在

解: $\lim_{x\to 0}\lim_{y\to 0}\dfrac{x-y}{x+y}=\lim_{x\to 0}\dfrac{x}{x}=\lim_{x\to 0}1=1$

$\lim_{y\to 0}\lim_{x\to 0}\dfrac{x-y}{x+y}=\lim_{y\to 0}\dfrac{-y}{y}=\lim_{y\to 0}-1=-1$

$\lim_{(x,y)\to(0,0)} f(x,y)$不存在

选择路径$y=2x$, $f(x,2x)=\dfrac{x-2x}{x+2x}=-\dfrac{1}{3}$

选择路径$y=3x$, $f(x,3x)=\dfrac{x-3x}{x+3x}=-\dfrac{1}{2}$

$\therefore$ 极限不存在

在(x_0,y_0)连续 $\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$

4. 向量值函数的连续

Def. 设$f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m, x_0\in\Omega$,若$\lim\limits_{x\to x_0} f(x)=f(x_0)$,也即

$\forall\varepsilon>0,\exists\delta>0, s.t.$

$$\|f(x)-f(x_0)\|<\varepsilon,\quad \forall x\in\Omega\bigcap B(x_0,\delta),$$

则称f在点x_0处连续, 称f的不连续点为间断点.

Def. 设$f:\Omega\subset\mathbb{R}^n\to\mathbb{R}$,若$f$在$\Omega$上点点连续,则称$f$在$\Omega$上连续,记作$f\in C(\Omega)$.

Remark. $f=(f_1,f_1,\ldots,f_m):\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$,则

$$f\text{在点}x_0\text{连续}\Leftrightarrow f_i\text{在点}x_0\text{连续}, i=1,2,\cdots m.$$

例：讨论 $f(x,y)=\begin{cases}\dfrac{x^2y^2}{\left(x^2+y^2\right)^{3/2}} & (x,y)\neq(0,0)\\ 0 & 其它情形\end{cases}$ 的连续性.

解：只需研究 $\lim_{(x,y)\to(0,0)}\dfrac{x^2y^2}{\left(x^2+y^2\right)^{3/2}}$ 是否为0.

$$0\le\frac{x^2y^2}{\left(x^2+y^2\right)^{3/2}}=\frac{(xy)^2}{\left(x^2+y^2\right)^{3/2}}\le\frac{\left(\dfrac{x^2+y^2}{2}\right)^2}{\left(x^2+y^2\right)^{3/2}}=\frac{1}{4}\sqrt{x^2+y^2}$$

$$\therefore\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{\left(x^2+y^2\right)^{3/2}}=0$$

例：讨论 $f(x,y)=\begin{cases}1 & y=x^2, x>0\\ 0 & \text{其它情形}\end{cases}$ 的连续性.

解：f 在开区域 $\{(x,y)\,|\,x\neq\sqrt{y}\}$ 中为初等函数，故处处连续. 而 f 在曲线 $x=\sqrt{y}$ 上每一点都不连续.

事实上，任取 (x_0,y_0)，$x_0=\sqrt{y_0}$，当点列 $\{P_k(x_k,y_k)\}$ 沿曲线 $x=\sqrt{y}$ 趋于 (x_0,y_0) 时，$f(x_k,y_k)\to 1$；当点列 $\{P_k\}$ 沿直线 $x=x_0$ 趋于 (x_0,y_0) 时，$f(x_k,y_k)\to 0$. □

Thm.(介值定理) 设$\Omega\subset\mathbb{R}^n$为连通区域, $f\in C(\Omega), x_1, x_2\in\Omega$, $f(x_1)=\lambda\le\mu=f(x_2)$, 则$\forall\sigma\in[\lambda,\mu], \exists x\in\Omega, s.t. f(x)=\sigma$.

Thm.(最值定理) 设$\Omega\subset\mathbb{R}^n$为有界闭集, $f\in C(\Omega)$, 则f在Ω上存在最大值M和最小值m, 即$\exists\xi,\eta\in\Omega, s.t.\forall x\in\Omega$, 都有

$$m=f(\xi)\le f(x)\le f(\eta)=M.$$

例(P24−T8): $\lim_{x^2+y^2\to+\infty} f(x,y)=+\infty \Rightarrow f(x,y)$有最小值

证明: $\lim_{x^2+y^2\to+\infty} f(x,y)=+\infty \Rightarrow$

$\forall M>0, \exists R>0, s.t. \forall (x,y)$, 满足$x^2+y^2\ge R^2$, $f(x,y)\ge M$

$x^2+y^2\le R^2$是有界闭集, 故$f(x,y)$在$x^2+y^2\le R^2$有最小值

取$M=f(0,0)$,

$\exists R>0, s.t. \forall (x,y)$, 满足$x^2+y^2\ge R^2$, $f(x,y)\ge f(0,0)$

$f(x,y)$在$x^2+y^2\le R^2$有最小值

$$f(x_0,y_0)=\min_{x^2+y^2\le R^2} f(x,y)\le f(0,0)\le f(x,y),\quad \forall x^2+y^2\ge R^2$$

5. 偏导数

Def. $u=f(\mathrm{x})=f(x_1,x_2,\cdots,x_n)$在$\mathrm{x}_0=(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})\in\mathbb{R}^n$的某个邻域中有定义, 若极限

$$\lim_{\Delta x_i\to 0}\frac{\Delta_{x_i}u}{\Delta x_i}=\lim_{\Delta x_i\to 0}\frac{f(x_0^{(1)},\cdots,x_0^{(i-1)},x_0^{(i)}+\Delta x_i,x_0^{(i+1)},\cdots,x_0^{(n)})-f(\mathrm{x}_0)}{\Delta x_i}$$

存在, 则称之为$f(\mathrm{x})$在x_0关于x_i的偏导数, 记作$\frac{\partial f}{\partial x_i}(\mathrm{x}_0)$, $\frac{\partial u}{\partial x_i}(\mathrm{x}_0)$, $\left.\frac{\partial f}{\partial x_i}\right|_{\mathrm{x}_0}$, $\left.\frac{\partial u}{\partial x_i}\right|_{\mathrm{x}_0}$, $u'_{x_i}(\mathrm{x}_0)$或$f'_{x_i}(\mathrm{x}_0)$.

5. 偏导数

$$f_x'(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}.$$

Remark: 1) 对某个变量求偏导数时, 视其余变量为常数, 按一元函数求导法则和公式去求.

2)求分段函数的偏导函数时, 用定义求分界点处的偏导数, 用1) 中方法求其它点处的偏导数. 一般地, 分段函数的偏导函数仍为分段函数.

3)求某一点的偏导数时, 可以先带入其他变量的值, 使之完全退化为一元函数, 再求导

例. $f(x,y)=x^2e^y+(x-1)\arctan\dfrac{y}{x}$, 求$f_x'(1,0)$.

解法一: $f(x,0)=x^2$, 所以$f_x'(1,0)=2$.

解法二:

$$f_x'(x,y)=2xe^y+\arctan\frac{y}{x}+(x-1)\cdot\frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$

$$=2xe^y+\arctan\frac{y}{x}+\frac{y(1-x)}{x^2+y^2}.$$

所以$f_x'(1,0)=2$.□

Remark: 求具体点处的偏导数时, 第一种方法较好.

5. 偏导数

4)偏导数仅仅说明了沿着坐标轴方向,函数是光滑的, 因此和连续性互不蕴含

例： $f(x,y)=\begin{cases}1 & y=x^2, x>0\\ 0 & \text{其它情形}\end{cases}$ 在(0, 0)处不连续, 俩偏导数都为0

偏导数的局限性:只看坐标轴方向, 不全面

——引出方向导数、可微两个概念

5. 偏导数

$(x+1)\sin y+\sin x$

例. $z=f(x,y)$偏导数存在,$\dfrac{\partial z}{\partial x}=\sin y+\cos x,\ f(0,y)=\sin y$,求$f(x,y)=$________.

$\dfrac{\partial z}{\partial x}$的得出:视$y$为常数,对$x$求导 $\therefore f(x,y)=\int \sin y+\cos x dx$

$\therefore f(x,y)=x\sin y+\sin x+g(y)$

$\because g(y)=f(0,y)\ =\sin y$

$\therefore f(x,y)=(x+1)\sin y+\sin x$

6. 可微

$$f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)=\frac{\partial f}{\partial x_1}(\mathrm{x}_0)\Delta x_1+\frac{\partial f}{\partial x_2}(\mathrm{x}_0)\Delta x_2+\cdots+\frac{\partial f}{\partial x_n}(\mathrm{x}_0)\Delta x_n+o(\|\Delta\mathrm{x}\|)$$

$$\Leftrightarrow \lim_{\Delta\mathrm{x}\to O}\frac{f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)-\left(\frac{\partial f}{\partial x_1}(\mathrm{x}_0)\Delta x_1+\frac{\partial f}{\partial x_2}(\mathrm{x}_0)\Delta x_2+\cdots+\frac{\partial f}{\partial x_n}(\mathrm{x}_0)\Delta x_n\right)}{\|\Delta\mathrm{x}\|}=0$$

二元函数特殊情况

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)-f_x'(x_0,y_0)(x-x_0)-f_y'(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

可微一定连续, 偏导数也一定存在.

7. 总结(二元函数版本的连续可偏导可微)

连续：$f(x_0,y_0)=\lim\limits_{(x,y)\to(0,0)} f(x,y)$

可偏导：$f_x'(x_0,y_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$,

$f_y'(x_0,y_0)=\lim\limits_{\Delta y\to 0}\dfrac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$.

可微 $\Leftrightarrow$

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)-f_x'(x_0,y_0)(x-x_0)-f_y'(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

例. $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ 在原点的可微性

解：.Step1. 计算偏导数 $f(x,0)=\begin{cases}x^2\sin\dfrac{1}{x^2}, & x\neq 0\\ 0, & x=0\end{cases}$,

$$f_x'(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}x\sin\frac{1}{x^2}=0;$$

同理$f_y'(0,0)=0.$

Step2. 考察 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=0$是否成立

本题中 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)}{\sqrt{x^2+y^2}}=0$是否成立?

例. $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ 在原点的可微性

解: . $\lim\limits_{(x,y)\to(0,0)}\dfrac{(x^2+y^2)\sin\dfrac{1}{x^2+y^2}}{\sqrt{x^2+y^2}}=0$ $\therefore$ 可微

Hint. 分段函数分析可微性:

(1)用定义计算偏导数;(不是用求导法则)

(2)用定义验证可微:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)-f_x'(x_0,y_0)(x-x_0)-f_y'(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

例. P42-2(4)

$f(x,y)=|x-y|\varphi(x,y)$, $\varphi(x,y)$在(0,0)的邻域内连续, $\varphi(0,0)=0$

问: $f(x,y)=|x-y|\varphi(x,y)$是否可微

解. P42-2(4)

$$\left|\frac{|x|\varphi(x,0)}{x}\right|=|\varphi(x,0)|$$

$$x\to 0, |\varphi(x,0)|\to|\varphi(0,0)|=0$$

Step1. 计算偏导数

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)}=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{|x|\varphi(x,0)}{x}=0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)}=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{x\to 0}\frac{|y|\varphi(0,y)}{y}=0$$

Step2. 考察 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=0$是否成立

例. P42-2(4)

$f(x,y)=|x-y|\varphi(x,y)$, $\varphi(x,y)$在$(0,0)$的邻域内连续, $\varphi(0,0)=0$

问: $f(x,y)=|x-y|\varphi(x,y)$是否可微

解. P42-2(4)

Step2. 考察 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=0$是否成立

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}$$

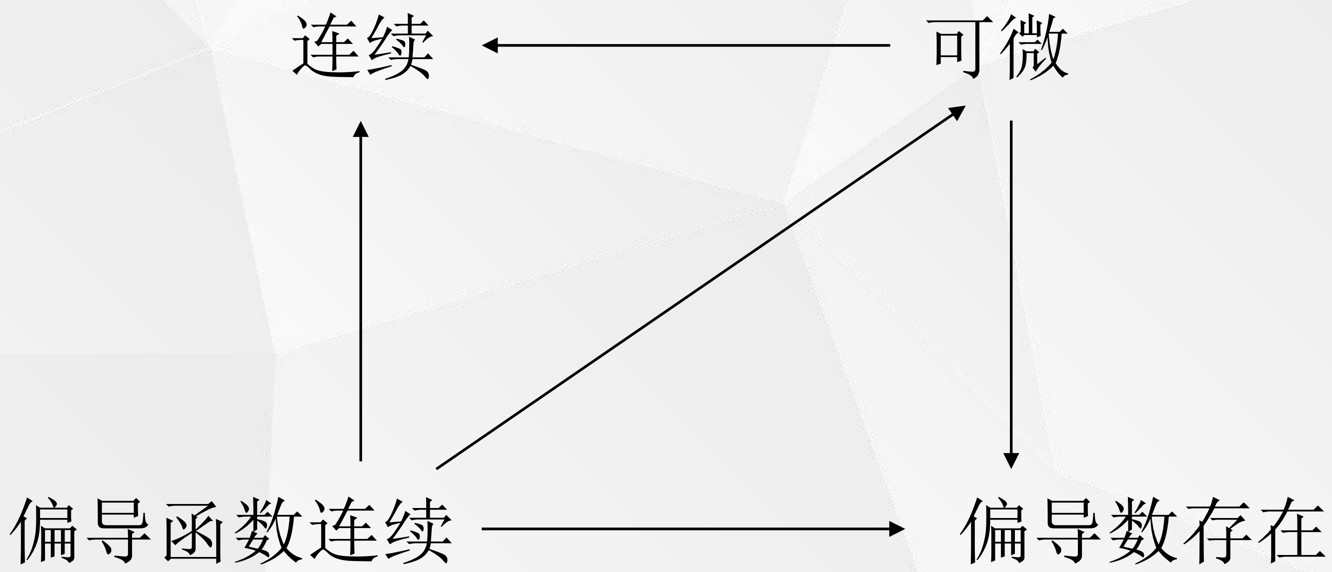
$$=\lim_{(x,y)\to(0,0)}\frac{|x-y|\varphi(x,y)}{\sqrt{x^2+y^2}}=0$$

$$\left|\frac{|x-y|\varphi(x,y)}{\sqrt{x^2+y^2}}\right|\le|\varphi(x,y)|\frac{|x|+|y|}{\sqrt{x^2+y^2}}$$

$$\le 2|\varphi(x,y)|$$

$2|\varphi(x,y)|\to 0$, 当$x,y\to(0,0)$

Remark: 函数的连续性、可微性、偏导数存在性与偏导数连续性之间的蕴含关系图.



例: $f(A)=A^2, A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, f在$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$处的微分$df=$ ________

$$f(A)=A^2=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & cb+d^2 \end{pmatrix},$$

$$Jf(A)=\begin{pmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{pmatrix}=\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad df=\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} da \\ db \\ dc \\ dd \end{pmatrix}$$

Def. f在$\mathrm{x}_0 \in \mathbb{R}^n$的邻域中有定义, $\vec{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ 为非零向量, l为过x_0沿$\vec{v}$方向的射线,若t的函数

$$g(t) = f(\mathrm{x}_0 + \frac{\vec{v}}{\|\vec{v}\|} t) = f(\mathrm{x}_0^{(1)} + \frac{v_1}{\|\vec{v}\|} t, \cdots, \mathrm{x}_0^{(n)} + \frac{v_n}{\|\vec{v}\|} t)$$

在$t = 0$存在右导数,即极限

$$\lim_{\substack{\mathrm{x} \to \mathrm{x}_0 \\ \mathrm{x} \in l}} \frac{f(\mathrm{x}) - f(\mathrm{x}_0)}{\|\mathrm{x} - \mathrm{x}_0\|} = \lim_{t \to 0^+} \frac{g(t) - g(0)}{t}$$

存在,则称该极限为$f(\mathrm{x})$在x_0沿方向$\vec{v}$的方向导数, 记作

$$\frac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}, \left.\frac{\partial f}{\partial \vec{v}}\right|_{\mathrm{x}_0} \text{ 或} f'_{\vec{v}}(\mathrm{x}_0).$$

Remark. $\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}$是函数$f(\mathrm{x})$在点$\mathrm{x}_0$沿方向$\vec{v}$的变化率.

Remark. $\dfrac{\partial f(\mathrm{x}_0)}{\partial x_i}$为$f$在$\mathrm{x}_0$沿$e_i=(0,\cdots 0,\underset{\text{第}i\text{个分量}}{\overset{\downarrow}{1}},0,\cdots 0)$的方向导数.

Thm. 设f在$\mathrm{x}_0\in\mathbb{R}^n$可微, $\vec{v}=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$为非零向量,

则方向导数$\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}$存在, 且

$$\frac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}=\frac{\partial f(\mathrm{x}_0)}{\partial x_1}\frac{v_1}{\|\vec{v}\|}+\frac{\partial f(\mathrm{x}_0)}{\partial x_2}\frac{v_2}{\|\vec{v}\|}+\cdots+\frac{\partial f(\mathrm{x}_0)}{\partial x_n}\frac{v_n}{\|\vec{v}\|}.$$

例. (1)计算$f(x,y)=\sin(x+2y)$在$(0,0)$处，沿着$I=(1,1)$方向的方向导数;

(2)求出方向导数最大的方向（单位化为单位向量）

解. (1) $\frac{\partial f}{\partial x}(x,y)=\cos(x+2y), \frac{\partial f}{\partial y}(x,y)=2\cos(x+2y) \therefore \frac{\partial f}{\partial x}(0,0)=1, \frac{\partial f}{\partial y}(0,0)=2$

$$\therefore \frac{\partial f}{\partial I}(0,0)=1\times\frac{1}{\sqrt{2}}+2\times\frac{1}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

(2)求出方向导数最大的方向

设这一方向为$I=(\cos\theta,\sin\theta)$

$\therefore \frac{\partial f}{\partial I}(0,0)=1\times\cos\theta+2\times\sin\theta$，由柯西-施瓦茨不等式，$\frac{\partial f}{\partial I}(0,0)\le\sqrt{5}$，

当$(\cos\theta,\sin\theta)=(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})$

1／多元连续函数、偏导数、全微分

例.（2020春模拟）$f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$

(1) $f(x,y)$在$(0,0)$处的连续性？ ;(2) $f(x,y)$在$(0,0)$处两个一阶偏导数的存在性?;

(3) $f(x,y)$在$(0,0)$处是否可微？

解:(1) $|x^3+y^3|=|x+y||x^2-xy+y^2|\leq|x^2+|xy|+y^2||x+y|\leq\frac{3}{2}|x^2+y^2||x+y|$

$\therefore\left|\dfrac{x^3+y^3}{x^2+y^2}\right|\leq\dfrac{3}{2}|x+y|$ $\therefore\lim_{(x,y)\to(0,0)}\dfrac{x^3+y^3}{x^2+y^2}=0=f(0,0)$ $\therefore$ 连续

(2) $f'_x(0,0)=\lim_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\dfrac{x-0}{x}=1$ $\quad f'_y(0,0)=1$

(3)不可微. $f(x,y)-f(0,0)-xf'(0,0)-yf'(0,0)=\dfrac{x^3+y^3}{x^2+y^2}-x-y=-\dfrac{xy(x+y)}{x^2+y^2}$

考虑极限$\lim_{(x,y)\to(0,0)}\dfrac{-\dfrac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}}$是否存在,并且是否为0

1 / 多元连续函数、偏导数、全微分

例. (2020春模拟) $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$

(1) $f(x,y)$在$(0,0)$处的连续性？ ;(2) $f(x,y)$在$(0,0)$处两个一阶偏导数的存在性?;

(3) $f(x,y)$在$(0,0)$处是否可微？

(3)不可微. 考虑极限 $\lim\limits_{(x,y)\to(0,0)}\dfrac{-\dfrac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}}$ 是否存在,并且是否为0

取$y=x$, $\dfrac{-\dfrac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}}=\dfrac{-\dfrac{2x^3}{2x^2}}{\sqrt{2x^2}}=\dfrac{-x}{\sqrt{2}\,|x|}$ $\quad\lim\limits_{x\to0+}\dfrac{-x}{\sqrt{2}\,|x|}=-\dfrac{1}{\sqrt{2}}$

2)求分段函数的偏导函数时,用定义求**分界点**处的偏导数,用1)中方法求其它点处的偏导数. 一般地,分段函数的偏导函数仍为分段函数.

2 / 链锁法则和隐函数定理

- Chain Rule

$u=g(x):\Omega\subset\mathbb{R}^n\to\mathbb{R}^m, y=f(u):g(\Omega)\subset\mathbb{R}^m\to\mathbb{R}^k,$

$g(x)$在$x_0\in\Omega$可微, $f(u)$在$u_0=g(x_0)$可微, 则

$$J(f\circ g)\big|_{x_0}=J(f)\big|_{u_0}\cdot J(g)\big|_{x_0},$$

即
$$\left.\frac{\partial(y_1,y_2,\cdots,y_k)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0}=\left.\frac{\partial(y_1,y_2,\cdots,y_k)}{\partial(u_1,u_2,\cdots,u_m)}\right|_{u_0}\cdot\left.\frac{\partial(u_1,u_2,\cdots,u_m)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0},$$

简记为
$$\left.\frac{\partial y}{\partial x}\right|_{x_0}=\left.\frac{\partial y}{\partial u}\right|_{u_0}\cdot\left.\frac{\partial u}{\partial x}\right|_{x_0}.$$

$k=1$时, $\dfrac{\partial y}{\partial x_i}=\dfrac{\partial y}{\partial u_1}\dfrac{\partial u_1}{\partial x_i}+\dfrac{\partial y}{\partial u_2}\dfrac{\partial u_2}{\partial x_i}+\cdots+\dfrac{\partial y}{\partial u_m}\dfrac{\partial u_m}{\partial x_i}, i=1,2,\cdots,n.$

2 链锁法则和隐函数定理

例. (2020期末) $f \in C^2, z = f(x^2+xy+y^2)$, 计算$z'_y, z''_{xy}$在(1,1)的值.

解. $z'_y = f'(x^2+xy+y^2)(2y+x) \quad \therefore z'_y(1,1) = 3f'(3)$

$\because z'_y(x,1) = f'(x^2+x+1)(2+x)$

$\therefore z''_{yx}(x,1) = (z'_y(x,1))' = f''(x^2+x+1)(2+x)^2 + f'(x^2+x+1)$

$\therefore z''_{yx}(1,1) = 9f''(3) + f'(3)$

2/ 链锁法则和隐函数定理

例. $z = f(xy, x^2 + y^2)$, 计算z'_x, z'_y

解. $z'_x = f'_1(xy, x^2 + y^2)y + f'_2(xy, x^2 + y^2)2x \quad z'_y = f'_1(xy, x^2 + y^2)x + f'_2(xy, x^2 + y^2)2y$

例. $u = u(x, y, z)$, u在全空间可微, u满足

$$u(tx, ty, tz) = t^k u(x, y, z), \forall t, x, y, z, \text{其中} k > 0$$

证明: $ku(x, y, z) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

证. $\because u(tx, ty, tz) = t^k u(x, y, z), \forall t, x, y, z$. 等式两边对$t$求导

$\because xu'_1(tx, ty, tz) + yu'_2(tx, ty, tz) + zu'_3(tx, ty, tz) = kt^{k-1}u(x, y, z)$

取$t = 1$, 得$ku(x, y, z) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

2/ 链锁法则和隐函数定理

例. $u=u(x,y,z)$,u在全空间可微, u满足

$$ku(x,y,z)=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$$

证明:$u(tx,ty,tz)=t^k u(x,y,z)$,, $\forall t,x,y,z$, 其中$k>0$

证. 构造辅助函数$F(t)=u(tx,ty,tz)-t^k u(x,y,z)$

$$F'(t)=xu_1'(tx,ty,tz)+yu_2'(tx,ty,tz)+zu_3'(tx,ty,tz)-kt^{k-1}u(x,y,z)$$

$$=\frac{1}{t}(txu_1'(tx,ty,tz)+tyu_2'(tx,ty,tz)+tzu_3'(tx,ty,tz)-kt^k u(x,y,z))$$

$$=\frac{1}{t}(ku(tx,ty,tz)-kt^k u(x,y,z))=\frac{k}{t}(u(tx,ty,tz)-t^k u(x,y,z))=\frac{k}{t}F(t)$$

$$\therefore F'(t)=\frac{k}{t}F(t)\Rightarrow F(t)=Ct^k \because F(1)=u(x,y,z)-u(x,y,z)=0 \quad \therefore C=0 \therefore F(t)=0$$

Thm. $F(x,y):\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$在$(x_0,y_0)$的邻域W中有定义,且满足 (1)$F(x_0,y_0)=0$,

(2)$F\in C^q(W)$,即F的各分量函数在W中q阶连续可微,

(3) $\dfrac{\partial F}{\partial y}(x_0,y_0)$可逆,

则存在x_0的某个邻域$U\in\mathbb{R}^n$,以及定义在U上的向量值函数$y=y(x)$,满足

(1)$y(x_0)=y_0, F(x,y(x))=0, \forall x\in U$;

(2)$y(x)$在U上q阶连续可微;

(3) $\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$.

求$\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y}$时 x,y 相互独立!

Remark: $F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m,(x,y)\mapsto F(x,y)$,若$\dfrac{\partial F}{\partial y}$可逆,

则$F(x,y)=0$确定隐"函数"$y=y(x)$,求$\dfrac{\partial y}{\partial x}$有两种方法:

- 套用定理:$\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$.

这里求Jaccobi矩阵时 x,y 相互独立!

- 将$F(x,y)=0$中y视为$y=y(x)$,利用复合映射的链式法则,方程组 $F(x,y(x))=0$两边对x求Jaccobi矩阵.

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

例. (2020模拟)二阶连续可微函数$z=z(x,y)$满足：$x^3+y^3+z^3=x+y+z$

计算$\dfrac{\partial^2 z}{\partial x\partial y}$

解. $\because x^3+y^3+z^3(x,y)=x+y+z(x,y)$

$\therefore$ 对x偏导，$3x^2+3z^2\dfrac{\partial z}{\partial x}=1+\dfrac{\partial z}{\partial x}$ $\qquad \therefore \dfrac{\partial z}{\partial x}=\dfrac{1-3x^2}{3z^2-1}, \dfrac{\partial z}{\partial y}=\dfrac{1-3y^2}{3z^2-1}$

再对y偏导，$6z\dfrac{\partial z}{\partial y}\dfrac{\partial z}{\partial x}+3z^2\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial x\partial y}$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{6z}{1-3z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}=\frac{6z}{(1-3z^2)^3}(1-3x^2)(1-3y^2)$$

例. (2020真题)$u(t)\in C^2(\mathbb{R}), z=u(\sqrt{x^2+y^2})$满足:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=x^2+y^2(x^2+y^2>0)$$

证明:$u(t)$满足$u''+\frac{1}{t}u'=t^2$

证. $\because z=u(\sqrt{x^2+y^2})$ $\therefore z'_x=\frac{x}{\sqrt{x^2+y^2}}u'(\sqrt{x^2+y^2}), z'_y=\frac{y}{\sqrt{x^2+y^2}}u'(\sqrt{x^2+y^2})$

$$\therefore z''_{xx}=\frac{y^2}{(x^2+y^2)^{3/2}}u'(\sqrt{x^2+y^2})+\frac{x^2}{(x^2+y^2)}u''(\sqrt{x^2+y^2})$$

$$z''_{yy}=\frac{x^2}{(x^2+y^2)^{3/2}}u'(\sqrt{x^2+y^2})++\frac{y^2}{(x^2+y^2)}u''(\sqrt{x^2+y^2})$$

$\therefore z''_{xx}+z''_{yy}=\frac{1}{\sqrt{x^2+y^2}}u'(\sqrt{x^2+y^2})+u''(\sqrt{x^2+y^2})=x^2+y^2$ 令$t=\sqrt{x^2+y^2}$即可.

2 / 链锁法则和隐函数定理

例. (2020期末) $y=y(x), z=z(x)$由方程组$\begin{cases} x^3+y^3-z^3=10 \\ x+y+z=0 \end{cases}$在$(1,1,-2)$处确定隐函数

求$y=y(x), z=z(x)$在$x=1$处的导数

解. 在方程组$\begin{cases} x^3+y^3-z^3=10 \\ x+y+z=0 \end{cases}$两边对$x$求导 得$\begin{cases} 3x^2+3y^2y'-3z^2z'=0 \\ 1+y'+z'=0 \end{cases}$

按照$x=1, y=1, z=-2$带入得$\begin{cases} 3+3y'-12z'=0 \\ 1+y'+z'=0 \end{cases}$ $\therefore \begin{cases} y'(1)=-1 \\ z'(1)=0 \end{cases}$

例. $f\in C^2(\mathbb{R}^2)$, $f''_{xx}+f''_{yy}=0$, 令$p=\dfrac{x}{x^2+y^2}$, $q=\dfrac{y}{x^2+y^2}$

求证: $u(x,y)=f\left(\dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2}\right)$也满足$u''_{xx}+u''_{yy}=0$.

分析: 直接思路是强算, 但是可以预见到这种计算过于复杂

证明: $\because u(x,y)=f(p(x,y),q(x,y))$. $\therefore u'_x=f'_1p'_x+f'_2q'_x$, $u'_y=f'_1p'_y+f'_2q'_y$

$$\therefore u''_{xx}=\left(f_{11}''p'_x+f_{12}''q'_x\right)p'_x+f'_1p''_{xx}+\left(f_{21}''p'_x+f_{22}''q'_x\right)q'_x+f'_2q''_{xx}$$

$$=f_{11}''p'^2_x+2f_{12}''q'_xp'_x+f_{22}''q'^2_x+f'_1p''_{xx}+f'_2q''_{xx}$$

$$\therefore u''_{yy}=f_{11}''p'^2_y+2f_{12}''q'_yp'_y+f_{22}''q'^2_y+f'_1p''_{yy}+f'_2q''_{yy}$$

$$\therefore u''_{xx}+u''_{yy}=f_{11}''\left(p'^2_y+p'^2_x\right)+2f_{12}''\left(q'_xp'_x+q'_yp'_y\right)+f_{22}''\left(q'^2_y+q'^2_x\right)$$

$$+f'_1\left(p''_{yy}+p''_{xx}\right)+f'_2\left(q''_{yy}+q''_{xx}\right)$$

令$p=\dfrac{x}{x^2+y^2}, q=\dfrac{y}{x^2+y^2}$ 则$p'_x=\dfrac{y^2-x^2}{\left(x^2+y^2\right)^2}, q'_y=\dfrac{x^2-y^2}{\left(x^2+y^2\right)^2}$

则$p'_y=\dfrac{-2xy}{\left(x^2+y^2\right)^2}, q'_x=\dfrac{-2xy}{\left(x^2+y^2\right)^2}$ $\therefore p'_y=q'_x,\ \ p'_x=-q'_y$

$$\therefore p''_{yy}=q''_{xy},\ \ p''_{yx}=q''_{xx},\ \ p''_{xy}=-q''_{yy},\ \ p''_{xx}=-q''_{xy}$$

$$\therefore u''_{xx}+u''_{yy}=f_{11}''\left(p_y'^2+p_x'^2\right)+2f_{12}''\left(q'_xp'_x+q'_yp'_y\right)+f_{22}''\left(q_y'^2+q_x'^2\right)$$
$$+f_1'\left(p''_{yy}+p''_{xx}\right)+f_2'\left(q''_{yy}+q''_{xx}\right)$$

$$=f_{11}''\left(q_x'^2+q_y'^2\right)+f_{22}''\left(q_y'^2+q_x'^2\right)=\left(f_{11}''+f_{22}''\right)\left(q_x'^2+q_y'^2\right)=0.\square$$

例. (2020期中-类似)$f\in C^2(\mathbb{R}^2),\ f>0,\ f''_{xy}f=f'_xf'_y$,

求证:存在一元函数$u(x),v(y),s.t.\ f(x,y)=u(x)v(y)$

分析:

$$\ln f(x,y)=\ln u(x)+\ln v(y)\Leftrightarrow\frac{\partial\ln f(x,y)}{\partial x}=\frac{u'(x)}{u(x)}\Leftrightarrow\frac{\partial^2\ln f(x,y)}{\partial x\partial y}=0$$

证明:

$$\frac{\partial\ln f(x,y)}{\partial x}=\frac{1}{f}\frac{\partial f}{\partial x},$$

$$\frac{\partial^2\ln f(x,y)}{\partial x\partial y}=\frac{\partial(\frac{f'_x}{f})}{\partial y}=\frac{f''_{xy}f-f'_yf'_x}{f^2}=0$$

例. $f \in C^1(\mathbb{R}^2), a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=0$,选取合适的变量替换$\begin{cases}u=x+py\\ v=x+qy\end{cases}$, p,q为常数,

将原方程化为$\frac{\partial f}{\partial u}=0$,从而解为$f=g(x+qy)$

解. $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v} \qquad \frac{\partial f}{\partial y}=p\frac{\partial f}{\partial u}+q\frac{\partial f}{\partial v}$

$$a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=(a+bp)\frac{\partial f}{\partial u}+(a+bq)\frac{\partial f}{\partial v}$$

$$\therefore a+bp=1, a+bq=0 \Rightarrow p=\frac{1-a}{b}, q=-\frac{a}{b} \quad \therefore f=g(x-\frac{a}{b}y)$$

例. (2020模拟)$f\in C^2(\mathbb{R}^2)$,满足(1) $f'_x=f'_y$,(2) $f(x,0)>0$;

证明:$f(x,y)>0$.

分析.$f'_x-f'_y=0\Rightarrow$

$u=f(x+h,y-h),u'_h=f'_x(x+h,y-h)-f'_y(x+h,y-h)=0$

$\therefore f(x,y)=f(x+y,0)>0$

3 曲线的切线、曲面的切平面

1.空间曲线的表示

(1)参数方程表示: $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

切线的方向向量. $\mathrm{r}'(t) = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \mathrm{r}}{\Delta t} = \lim\limits_{\Delta t \to 0} \left(\dfrac{\Delta x}{\Delta t}, \dfrac{\Delta y}{\Delta t}, \dfrac{\Delta z}{\Delta t} \right) = \left(x'(t), y'(t), z'(t) \right).$

切线方程.

$$\begin{cases} x = x_0 + x'(t_0)t \\ y = y_0 + y'(t_0)t \\ z = z_0 + z'(t_0)t \end{cases}$$

法平面方程.

$$x'(t_0)(x - x_0) + y'(t_0)(y - y_0) + z'(t_0)(z - z_0) = 0$$

3/ 曲线的切线、曲面的切平面

1.空间曲线的表示

例. 求曲线$\begin{cases} x = e^t \\ y = t \\ z = t^2 \end{cases}$在(1,0,0)处的切线方程

$\because t = 0$,方向向量为$\left(e^t, 1, 2t\right)$,$\therefore$ 带入数字得到$(1,1,0)$

$\therefore$ 切线方程为$\begin{cases} x = 1 + t \\ y = t \\ z = 0 \end{cases}$

3 曲线的切线、曲面的切平面

2.空间曲面的表示

(1)一般方程表示：$F(x,y,z)=0$

曲面的法向量. $\nabla F=\left(F_x',F_y',F_z'\right).$

切平面方程.

$$F_x'(x_0,y_0,z_0)\left(x-x_0\right)+F_y'(x_0,y_0,z_0)\left(y-y_0\right)+F_z'(x_0,y_0,z_0)\left(z-z_0\right)=0$$

法线方程.

$$\begin{cases} x=x_0+F_x'(x_0,y_0,z_0)t \\ y=y_0+F_y'(x_0,y_0,z_0)t \\ z=z_0+F_z'(x_0,y_0,z_0)t \end{cases}$$

3 曲线的切线、曲面的切平面

2.空间曲面的表示

例. 求曲面$3x^2+2y^2-2z-1=0$在$(1,1,2)$处的法向量和切平面方程

第一步. 计算梯度向量为$(6x,4y,-2)$

代入数字, 得到$(6,4,-2)$ $\leftarrow$ 法向量

$6(x-1)+4(y-1)-2(z-2)=0$ $\leftarrow$ 切平面

6. 设曲面 $z=x^2-y^2$ 在点 $(1,0,1)$ 的切平面方程为 $z=f(x,y)$，则 $f(2,1)=$__________.

3/ 曲线的切线、曲面的切平面

1.空间曲线的表示

(2)两个曲面的交线: $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

切线的方向向量.

$$\nabla F\times\nabla G=\begin{vmatrix} i & j & k \\ F_x'(x_0,y_0,z_0) & F_y'(x_0,y_0,z_0) & F_z'(x_0,y_0,z_0) \\ G_x'(x_0,y_0,z_0) & G_y'(x_0,y_0,z_0) & G_z'(x_0,y_0,z_0) \end{vmatrix}$$

切线方程和法平面方程怎么写?

3 曲线的切线、曲面的切平面

1.空间曲线的表示

4. 曲线 $\begin{cases} x^2+y^2=2, \\ x^2+z^2=2 \end{cases}$ 在点 (1,1,1) 的法平面方程是

(A) $x-y-z=-1$

(B) $y-x-z=-1$

(C) $z-x-y=-1$

(D) $x-1=1-y=1-z$

$$\nabla F\times\nabla G=\begin{vmatrix} i & j & k \\ 2x & 2y & 0 \\ 2x & 0 & 2z \end{vmatrix}=\begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix}=(4,-4,-4) \qquad 4(x-1)-4(y-1)-4(z-1)=0$$

2. 参数方程下曲面的切平面

设曲面S的参数方程为$\mathbf{r}=\mathbf{r}(u,v)$，即

$$S:\begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases}$$

•S在r_0的法向量： $\vec{n}=\left(\mathbf{r}'_u\times\mathbf{r}'_v\right)\Big|_{(u_0,v_0)}$.

其中$r_0=r\left(u_0,v_0\right)$

例:求球面$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$ $\begin{pmatrix} 0 \le \varphi \le \pi \\ 0 \le \theta < 2\pi \end{pmatrix}$在$\varphi = \pi/6$,

$\theta = \pi/3$的切平面和法向量.

解: $r'_\theta = (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0)$

$r'_\varphi = (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi)$.

当$\varphi = \pi/6, \theta = \pi/3$时,

$$(x, y, z) = (a/4, \sqrt{3}a/4, \sqrt{3}a/2),$$

$$r'_\varphi = (\sqrt{3}a/4, 3a/4, -a/2),$$

$$r'_\theta = (-\sqrt{3}a/4, a/4, 0).$$

$$\vec{n} // \left(r'_\varphi \times r'_\theta\right) = \det\begin{pmatrix} i & j & k \\ \sqrt{3}a/4 & 3a/4 & -a/2 \\ -\sqrt{3}a/4 & a/4 & 0 \end{pmatrix}$$

$$\vec{n} // \left(1/8, \sqrt{3}/8, \sqrt{3}/4\right).$$

切平面方程为

$$(x - a/4, y - \sqrt{3}a/4, z - \sqrt{3}a/2) \cdot \vec{n} = 0,$$

即

$$x + \sqrt{3}y + 2\sqrt{3}z - 4a = 0 .\square$$

3/ 多元泰勒公式和极值原理

Thm. 设n元函数f在$B(x_0,\delta)$中二阶连续可微，则

$\forall x_0+\Delta x\in B(x_0,\delta),\exists\theta\in(0,1),s.t.$

$$f(x_0+\Delta x)=f(x_0)+J_f(x_0)\Delta x$$
$$+\frac{1}{2}(\Delta x)^T H_f(x_0+\theta\Delta x)\Delta x$$

(称为带*Lagrange*余项的一阶*Taylor*公式)，且

$$f(x_0+\Delta x)=f(x_0)+J_f(x_0)\Delta x$$
$$+\frac{1}{2}(\Delta x)^T H_f(x_0)\Delta x+o(\|\Delta x\|^2),\Delta x\to 0\text{时}$$

(称为带*Peano*余项的二阶*Taylor*公式).

Thm. 设函数$f(x,y)$在区域D中$n+1$阶连续可微, $M_0(x_0,y_0)\in D, M(x,y)\in D$, 且线段$\overline{M_0M}$完全包含在$D$中. 记

$$h=x-x_0, k=y-y_0,$$

记算子

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m \triangleq \sum_{i=0}^{m} C_m^i h^i k^{m-i}\frac{\partial^m}{\partial x^i \partial y^{m-i}},$$

则f在点(x_0,y_0)有

(1)带$Lagrange$余项的n阶$Taylor$公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)$$

$$(0<\theta<1)$$

(2)带$Peano$余项的$n+1$阶$Taylor$公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1}f(x_0,y_0)$$

$$+o\left(\left(\sqrt{h^2+k^2}\right)^{n+1}\right).$$

3/ 多元泰勒公式和极值原理

Note.一般来说, 我们不用如此复杂的公式, 而是设法化为一元函数的泰勒公式

例. $\cos(x^2+y^2)$在(0,0)的8阶带Peano余项的Taylor展开式.

解: $\cos t = 1-\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots+(-1)^n\frac{t^{2n}}{(2n)!}+o(t^{2n}),\ t\to 0$时.

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\cdots+(-1)^n\frac{(x^2+y^2)^{2n}}{(2n)!}$$

$$+o((x^2+y^2)^{2n}),\ x^2+y^2\to 0\text{时}.$$

令$n=2$得

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\frac{(x^2+y^2)^4}{4!}+o((x^2+y^2)^4),$$

$$x^2+y^2\to 0\text{时}.\square$$

例：求$f(x,y)=x^y$在点(1,1)的邻域内带Peano余项的3阶Taylor公式

$$f(x,y)=x^y=(x-1+1)^y \qquad \rho=\sqrt{(x-1)^2+(y-1)^2}$$

解：$$(x-1+1)^y=1+y(x-1)+\frac{y(y-1)}{2!}(x-1)^2+\frac{y(y-1)(y-2)}{3!}(x-1)^3+o(\rho^3)$$

$$y(x-1)=(y-1)(x-1)+(x-1)$$

$$\frac{y(y-1)}{2!}(x-1)^2=\frac{1}{2}(x-1)^2(y-1)+\frac{1}{2}(x-1)^2(y-1)^2$$

$$\frac{1}{6}y(y-1)(y-2)(x-1)^3=\frac{1}{6}(y-1)(y^2-2y)(x-1)^3=\frac{1}{6}(y-1)^3(x-1)^3-\frac{1}{6}(y-1)(x-1)^3$$

$$\therefore 原式=1+(x-1)+(y-1)(x-1)+\frac{(y-1)}{2!}(x-1)^2+o(\rho^3)$$

例. $\ln(2+x+y+xy)$在$(0,0)$带Peano余项的2阶Taylor展开.

解：$x+y+xy\to 0$ 时,

$$\ln(2+x+y+xy)=\ln 2+\ln(1+\frac{x+y+xy}{2})$$

$$=\ln 2+\frac{x+y+xy}{2}-\frac{1}{2}\left(\frac{x+y+xy}{2}\right)^2+o\left((x+y+xy)^2\right)$$

$x^2+y^2\to 0$时, 必有$x+y+xy\to 0$ 时, 因此

$$\frac{o((x+y+xy)^2)}{x^2+y^2}=\frac{o((x+y+xy)^2)}{(x+y+xy)^2}\cdot\frac{(x+y+xy)^2}{x^2+y^2}\to 0,$$

$$\ln(2+x+y+xy)$$

$$=\ln 2+\frac{x+y}{2}-\frac{x^2+y^2-2xy}{8}+o(x^2+y^2).$$ □

例. $\sin(x+y)+ze^z-ye^x=0$确定了隐函数$z=z(x,y)$, 求$z=z(x,y)$在$(0,0)$带Peano余项的2阶Taylor展开.

解： 计算 $\frac{\partial z}{\partial x}(0,0), \frac{\partial z}{\partial y}(0,0), \frac{\partial^2 z}{\partial x^2}(0,0), \frac{\partial^2 z}{\partial y^2}(0,0), \frac{\partial^2 z}{\partial y\partial x}(0,0)$

在$\sin(x+y)+ze^z-ye^x=0$两边同时对x求偏导, 得

$$\cos(x+y)+\frac{\partial z}{\partial x}e^z+\frac{\partial z}{\partial x}ze^z-ye^x=0$$

在$\sin(x+y)+ze^z-ye^x=0$两边同时对y求偏导, 得

$$\cos(x+y)+\frac{\partial z}{\partial y}e^z+\frac{\partial z}{\partial y}ze^z-e^x=0$$

$x=0, y=0$时, $z=0$

$$\therefore \frac{\partial z}{\partial x}=-1 \qquad \frac{\partial z}{\partial y}=0$$

例. $\sin(x+y)+ze^z-ye^x=0$确定了隐函数$z=z(x,y)$, 求$z=z(x,y)$在$(0,0)$带Peano余项的2阶Taylor展开.

解：计算$\frac{\partial z}{\partial x}(0,0), \frac{\partial z}{\partial y}(0,0), \frac{\partial^2 z}{\partial x^2}(0,0), \frac{\partial^2 z}{\partial y^2}(0,0), \frac{\partial^2 z}{\partial y \partial x}(0,0)$

在$\cos(x+y)+\frac{\partial z}{\partial y}e^z+\frac{\partial z}{\partial y}ze^z-e^x=0$两边再对$x$求偏导, 得

$$-\sin(x+y)+\frac{\partial^2 z}{\partial y \partial x}e^z+\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}e^z+\frac{\partial^2 z}{\partial y \partial x}ze^z+\frac{\partial z}{\partial y}\frac{\partial ze^z}{\partial x}-e^x=0$$

取$x=y=z=0$ $\quad \therefore \frac{\partial^2 z}{\partial y \partial x}=1$

在$\cos(x+y)+\frac{\partial z}{\partial y}e^z+\frac{\partial z}{\partial y}ze^z-e^x=0$两边再对$y$求偏导, 得

$$-\sin(x+y)+\frac{\partial^2 z}{\partial y^2}e^z+(\frac{\partial z}{\partial y})^2e^z+\frac{\partial^2 z}{\partial y^2}ze^z+\frac{\partial z}{\partial y}\frac{\partial ze^z}{\partial y}=0 \qquad \therefore \frac{\partial^2 z}{\partial y^2}=0$$

例. $\sin(x+y)+ze^z-ye^x=0$确定了隐函数$z=z(x,y)$,求$z=z(x,y)$在(0,0)带Peano余项的2阶Taylor展开.

解： 计算$\frac{\partial z}{\partial x}(0,0),\frac{\partial z}{\partial y}(0,0),\frac{\partial^2 z}{\partial x^2}(0,0),\frac{\partial^2 z}{\partial y^2}(0,0),\frac{\partial^2 z}{\partial y\partial x}(0,0)$

在$\cos(x+y)+\frac{\partial z}{\partial x}e^z+\frac{\partial z}{\partial x}ze^z-ye^x=0$两边再对$x$求偏导, 得

$$-\sin(x+y)+\frac{\partial^2 z}{\partial x^2}e^z+(\frac{\partial z}{\partial x})^2e^z+\frac{\partial^2 z}{\partial x^2}ze^z+(\frac{\partial z}{\partial x})^2e^z+(\frac{\partial z}{\partial x})^2ze^z-ye^x=0$$

$$\therefore \frac{\partial^2 z}{\partial x^2}=-2$$

$$\therefore z(x,y)=-x+\frac{1}{2!}(2xy-2x^2)+o(x^2+y^2)=-x+xy-x^2+o(x^2+y^2)$$

例. (2020真题) f 二阶连续可微, 求证: $\lim_{h\to 0+}\dfrac{f(2h,e^{-\frac{1}{2h}})-2f(h,e^{-\frac{1}{h}})+f(0,0)}{h^2}=f''_{xx}(0,0)$

解:

$$\because f\text{二阶连续可微}\ \therefore f(x,y)=f(0,0)+xf'_x(0,0)+yf'_y(0,0)+\frac{1}{2}x^2f''_{xx}(0,0)+\frac{1}{2}y^2f''_{yy}(0,0)$$

$$+xyf''_{xy}(0,0)+o(x^2+y^2)$$

$$f(2h,e^{-\frac{1}{2h}})=f(0,0)+2hf'_x(0,0)+e^{-\frac{1}{2h}}f'_y(0,0)+2h^2f''_{xx}(0,0)+\frac{1}{2}e^{-\frac{1}{h}}f''_{yy}(0,0)+2he^{-\frac{1}{2h}}f''_{xy}(0,0)+o(h^2)$$

$$=f(0,0)+2hf'_x(0,0)+2h^2f''_{xx}(0,0)+o(h^2)$$

$$f(h,e^{-\frac{1}{h}})=f(0,0)+hf'_x(0,0)+e^{-\frac{1}{h}}f'_y(0,0)+\frac{1}{2}h^2f''_{xx}(0,0)+\frac{1}{2}e^{-\frac{2}{h}}f''_{yy}(0,0)+he^{-\frac{1}{h}}f''_{xy}(0,0)+o(h^2)$$

$$=f(0,0)+hf'_x(0,0)+\frac{1}{2}h^2f''_{xx}(0,0)+o(h^2)$$

$e^{-\frac{1}{h}}=o(h^2)$!

$$\therefore f(2h,e^{-\frac{1}{2h}})-2f(h,e^{-\frac{1}{h}})+f(0,0)=h^2f''_{xx}(0,0)+o(h^2)$$

3 多元泰勒公式和极值原理

Thm. n元函数f在x_0的某个邻域中可微, x_0为f的极值点,则x_0为f的驻点, 即$\mathrm{grad} f(x_0)=0$.

Thm. n元函数f在x_0的邻域中二阶连续可微,

$\mathrm{grad} f(x_0)=0$,

(1)若$H_f(x_0)$正定,则$f(x_0)$严格极小.

(2)若$H_f(x_0)$负定,则$f(x_0)$严格极大.

(3)若$H_f(x_0)$不定,则$f(x_0)$不是极值.

3 / 多元泰勒公式和极值原理

Thm. n元函数f在x_0的邻域中二阶连续可微，x_0为极值点，则

(1) $f(x_0)$极小，则$H_f(x_0)$半正定

(2) $f(x_0)$极大，则$H_f(x_0)$半负定

例: 求$z = x^4 + y^4 - 2x^2 + 4xy - 2y^2$的极值.

解: $z'_x = 4x^3 - 4x + 4y$, $z'_y = 4y^3 + 4x - 4y$.

得驻点$(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (0, 0)$.

$$z''_{xx} = 12x^2 - 4, \quad z''_{xy} = 4, \quad z''_{yy} = 12y^2 - 4.$$

(1)在$(\sqrt{2}, -\sqrt{2})$,

$$A = C = 20, B = 4, \ AC - B^2 > 0,$$

取得极小值.

(2)同理$z(x, y)$在$(-\sqrt{2}, \sqrt{2})$取得极小值.

例: 求$z=x^4+y^4-2x^2+4xy-2y^2$的极值.

(3)在$(0,0)$,

$$A=C=-4, B=4, \quad AC-B^2=0,$$

判别法失效. 注意到

$$z(x,x)=2x^4>0, \text{当} x\neq 0 \text{时}.$$

$$z(x,0)=x^4-2x^2$$
$$=x^2(x^2-2)<0, \text{当} 0<x^2<2 \text{时}.$$

故$(0,0)$不是极值点.□

问:以上方法的局限性?

例: $2x^2+2y^2+z^2+8xz-z+8=0$确定隐函数$z=z(x,y)$.求$z(x,y)$的极值.

解: 在$2x^2+2y^2+z^2+8xz-z+8=0$两边分别对x,y求偏导数.

$$4x+2z\frac{\partial z}{\partial x}+8z+8x\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}=0(1)$$

$$4y+2z\frac{\partial z}{\partial y}+8x\frac{\partial z}{\partial y}-\frac{\partial z}{\partial y}=0(2)$$

先计算驻点,即$\frac{\partial z}{\partial x}=0,\frac{\partial z}{\partial y}=0$ $\quad\therefore 4x+8z=0,4y=0$

结合$2x^2+2y^2+z^2+8xz-z+8=0.$ $\quad\therefore -7z^2-z+8=0,\therefore z=1$或$-\frac{8}{7}$

$\therefore$ 两个驻点为$(-2,0)$和$(16/7,0)$

3/ 多元泰勒公式和极值原理

例: $2x^2+2y^2+z^2+8xz-z+8=0$确定隐函数$z=z(x,y)$.求$z(x,y)$的极值.

$\therefore$两个驻点为$(-2,0)$和$(16/7,0)$ 下面计算$(-2,0)$和$(16/7,0)$处的海塞矩阵

对$4x+2z\frac{\partial z}{\partial x}+8z+8x\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}=0$两边对$x,y$求导:

$$4+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}+8\frac{\partial z}{\partial x}+8\frac{\partial z}{\partial x}+8x\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x^2}=0$$

$$2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+2z\frac{\partial^2 z}{\partial x\partial y}+8\frac{\partial z}{\partial y}+8x\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial x\partial y}=0$$

对$4y+2z\frac{\partial z}{\partial y}+8x\frac{\partial z}{\partial y}-\frac{\partial z}{\partial y}=0$两边同时对$y$求导 得$4+2\left(\frac{\partial z}{\partial y}\right)^2+2z\frac{\partial^2 z}{\partial y^2}+8x\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial y^2}=0$

例: $2x^2+2y^2+z^2+8xz-z+8=0$确定隐函数$z=z(x,y)$.求$z(x,y)$的极值.

$\therefore$两个驻点为$(-2,0)$和$(16/7,0)$ 下面计算$(-2,0)$和$(16/7,0)$处的海塞矩阵

在$(-2,0)$处, $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0,\ z=1$

$\frac{\partial^2 z}{\partial x^2}=\frac{4}{15},\frac{\partial^2 z}{\partial x\partial y}=0,\frac{\partial^2 z}{\partial y^2}=\frac{4}{15}$

极小

在$(16/7,0)$处, $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0,\ z=1$

$\frac{\partial^2 z}{\partial x^2}=-\frac{4}{15},\frac{\partial^2 z}{\partial x\partial y}=0,\frac{\partial^2 z}{\partial y^2}=-\frac{4}{15}$

极大

例: f 连续, $\lim_{(x,y)\to(0,0)} \frac{f(x,y)-xy}{(x^2+y^2)^2}=1$. $f(0,0)$是否极值?

解: $\lim_{(x,y)\to(0,0)} \left(f(x,y)-xy\right)=0,\ f(0,0)=0.$

存在$\varepsilon>0$, 当$x^2+y^2<\varepsilon$时,

$$\frac{3}{2}(x^2+y^2)^2 > f(x,y)-xy > \frac{1}{2}(x^2+y^2)^2.$$

于是对充分大的n, $f\left(\frac{1}{n},\frac{1}{n}\right) > \frac{1}{n^2}+\frac{2}{n^4} > 0,$

$$f\left(\frac{1}{n},-\frac{1}{n}\right) < -\frac{1}{n^2}+\frac{6}{n^4} = -\frac{1}{n^2}\left(1-\frac{6}{n^2}\right) < 0.$$

故$f(0,0)$不是极值. □

3 多元泰勒公式和极值原理

Note: 无条件极值问题求解步骤:

(1)计算驻点,即偏导数为0的点;

(2)计算驻点处的*Hessen*矩阵,正定极小, 负定极大, 不定不是极值点

(3)如果(2)失效, 要考虑其他方法.

3/ 多元泰勒公式和极值原理

Ex. (巧妙运用极值原理 – $P94T5$)

(1) $f(x,y)$在$x^2+y^2\le 1$上连续, 在$x^2+y^2<1$内可导,

满足方程$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=kf(x,y)(k>0)$, 若在$x^2+y^2=1$上$f(x,y)=0$,

求证f在$x^2+y^2\le 1$内部恒为0.

证明　反证, 如果f在$x^2+y^2\le 1$内部不恒为0

即存在(x_0,y_0), $s.t.\ f(x_0,y_0)\ne 0$

1° $f(x_0,y_0)>0$, 则$f(x,y)$在$x^2+y^2\le 1$上有大于0的最大值

如果最大值在(x_1,y_1)处取, 有$f(x_1,y_1)>0$, 并且$x_1^2+y_1^2<1$

$\therefore f(x_1,y_1)$为极大值, $\therefore \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$　$\because \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=kf(x,y)$, $\therefore f(x_1,y_1)=0$, 矛盾!

2° $f(x_0,y_0)<0$, 取$-f$带入上面证明即可.□

•条件极值与Lagrange乘子法

$$\max(\min)\ f(\mathrm{x})=f(x_1,\cdots,x_n)$$

$$s.t.\quad g_i(\mathrm{x})=g_i(x_1,\cdots,x_n)=0,\quad i=1,\cdots,m.$$

其中rank $\dfrac{\partial(g_1,\cdots,g_m)}{\partial(x_1,\cdots,x_n)}=m$ (正则性条件).

结论: x_0是条件极值问题的最大(小)值点, 则$\exists\lambda_0$,s.t. (x_0,λ_0)是

$$L(\mathrm{x},\lambda)=L(x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_m)$$

$$=f(x_1,\cdots,x_n)+\sum_{i=1}^{m}\lambda_i g_i(x_1,\cdots,x_n)$$

的驻点.

3/ 多元泰勒公式和极值原理

例. (2020春模拟) 在椭球$x^2+y^2+\frac{z^2}{4}=1$上找一点, 位于$x>0, y>0, z>0$.

使得切平面与三个坐标轴的交点到原点距离的平方和最小

解. 设该点坐标为(a,b,c), 法向量为$(2a,2b,\frac{c}{2})$

切平面为$2a(x-a)+2b(y-b)+\frac{c}{2}(z-c)=0$　即$ax+by+\frac{c}{4}z=1$

解得三个交点坐标为$(\frac{1}{a},0,0),(0,\frac{1}{b},0),(0,0,\frac{4}{c})$

求解如下条件极值问题

$$\min:\frac{1}{a^2}+\frac{1}{b^2}+\frac{16}{c^2}$$

$$s.t. a^2+b^2+\frac{c^2}{4}=1;\quad a>0,b>0,c>0$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{16}{c^2}=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{16}{c^2}\right)\left(a^2+b^2+\frac{c^2}{4}\right)$$

$$\geq\left(\frac{1}{a}a+\frac{1}{b}b+\frac{4}{c}\frac{c}{2}\right)^2=4^2$$

3 / 多元泰勒公式和极值原理

例. (2020春模拟) 在椭球$x^2+y^2+\frac{z^2}{4}=1$上找一点, 位于$x>0, y>0, z>0$.

使得切平面与三个坐标轴的交点到原点距离的平方和最小

$$L(a,b,c,\lambda)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{16}{c^2}+\lambda(a^2+b^2+\frac{c^2}{4}-1)$$

$$L'_a(a,b,c,\lambda)=-\frac{2}{a^3}+2\lambda a=0$$

$$L'_b(a,b,c,\lambda)=-\frac{2}{b^3}+2\lambda b=0$$

$$L'_b(a,b,c,\lambda)=-\frac{32}{c^3}+\frac{\lambda c}{2}=0$$

$$\therefore \lambda=\frac{1}{a^4}=\frac{1}{b^4}=\frac{64}{c^4} \qquad \therefore a=b=\frac{c}{2\sqrt{2}}$$

结合$a^2+b^2+\frac{c^2}{4}=1$

$$\therefore a=1/2, b=1/2, c=\sqrt{2}$$

3/ 多元泰勒公式和极值原理

例. (2020春期末) 求函数$u=\sin x\sin y\sin z$在条件$x+y+z=\frac{\pi}{2}, x>0, y>0, z>0$

下的极值,并说明是极大值还是极小值.

解: (化为无条件极值) 考虑$v(x,y)=\sin x\sin y\sin(\frac{\pi}{2}-x-y)=\sin x\sin y\cos(x+y)$

在$\{(x,y): x>0, y>0, \frac{\pi}{2}-x-y>0\}$上的极值

$$\frac{\partial v(x,y)}{\partial x}=\cos x\sin y\cos(x+y)-\sin x\sin y\sin(x+y)=\sin y(\cos x\cos(x+y)-\sin x\sin(x+y))=$$

$\sin y\cos(2x+y)=0$ $\quad\because y>0, \therefore 2x+y=\frac{\pi}{2}$

$\frac{\partial v(x,y)}{\partial y}=\sin x\cos(x+2y)=0\Rightarrow\ x+2y=\frac{\pi}{2}$ $\quad\therefore x=y=z=\frac{\pi}{6}$ 是唯一驻点. $v(\frac{\pi}{6},\frac{\pi}{6})=\left(\frac{1}{2}\right)^3=\frac{1}{8}$

进一步考虑$D=\{(x,y): x\geq 0, y\geq 0, \frac{\pi}{2}-x-y\geq 0\}$ v在D上有最大值和最小值

D的边界上, $v(x,y)=0$.可知上面所求为最大值.

3 / 多元泰勒公式和极值原理

例. (2020春期末) 求函数$u=\sin x\sin y\sin z$在条件$x+y+z=\frac{\pi}{2}, x>0, y>0, z>0$

下的极值,并说明是极大值还是极小值.

解:(化为无条件极值) 考虑$v(x,y)=\sin x\sin y\sin(\frac{\pi}{2}-x-y)=\sin x\sin y\cos(x+y)$

在$\{(x,y): x>0, y>0, \frac{\pi}{2}-x-y>0\}$上的极值

$\therefore x=y=z=\frac{\pi}{6}$ 是唯一驻点. $\because v'_x=\sin y\cos(2x+y), v'_y=\sin x\cos(x+2y)$

$$\because \begin{cases} v''_{xx}=-2\sin y\sin(2x+y) \\ v''_{xy}=\cos y\cos(x+2y)-\sin y\sin(2x+y)=\cos(2x+2y) \\ v''_{yy}=-2\sin x\sin(x+2y) \end{cases} \quad \because \begin{cases} v''_{xx}=-1 \\ v''_{xy}=\cos(\frac{2\pi}{3})=-\frac{1}{2} \\ v''_{yy}=-1 \end{cases}$$ 海塞矩阵负定.

4/ 含参数积分

- 含参数定积分：$\int_{\alpha}^{\beta} g(t,x)dx$
- 含参数广义积分：$\int_{\alpha}^{+\infty} g(t,x)dx$
- 无论是含参数定积分还是含参数的广义积分，本质上都是关于于参数t的函数
  - 对于一个函数来讲，主要研究其连续性、可导性、可积性
  - 连续性：$\lim\limits_{t\to t_0}\int_{\alpha}^{\beta} g(t,x)dx = \int_{\alpha}^{\beta}\lim\limits_{t\to t_0} g(t,x)dx = \int_{\alpha}^{\beta} g(t_0,x)dx$
  - 可导性：$f'(t) = \frac{d}{dt}\int_{\alpha}^{\beta} g(t,x)dx = \int_{\alpha}^{\beta} g_t'(t,x)dx.$
  - 可积性：$\int_{a}^{b}\left(\int_{\alpha}^{\beta} g(t,x)dx\right)dt = \int_{\alpha}^{\beta}\left(\int_{a}^{b} g(t,x)dt\right)dx$
- 对于含参数定积分，一般只要求被积函数$g(t,x)$及g_t'的连续性即可.
- 对于含参数广义积分，除了含参数定积分的条件外，还需要更强的条件.

4/ 含参数积分

Thm1.(连续性) 设二元函数$g(t,x)$在$D=[a,b]\times[\alpha,\beta]$上连续,则

$f(t)=\int_\alpha^\beta g(t,x)dx$在$[a,b]$上一致连续.

$$\text{也即}\ \lim_{t\to t_0}\int_\alpha^\beta g(t,x)dx=\int_\alpha^\beta \lim_{t\to t_0} g(t,x)dx.$$

Thm2.(在积分号下求导)设$D=[a,b]\times[\alpha,\beta]$,且$g(t,x),g_t'(t,x)\in C(D)$,则

$f(t)=\int_\alpha^\beta g(t,x)dx$在$[a,b]$上连续可导,且

$$f'(t)=\frac{\mathrm{d}}{dt}\int_\alpha^\beta g(t,x)dx=\int_\alpha^\beta g_t'(t,x)dx.$$

例. 求a,b, s.t. $\int_1^3 (ax+b-x^2)^2dx$取最小值

解. $I(a,b)=\int_1^3 (ax+b-x^2)^2dx$

$$\frac{\partial I(a,b)}{\partial a}=\int_1^3 2(ax+b-x^2)xdx=2\int_1^3 ax^2+bx-x^3dx=2(\frac{26}{3}a+4b-20)=0$$

$$\frac{\partial I(a,b)}{\partial b}=\int_1^3 2(ax+b-x^2)dx=2\int_1^3 ax+b-x^2dx=2(4a+2b-\frac{26}{3})=0$$

解方程$\begin{cases}\frac{26}{3}a+4b=20\\ 4a+2b=\frac{26}{3}\end{cases}\Rightarrow\begin{cases}a=4\\ b=-\frac{11}{3}\end{cases}$

例. 计算$\int_0^{\pi/2}\frac{\arctan(a\tan x)}{\tan x}dx$ 解. 记$I(a)=\int_0^{\pi/2}\frac{\arctan(a\tan x)}{\tan x}dx$

$$I'(a)=\int_0^{\pi/2}\left(\frac{\arctan(a\tan x)}{\tan x}\right)'_a dx=\int_0^{\pi/2}\frac{1}{1+a^2\tan^2 x}dx\overset{y=\tan x}{=}\int_0^{+\infty}\frac{1}{(1+y^2)(1+a^2y^2)}dy$$

$$=\frac{1}{1-a^2}\left(\int_0^{+\infty}\frac{1}{(1+y^2)}dy-\int_0^{+\infty}\frac{a^2}{(1+a^2y^2)}dy\right)=\frac{1}{1-a^2}\left(\frac{\pi}{2}-a\int_0^{+\infty}\frac{1}{(1+a^2y^2)}d(ay)\right)$$

$$=\frac{1}{1-a^2}\left(\frac{\pi}{2}-\frac{\pi}{2}a\operatorname{sgn}(a)\right)$$

$a>0$时：$=\frac{1}{1-a^2}\left(\frac{\pi}{2}-\frac{\pi}{2}a\right)=\frac{\pi}{2}\frac{1}{1+a}$ $I(a)=\frac{\pi}{2}\ln(1+a)+C,\ I(0)=C=0$

$\therefore I(a)=\frac{\pi}{2}\ln(1+a)$ $a<0$时：奇函数！

课后习题. 计算(1)$\int_0^{\pi/2}\ln\frac{1+a\cos x}{1-a\cos x}\frac{dx}{\cos x}\,(|a|<1)$;

(2)$\int_0^{\pi/2}\ln\left(a^2\cos^2 x+b^2\sin^2 x\right)dx$;

答案. (1)$\arcsin a$

(2)$\pi\ln\frac{a+b}{2}$

4/ 含参数积分

Thm3. 设$g(t,x), g_t'(t,x) \in C([a,b]\times[c,d]), \alpha(t), \beta(t)$在$[a,b]$上可导,且

$$c \le \alpha(t), \beta(t) \le d, \quad \forall t \in [a,b],$$

则

$$f(t) = \int_{\alpha(t)}^{\beta(t)} g(t,x)dx$$

在区间$[a,b]$上可导,且

$$f'(t) = \frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} g(t,x)dx$$

$$= \int_{\alpha(t)}^{\beta(t)} g_t'(t,x)dx + g(t,\beta(t))\beta'(t) - g(t,\alpha(t))\alpha'(t).$$

例. $f(x)=\int_x^{x^2} e^{-x^2u^2}du,\ f'(x)=$ ______________

解. $$f'(x)=2xe^{-x^6}-e^{-x^4}+\int_x^{x^2} e^{-x^2u^2}\frac{d(-x^2u^2)}{dx}du$$

$$=2xe^{-x^6}-e^{-x^4}-2\int_x^{x^2} e^{-x^2u^2}xu^2du$$

要点. 上限替代被积变量*上限的导数–下限替代被积变量*下限的导数+积分号下求导的部分

例. $f(y)=\int_y^{y^2}\frac{\sin(xy)}{x}dx,\ f'(y)=$ $\underline{\frac{3\sin(y^3)}{y}-\frac{2\sin(y^2)}{y}}$

解. $$f(y)=\int_y^{y^2}\frac{\sin(xy)}{x}dx,\ f'(y)=\frac{\sin(y^3)}{y^2}2y-\frac{\sin(y^2)}{y}+\int_y^{y^2}\cos(xy)dx$$

$$=\frac{2\sin(y^3)}{y}-\frac{\sin(y^2)}{y}+\frac{\sin(y^3)-\sin(y^2)}{y}$$

4. 含参积分的可积性

Thm4. (累次积分交换次序的充分条件)

设$g(t,x)$在$(t,x)\in D=[a,b]\times[\alpha,\beta]$上连续, 则$\int_\alpha^\beta g(t,x)dx$在$t\in[a,b]$上可积, $\int_a^b g(t,x)dt$在$x\in[\alpha,\beta]$上可积, 且

$$\int_a^b\left(\int_\alpha^\beta g(t,x)dx\right)dt=\int_\alpha^\beta\left(\int_a^b g(t,x)dt\right)dx,$$

简记为 $\int_a^b dt\int_\alpha^\beta g(t,x)dx=\int_\alpha^\beta dx\int_a^b g(t,x)dt.$

Proof. 由$g(t,x)$的连续性及Thm1, $\int_\alpha^\beta g(t,x)dx$在$t\in[a,b]$上连续, 从而可积. 同理, $\int_a^b g(t,x)dt$在$x\in[\alpha,\beta]$上可积.

例. 计算$\int_0^1 \frac{x^b-x^a}{\ln x}\sin(\ln\frac{1}{x})dx (a,b>0)$

要点. 交换积分次序,会让难算的积分变得好算.

如果给你的是定积分,需要先变出两重积分号!

解. $\frac{x^b-x^a}{\ln x}=\int_a^b x^y dy$

$$原式=\int_0^1\int_a^b x^y\sin(\ln\frac{1}{x})dydx (a,b>0) \quad =\int_a^b(\int_0^1 x^y\sin(\ln\frac{1}{x})dx)dy$$

$$=\int_a^b\frac{1}{(y+1)^2+1}dy=\arctan(b+1)-\arctan(a+1)$$

$x^y\sin(\ln\frac{1}{x})$在$\{(x,y):a\le y\le b, 0\le x\le 1\}$上连续

注:$\lim_{(x,y)\to(0,y_0)} x^y\sin(\ln\frac{1}{x})=0 \quad \because |x^y\sin(\ln\frac{1}{x})|\le x^y, 0<a\le y_0\le b$

$$\int_0^1 x^y\sin(\ln\frac{1}{x})dx \overset{\ln\frac{1}{x}=t, x=e^{-t}}{=} \int_{+\infty}^0 e^{-ty}\sin(t)d(e^{-t})=\int_0^{+\infty}e^{-t(y+1)}\sin(t)dt=\frac{1}{(y+1)^2+1}$$

4/ 含参数广义积分

Note.对于含参数广义积分而言，需要更强的条件以满足以上定理

Def. $\forall t \in \Omega \subset \mathbb{R}, \int_a^{+\infty} f(t,x)dx$ 收敛. 若$\forall \varepsilon > 0, \exists \mathrm{M}(\varepsilon), s.t.$

$$\left| \int_a^{\mathrm{A}} f(t,x)dx - \int_a^{+\infty} f(t,x)dx \right| < \varepsilon, \quad \forall \mathrm{A} > \mathrm{M}, \forall t \in \Omega,$$

则称含参广义积分$\int_a^{+\infty} f(t,x)dx$ 关于$t \in \Omega$一致收敛.

一致性体现在，一旦ε被指定，

则$\forall t \in \Omega$，$\exists$同一个$M, s.t.$ $\left| \int_a^{\mathrm{A}} f(t,x)dx - \int_a^{+\infty} f(t,x)dx \right| < \varepsilon, \quad \forall \mathrm{A} > \mathrm{M}$

4 / 含参数广义积分

Thm.(Weirstrass判别法) $\forall t\in\Omega\subset\mathbb{R},\int_a^{+\infty}f(t,x)dx$收敛,

若存在$[a,+\infty)$上的广义可积函数$g(x)$, s.t.

$$|f(t,x)|\le g(x),\quad \forall (t,x)\in\Omega\times[a,+\infty),$$

则$\int_a^{+\infty}f(t,x)dx$在$t\in\Omega$上一致收敛.

问:如何证明不一致收敛?

Thm.(Cauchy收敛原理)

$$\int_a^{+\infty}f(t,x)dx \text{ 关于} t\in\Omega\text{一致收敛} \Leftrightarrow \forall\varepsilon>0,\exists \mathrm{M}(\varepsilon), s.t.$$

$$\left|\int_{\mathrm{A}}^{\mathrm{A}'}f(t,x)dx\right|<\varepsilon,\quad \forall \mathrm{A},\mathrm{A}'>\mathrm{M},\forall t\in\Omega.$$

4/ 含参数广义积分

问:如何证明不一致收敛?

Remark.(Cauchy收敛原理逆否)

$$\int_a^{+\infty} f(t,x)dx \text{ 关于} t\in\Omega \text{不一致收敛} \Leftrightarrow \exists\varepsilon_0>0, \forall M, s.t.$$

$$\left|\int_A^{A'} f(t_0,x)dx\right| > \varepsilon_0, \quad \exists A, A' > M, \exists t_0 \in \Omega.$$

例. (1)设$c>0$, $\int_0^{+\infty} e^{-xy}dx$ 在$y\in[c,+\infty)$上是否一致收敛?

(2)$\int_0^{+\infty} e^{-xy}dx$ 在$y\in(0,+\infty)$上是否一致收敛?

解: (1) $c>0$, 则$\int_0^{+\infty} e^{-cx}dx=-\frac{1}{c}e^{-cx}\Big|_{x=0}^{+\infty}=\frac{1}{c}$收敛, 且

$$e^{-xy}\le e^{-cx},\quad \forall(x,y)\in[0,+\infty)\times[c,+\infty).$$

故$\int_0^{+\infty} e^{-xy}dx$ 在$y\in[c,+\infty)$上一致收敛(Weirstrass).

(2) $\exists\varepsilon_0=e^{-1}-e^{-2}$, $\forall \mathrm{M}>0$, $\exists \mathrm{A}=\mathrm{M}+1$, $\mathrm{A}'=2\mathrm{A}$, $y_0=\frac{1}{\mathrm{A}}$, s.t.

$$\left|\int_{\mathrm{A}}^{\mathrm{A}'} e^{-xy_0}dx\right|=-\frac{1}{y_0}e^{-xy_0}\Big|_{x=\mathrm{A}}^{\mathrm{A}'}=\frac{1}{y_0}(e^{-\mathrm{A}y_0}-e^{-\mathrm{A}'y_0})=\mathrm{A}\varepsilon_0>\varepsilon_0,$$

故$\int_0^{+\infty} e^{-xy}dx$ 在$y\in[0,+\infty)$上不一致收敛(Cauchy).□

4/ 含参数广义积分

Thm1. $f(t,x)\in C\left([\alpha,\beta]\times[a,+\infty)\right), I(t)=\int_a^{+\infty} f(t,x)dx$ 关于$t\in[\alpha,\beta]$一致收敛,则$I(t)\in C[\alpha,\beta]$.

Thm1(逆否). $f(t,x)\in C\left([\alpha,\beta]\times[a,+\infty)\right), I(t)\notin C[\alpha,\beta]$.则$I(t)=\int_a^{+\infty} f(t,x)dx$ 关于$t\in[\alpha,\beta]$不一致收敛,则

例. 证明$\int_0^{+\infty}\frac{\sin tx}{x}dx$在$t\in[0,+\infty)$上不一致收敛.

解: $I(t)=\int_0^{+\infty}\frac{\sin tx}{x}dx=\begin{cases}\int_0^{+\infty}\frac{\sin x}{x}dx, & t>0\\ 0, & t=0\end{cases}.$

若$\int_0^{+\infty}\frac{\sin tx}{x}dx$在$t\in[0,+\infty)$上一致收敛,则$I(t)$ $\in C[0,+\infty)$,矛盾.□

Remark. 证明含参积分不一致收敛的方法:

定义、Cauchy准则、含参积分不连续.

Thm2. 设(1) $f(t,x),\ f_t'(t,x)\in C\big([\alpha,\beta]\times[a,+\infty)\big)$;

(2) $\forall t\in[\alpha,\beta], I(t)=\int_a^{+\infty} f(t,x)dx$ 收敛;

(3) $\int_a^{+\infty} f_t'(t,x)dx$ 关于 $t\in[\alpha,\beta]$ 一致收敛;

则 $I(t)\in C^1[\alpha,\beta]$, 且

$$I'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{+\infty} f(t,x)dx=\int_a^{+\infty} f_t'(t,x)dx.$$

注意. 是 $\int_a^{+\infty} f_t'(t,x)dx$ 一致收敛！不是 $\int_a^{+\infty} f(t,x)dx$ 一致收敛

例(2020样题): $I=\int_0^{+\infty}\frac{\arctan bx-\arctan ax}{x}dx \quad (b>a>0).$

解:引入参数 $t\in[a,b]$, 令 $I(t)=\int_0^{+\infty}\frac{\arctan tx-\arctan ax}{x}dx.$

$$\because \left|\frac{1}{1+t^2x^2}\right|\le\frac{1}{1+a^2x^2},\forall t\in[a,b]$$

$$\therefore I'(t)=\int_0^{+\infty}f_t'(t,x)dx=\int_0^{+\infty}\frac{1}{1+t^2x^2}dx=\frac{\pi}{2t}$$

$$\therefore I(a)=0, I(b)=I(a)+\int_a^b I'(t)dt=\int_a^b I'(t)dt=\frac{\pi}{2}\ln(b/a)$$

例.(2019) 计算 $\int_0^{+\infty}\frac{1-e^{-ax}}{xe^x}dx, a\geq 0$

解: 令 $I(a)=\int_0^{+\infty}\frac{1-e^{-ax}}{xe^x}dx$, 考虑积分号下求导.

$$\left(\frac{1-e^{-ax}}{xe^x}\right)_a'=\frac{xe^{-ax}}{xe^x}=e^{-(a+1)x}$$

$\int_0^{\infty}e^{-(a+1)x}dx$ 对 $a\geq 0$ 一致收敛(Weierstrass)

$$I'(a)=\int_0^{\infty}e^{-(a+1)x}dx=\frac{1}{1+a}, \therefore I(a)=\ln(1+a)+C,\quad I(0)=0\Rightarrow C=0,$$

$\therefore I(a)=\ln(1+a)$

例. 计算积分$\int_0^{+\infty} e^{-ax^2}\cos bx dx, a>0$

思想2:通过积分号下求导,

虽然仍旧不好算,但是构造了ODE

解. 视b为参数,定义$I(b)=\int_0^{+\infty} e^{-ax^2}\cos bx dx$,

$\because |xe^{-ax^2}\sin bx| \le xe^{-ax^2}, \forall a>0, \int_0^{+\infty} xe^{-ax^2} dx$存在

$\therefore I'(b) = -\int_0^{+\infty} xe^{-ax^2}\sin bx dx$

$$\therefore I'(b) = -\int_0^{+\infty} xe^{-ax^2}\sin bx dx = -\frac{1}{2}\int_0^{+\infty} e^{-ax^2}\sin bx dx^2 = \frac{1}{2a}\int_0^{+\infty}\sin bx d(e^{-ax^2})$$

$$= \frac{1}{2a}\left(\sin bx e^{-ax^2}\Big|_0^{+\infty} - b\int_0^{+\infty}\cos bx e^{-ax^2} dx\right) = -\frac{b}{2a}\int_0^{+\infty}\cos bx e^{-ax^2} dx = -\frac{b}{2a} I(b)$$

即$I'(b) = -\frac{b}{2a} I(b)$ 结合初值$I(0)=\frac{1}{2}\sqrt{\pi/a}$,解出$I(b)=\frac{1}{2}\sqrt{\pi/a}e^{-b^2/4a}$

Ex. $I=\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\mathrm{d}x$.(瑕积分)　　　　引入参变量$t$

解: 令$I(t)=\int_0^1 \frac{\arctan(tx)}{x\sqrt{1-x^2}}\mathrm{d}x$, 则$I(0)=0$, 求$I(1)$.

$$I'(t)\xlongequal{\text{Abel}}\int_0^1 \frac{1}{(1+t^2x^2)\sqrt{1-x^2}}dx \xlongequal{x=\sin\theta} \int_0^{\pi/2}\frac{d\theta}{1+t^2\sin^2\theta}$$

$$=\int_0^{\pi/2}\frac{\csc^2\theta d\theta}{\csc^2\theta+t^2}=\int_0^{\pi/2}\frac{-d\cot\theta}{1+t^2+\cot^2\theta}$$

$$=\frac{-1}{\sqrt{1+t^2}}\arctan\frac{\cot\theta}{\sqrt{1+t^2}}\Bigg|_{\theta=0}^{\pi/2}=\frac{\pi}{2\sqrt{1+t^2}}.$$

思想3:引入参变量, 化定积分为含参积分

$$I(1)=\int_0^1\frac{\pi}{2\sqrt{1+t^2}}dt=\frac{\pi}{2}\ln(t+\sqrt{1+t^2})\Big|_{t=0}^{1}=\frac{\pi}{2}\ln(1+\sqrt{2}).\square$$

例. 计算积分$\int_0^{+\infty}\frac{\arctan ax\arctan bx}{x^2}dx, a>0, b>0$

解. $$I(a,b)=\int_0^{+\infty}\frac{\arctan ax\arctan bx}{x^2}dx$$

$$I_a'(a,b)=\int_0^{+\infty}\frac{x\arctan bx}{\left(1+a^2x^2\right)x^2}dx=\int_0^{+\infty}\frac{\arctan bx}{\left(1+a^2x^2\right)x}dx$$

$$I_{ab}''(a,b)=\int_0^{+\infty}\frac{1}{\left(1+a^2x^2\right)\left(1+b^2x^2\right)}dx=\frac{1}{b^2-a^2}\left(\int_0^{+\infty}\frac{b^2}{\left(1+b^2x^2\right)}dx-\int_0^{+\infty}\frac{a^2}{\left(1+a^2x^2\right)}dx\right)$$

$$=\frac{1}{b^2-a^2}\left(b\int_0^{+\infty}\frac{1}{\left(1+b^2x^2\right)}d(bx)-a\int_0^{+\infty}\frac{1}{\left(1+a^2x^2\right)}d(ax)\right)=\frac{1}{b+a}\frac{\pi}{2}$$

例. 计算积分$\int_0^{+\infty}\frac{\arctan ax\arctan bx}{x^2}dx, a>0, b>0$

$$I_a'(a,b)=\frac{\pi}{2}\ln(a+b)+C(a)\quad 0=I_a'(a,0)=\frac{\pi}{2}\ln(a)+C(a)\quad\Rightarrow C(a)=-\frac{\pi}{2}\ln(a)$$

$$\therefore I_a'(a,b)=\frac{\pi}{2}\left(\ln(a+b)-\ln(a)\right)\quad\therefore 0=I(0,b)=\frac{\pi}{2}\left(b\ln(b)-b\right)+C(b)$$

$$\therefore I(a,b)=\frac{\pi}{2}\left((a+b)\ln(a+b)-(a+b)-a\ln(a)+a\right)+C(b)$$

$$\therefore 0=I(0,b)=\frac{\pi}{2}\left(b\ln b-b\right)+C(b)\quad\therefore C(b)=\frac{\pi}{2}\left(b-b\ln b\right)$$

$$\therefore I(a,b)=\frac{\pi}{2}\left((a+b)\ln(a+b)-(a+b)-a\ln(a)+a\right)+\frac{\pi}{2}\left(b-b\ln b\right)$$

$$\therefore I(a,b)=\frac{\pi}{2}\left((a+b)\ln(a+b)-a\ln(a)-b\ln b\right)$$

Thm3. $f(x,y)\in C\left([a,+\infty)\times[\alpha,\beta]\right), I(y)=\int_a^{+\infty} f(x,y)dx$

关于$y\in[\alpha,\beta]$一致收敛,则$I(y)$在$[\alpha,\beta]$上可积,且

$$\int_\alpha^\beta I(y)dy=\int_a^{+\infty}\left(\int_\alpha^\beta f(x,y)dy\right)dx,$$

即

$$\int_\alpha^\beta\left(\int_a^{+\infty} f(x,y)dx\right)dy=\int_a^{+\infty}\left(\int_\alpha^\beta f(x,y)dy\right)dx,$$

也记为

$$\int_\alpha^\beta dy\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty} dx\int_\alpha^\beta f(x,y)dy.$$

Thm4. $f(x,y)\in C\left([a,+\infty)\times[\alpha,+\infty]\right)$, 且满足

(1) $\forall \beta>\alpha, \int_a^{+\infty} f(x,y)dx$ 在 $y\in[\alpha,\beta]$上一致收敛;

$\forall b>a, \int_\alpha^{+\infty} f(x,y)dy$ 在 $x\in[a,b]$上一致收敛;

(2) $\int_\alpha^{+\infty} dy\int_a^{+\infty}\left|f(x,y)\right|dx$ 与 $\int_a^{+\infty} dx\int_\alpha^{+\infty}\left|f(x,y)\right|dy$ 中至少有一个存在;

则 $I(y)=\int_a^{+\infty} f(x,y)dx$ 在$[\alpha,+\infty]$上可积, 且

$$\int_\alpha^{+\infty} dy\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty} dx\int_\alpha^{+\infty} f(x,y)dy.$$

例 $I=\int_0^{+\infty}\frac{\arctan bx-\arctan ax}{x}dx \quad (b>a>0).$

解: $I=\int_0^{+\infty}\frac{\arctan bx-\arctan ax}{x}dx.\ \because \int_a^b\frac{1}{1+t^2x^2}dt=\frac{\arctan bx-\arctan ax}{x}$

$$\therefore I=\int_0^{+\infty}\frac{\arctan bx-\arctan ax}{x}dx=\int_0^{+\infty}\int_a^b\frac{1}{1+t^2x^2}dtdx$$

$\int_0^{+\infty}\frac{1}{1+t^2x^2}dx$ 对 $t\in[a,b]$ 一致收敛(Weierstrass)

$$\therefore I=\int_0^{+\infty}\int_a^b\frac{1}{1+t^2x^2}dtdx=\int_a^b dt\int_0^{+\infty}\frac{1}{1+t^2x^2}dx=\int_a^b\frac{1}{t}\frac{\pi}{2}dt=\frac{\pi}{2}\ln(b/a)$$

课后作业$I=\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}\cos cx dx \quad (b>a>0).$

答案：$I=\frac{1}{2}\ln\left(\frac{b^2+c^2}{a^2+c^2}\right)$

4/ 含参数积分

- 含参数定积分：$\int_{\alpha}^{\beta} g(t,x)dx$
- 含参数广义积分：$\int_{\alpha}^{+\infty} g(t,x)dx$
- 无论是含参数定积分还是含参数的广义积分，本质上都是关于参数t的函数
- 两大工具：
  - 积分号下求导：导数好积分；导数和原函数的关系；计算定积分用含参积分
  - 交换积分次序：换完之后好积分；给你一个定积分，要知道把它转化为含参积分
- 被积函数的连续性和可导性+一致收敛性是这两个工具能够使用的条件

例. $\alpha,\beta>0$, 计算Laplace积分

$$I(\beta)=\int_0^{+\infty}\frac{\cos\beta x}{\alpha^2+x^2}dx,\ J(\beta)=\int_0^{+\infty}\frac{x\sin\beta x}{\alpha^2+x^2}dx.$$

解: $\int_0^{+\infty}\frac{x\sin\beta x}{\alpha^2+x^2}dx$ 关于 $\beta\ge b>0$ 一致收敛(Dirichlet).

故 $I'(\beta)=-J(\beta)$.(再在积分下求导是不允许的?)

已知 $\beta>0$ 时, $\int_0^{+\infty}\frac{\sin\beta x}{x}dx=\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$.

两式相加,得 $I'(\beta)+\frac{\pi}{2}=\alpha^2\int_0^{+\infty}\frac{\sin\beta x}{x(\alpha^2+x^2)}dx.$

求导得 $$I''(\beta)=\alpha^2 I(\beta).$$

此微分方程通解为 $$I=c_1e^{-\alpha\beta}+c_2e^{\alpha\beta}.$$

因为 $$|I|\le\int_0^{+\infty}\frac{\mathrm{d}x}{\alpha^2+x^2}=\frac{\pi}{2\alpha},\ \lim_{\alpha\to+\infty}I=0,$$

所以 $$c_2=0,\ \ I=c_1e^{-\alpha\beta}.$$

又 $$\lim_{\beta\to0+}I=\lim_{\beta\to0+}\int_0^{+\infty}\frac{\cos\beta x}{\alpha^2+x^2}dx=\int_0^{+\infty}\frac{1}{\alpha^2+x^2}dx=\frac{\pi}{2\alpha}.$$

所以 $$c_1=\frac{\pi}{2\alpha},\quad I(\beta)=\frac{\pi}{2\alpha}e^{-\alpha\beta},$$

$$J(\beta)=-I'(\beta)=-\frac{\pi}{2}e^{-\alpha\beta}.$$ □