1. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$ 2. $\int \frac{dx}{x^4\sqrt{x^2+1}}$ 3. $\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}}$ 4. $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)}}$

5. $\int_0^{+\infty} \frac{dx}{(1+x^2)(1+x^\alpha)}$ (α为常数)

6. $I=\int_0^1 \frac{dx}{2x-\sqrt{1-x^2}}$ 7. $I=\int_0^2 \frac{x}{1-x^2}$

8. $\int \frac{\arcsin e^x}{e^x}dx$ 9. $\int \frac{2x}{(x+1)(x^2+1)^2}dx$ 10. $\int|x-1|dx$, $x\in R$

11. $\int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}dx$ 12. $\int \frac{\sin x\cos x}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}dx$ 13. $\int \frac{x^4}{(x+1)^{100}}dx$

14. $\int\left[\frac{f(x)}{f'(x)}-\frac{f^2(x)f''(x)}{[f'(x)]^3}\right]dx$ 15. $\int \frac{x}{\sqrt{(1+x^2)^3}}e^{-\frac{1}{\sqrt{1+x^2}}}dx$

16. $\int \frac{dx}{(x-a)\sqrt{(x-a)(x-b)}}$ 17. $\int \frac{1+\sin x}{1+\cos x}dx$ 18. $\int \frac{7\cos x-3\sin x}{5\cos x+2\sin x}$

19. $\int \frac{\sqrt{x(1+x)}}{\sqrt{x}+\sqrt{1+x}}dx$ 20. $\int \frac{\sqrt{2x^2+3}}{x}dx$ 21. $\int \frac{dx}{\sqrt{(x-2)(x-3)}}$

22. $\int \frac{e^{\arctan x}}{(1+x^2)\sqrt{1+x^2}}dx$ 23. $\int \frac{\cos x-\sin x}{1+\sin x\cos x}dx$ 24. $\int \frac{\sin x+\cos x}{1+\sin x\cos x}dx$

25. $\int \frac{\sin x}{\sqrt{2+\sin 2x}}dx$ 26. $\int \frac{\tan x}{a^2\sin^2 x+b^2\cos^2 x}dx$ 27. $\int \frac{1+x}{x(1+xe^x)}$

28. $\int \sqrt{\frac{e^x-1}{e^x+1}}dx$ 29. $\int \frac{\ln\tan x}{\sin 2x}dx$ 30. $\int \frac{dx}{x^4\sqrt{1+x^2}}$

31. $\int \frac{\sqrt{a^2-x^2}}{x^4}dx$ 32. $\int xe^x\sin x\,dx$ 33. $\int \frac{x^3}{(x+1)^2(x^2+x+1)}dx$

34. $\int_0^\pi \sqrt{\sin x-\sin^3 x}dx$ 35. $\int_0^{k\pi}\sqrt{1+\sin^2 x}dx$ 36. $\int_0^\pi \frac{\sin nx}{\sin x}dx$

37. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x^2}{(x\sin x+\cos x)^2}$ 38. $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$ 39. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{1+e^x}dx$

1. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

解：令 $x=\frac{1-t}{1+t}$，$dx=\frac{-2}{(1+t)^2}dt$　　原式 $=I$

$$I=\int_1^0 \frac{\ln(1+\frac{1-t}{1+t})}{1+(\frac{1-t}{1+t})^2}\cdot\frac{-2}{(1+t)^2}dt=\int_0^1 \frac{2}{(1-t)^2+(1+t)^2}\cdot\ln\frac{2}{1+t}dt$$

$$=\int_0^1 \frac{1}{1+t^2}[\ln 2-\ln(1+t)]dt=\int_0^1 \frac{\ln 2}{1+t^2}dt-I$$

于是 $2I=\int_0^1 \frac{\ln 2}{1+t^2}dt=\ln 2\cdot\arctan t\Big|_0^1=\frac{\pi}{4}\ln 2$　∴ $I=\frac{\pi}{8}\ln 2$

2. $\int \frac{dx}{x^4\sqrt{x^2+1}}$

此题要注意，可用 $x=\tan t$ 代换，但是十分复杂．分母次数又高于分子，倒代换

设 $x=\frac{1}{t}$，$dx=-\frac{1}{t^2}dt$　　原式 $=\int \frac{t^4}{\sqrt{1+\frac{1}{t^2}}}(-\frac{1}{t^2})dt=\int\frac{t^3}{\sqrt{1+t^2}}dt$

$$=\frac{1}{2}[\int\sqrt{1+t^2}d(t^2+1)-\int\frac{1}{\sqrt{1+t^2}}d(t^2+1)]$$

$$=-\frac{1}{2}\cdot\frac{2}{3}(1+t^2)^{\frac{3}{2}}+\sqrt{1+t^2}+C=-\frac{1}{3}(1+\frac{1}{x^2})^{\frac{3}{2}}+\sqrt{1+\frac{1}{x^2}}+C$$

3. 设 $\sqrt[6]{x}=t$，则 $x=t^6$，$dx=6t^5dt$

原式 $=\int\frac{6t^8}{t^3-t^2}dt=6\int(t^5+t^4+t^3+t^2+t+\frac{1}{t-1})dt$

$$=6(\frac{t^6}{6}+\frac{t^5}{5}+\frac{t^4}{4}+\frac{t^3}{3}+\frac{t^2}{2}+t+\ln|t-1|)+C=x+\frac{6}{5}x^{\frac{5}{6}}+\frac{3}{2}x^{\frac{2}{3}}+2x^{\frac{1}{3}}+6x^{\frac{1}{6}}+6\ln|x^{\frac{1}{6}}-1|+C$$

4. $\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$

解：设 $\sqrt[3]{\frac{x-1}{x+1}}=t$　则 $x=\frac{1+t^3}{1-t^3}$　$x+1=\frac{2}{1-t^3}$，$x-1=\frac{2t^3}{1-t^3}$，$dx=\frac{6t^2}{(1-t^3)^2}dt$

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原式 $=\int (x+1)(x-1)\sqrt[3]{\frac{x-1}{x+1}}\,dx=\int\left(\frac{2}{1-t^3}\cdot\frac{2t^3}{1-t^3}\cdot t\right)^{-1}\frac{6t^2}{(1-t^3)^2}dt=\frac{3}{2}\int\frac{1}{t^2}dt$

$=-\frac{3}{2t}+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$

5. 设 $x=\frac{1}{t}$，则 $dx=-\frac{1}{t^2}dt$ $\quad I=\int_0^{+\infty}\frac{dx}{(1+x^2)(1+x^\alpha)}=\int_{+\infty}^0\frac{(-\frac{1}{t^2})dt}{(1+\frac{1}{t^2})(1+\frac{1}{t^\alpha})}$

$=\int_0^{+\infty}\frac{t^\alpha}{(1+t^2)(1+t^\alpha)}dt$

于是 $I=\frac{1}{2}\left[\int_0^{+\infty}\frac{dx}{(1+x^2)(1+x^\alpha)}+\int_0^{+\infty}\frac{x^\alpha}{(1+x^2)(1+x^\alpha)}dx\right]=\frac{1}{2}\int_0^{+\infty}\frac{dx}{1+x^2}=\frac{1}{2}\arctan x\Big|_0^{+\infty}=\frac{\pi}{4}$

6. 令 $x=\sin t$，$I=\int_0^{\frac{\pi}{2}}\frac{\cos t}{2\sin t-\cos t}dt=\frac{1}{5}\left[\int_0^{\frac{\pi}{2}}dt+2\int_0^{\frac{\pi}{2}}\frac{d(2\sin t-\cos t)}{2\sin t-\cos t}\right]$

$=\frac{1}{5}(2\ln 2-\frac{\pi}{2})$

错误：将1，0代入发现分母不为0，则非广义 实际上令分母=0，得 $x=\frac{1}{\sqrt{5}}$

$I=\lim_{\varepsilon\to 0^+}\int_0^{\frac{1}{\sqrt{5}}-\varepsilon}\frac{dx}{2x-\sqrt{1-x^2}}+\lim_{\varepsilon\to 0^+}\int_{\frac{1}{\sqrt{5}}+\varepsilon}^1\frac{dx}{2x-\sqrt{1-x^2}}$

其中 $\lim_{\varepsilon\to 0^+}\int_0^{\frac{1}{\sqrt{5}}-\varepsilon}\frac{dx}{2x-\sqrt{1-x^2}}=\lim_{\varepsilon\to 0^+}\frac{1}{5}\left[2\ln\left|2(\frac{1}{\sqrt{5}}-\varepsilon)-\sqrt{1-(\frac{1}{\sqrt{5}}-\varepsilon)^2}\right|-\arcsin(\frac{1}{\sqrt{5}}-\varepsilon)\right]$

$\to\infty$

故原广义积分发散

7. 1. $I=-\frac{1}{2}\int_0^2\frac{d(1-x^2)}{1-x^2}=-\frac{1}{2}\ln|1-x^2|\Big|_0^2=-\frac{\ln 3}{2}$

2. $I=\int_0^1\frac{x}{1-x^2}dx+\int_1^2\frac{x}{1-x^2}dx=-\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\frac{d(1-x^2)}{1-x^2}-\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{1+\varepsilon}^2\frac{d(1-x^2)}{1-x^2}$

$=-\frac{1}{2}\lim_{\varepsilon\to 0^+}\left[\ln|1-x^2|\Big|_0^{1-\varepsilon}+\ln|1-x^2|\Big|_{1+\varepsilon}^2\right]=-\frac{\ln 3}{2}+\frac{1}{2}\lim_{\varepsilon\to 0^+}\ln\left|\frac{2+\varepsilon}{2-\varepsilon}\right|=-\frac{\ln 3}{2}$

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正确解法：$I=\int_0^1 \frac{x}{1-x^2}dx+\int_1^2 \frac{x}{1-x^2}dx$

其中 $\int_0^1 \frac{x}{1-x^2}dx=\lim_{\varepsilon\to 0^+}-\frac{1}{2}\int_0^{1-\varepsilon}\frac{d(1-x^2)}{1-x^2}=-\frac{1}{2}\lim_{\varepsilon\to 0^+}\ln|1-x^2|\Big|_0^{1-\varepsilon}\to+\infty$

故原广义积分发散

8. $\int \frac{\arcsin e^x}{e^x}dx=-\int \arcsin e^x d(e^{-x})=-e^{-x}\arcsin e^x+\int \frac{dx}{\sqrt{1-e^{2x}}}$

令 $e^x=\sin t$，$\int \frac{dx}{\sqrt{1-e^{2x}}}=\int \csc t\,dt=\ln|\csc t-\cot t|+C$

原式 $=-e^{-x}\arcsin e^x-x+\ln(1-\sqrt{1-e^{2x}})+C$

9. $\int \frac{2x}{(x+1)(x^2+1)^2}dx$ 令 $\frac{2x}{(x+1)(x^2+1)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$

由待定系数法，$A=C=-\frac{1}{2}$，$B=\frac{1}{2}$，$D=E=1$

于是 $\frac{2x}{(x+1)(x^2+1)^2}=-\frac{1}{2}\frac{1}{x+1}+\frac{1}{2}\frac{x+1}{x^2+1}+\frac{x+1}{(x^2+1)^2}$

$\int \frac{2x}{(x+1)(x^2+1)^2}dx=\int\left(-\frac{1}{2}\frac{1}{x+1}+\frac{1}{2}\frac{x+1}{x^2+1}+\frac{x+1}{(x^2+1)^2}\right)dx=+\frac{1}{4}\ln\frac{1+x^2}{(1+x)^2}+\frac{x+1}{2(x^2+1)}+C$

10. 当 $x\geq 1$ 时，$\int|x-1|dx=\int(x-1)dx=\frac{x^2}{2}-x+C_1$

当 $x<1$ 时，$\int|x-1|dx=-\int(x-1)dx=-\frac{x^2}{2}+x+C_2$

而 $\int|x-1|dx$ 在 $x=1$ 处连续，$C_1=1+C_2$ 故 $\int|x-1|dx=\begin{cases}\frac{x^2}{2}-x+C+1, & x\geq 1\\ -\frac{x^2}{2}+x+C, & x<1\end{cases}$

11.（凑微分）原式 $=\frac{1}{2}\int\ln\frac{1+x}{1-x}d\left(\ln\frac{1+x}{1-x}\right)=\frac{1}{4}\left(\ln\frac{1+x}{1-x}\right)^2+C$

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12. 原式 $= \frac{1}{2}\int \frac{d(\sin^2 x)}{\sqrt{(a^2-b^2)\sin^2 x + b^2}} = \begin{cases} \frac{\sqrt{(a^2-b^2)\sin^2 x + b^2}}{a^2-b^2} + C, & a^2 \neq b^2 \\ \frac{1}{2|b|}\sin^2 x + C, & a^2 = b^2 \end{cases}$

13. $\int \frac{x^4}{(x+1)^{100}}dx \xrightarrow{x+1=t} \int \frac{(t-1)^4}{t^{100}}dt = \int \left(\frac{1}{t^{96}} - \frac{4}{t^{97}} + \frac{6}{t^{98}} - \frac{4}{t^{99}} + \frac{1}{t^{100}}\right)dt$

$= -\frac{1}{95(x+1)^{95}} + \frac{1}{24(x+1)^{96}} - \frac{6}{97(x+1)^{97}} + \frac{2}{49(x+1)^{98}} - \frac{1}{99(x+1)^{99}} + C$

14. 原式 $= \int \frac{f(x)}{f'(x)} \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}dx = \int \frac{f(x)}{f'(x)} d\left[\frac{f(x)}{f'(x)}\right] = \frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2 + C$

15. $\int \frac{x}{\sqrt{(1+x^2)^3}} e^{-\frac{1}{\sqrt{1+x^2}}}dx = e^{-\frac{1}{\sqrt{1+x^2}}} + C$

16. $\int \frac{dx}{(x-a)\sqrt{(x-a)(x-b)}} = \int \frac{dx}{(x-a)\sqrt{(x-a)(x-a+a-b)}} = -\int \frac{d(\frac{1}{x-a})}{\sqrt{1+\frac{a-b}{x-a}}} = \begin{cases} \frac{2}{b-a}\sqrt{1+\frac{a-b}{x-a}} + C & a \neq b \\ \frac{1}{a-x} + C, & a = b \end{cases}$

17. $\int \frac{1+\sin x}{1+\cos x}dx = \int \frac{1}{2\cos^2\frac{x}{2}}dx + \int \frac{\sin x}{1+\cos x}dx = \tan\frac{x}{2} - \ln(1+\cos x) + C$

或者：$\int \frac{1+\sin x}{1+\cos x}dx = \int \frac{(1+\sin x)(1-\cos x)}{1-\cos^2 x}dx = \int (\csc^2 x + \csc x - \csc x\cot x - \cot x)dx$

18. $\int \frac{7\cos x - 3\sin x}{5\cos x + 2\sin x}dx = \int \frac{5\cos x + 2\sin x + (-5\sin x + 2\cos x)}{5\cos x + 2\sin x}dx = x + \ln|5\cos x + 2\sin x| + C$

19. $\int \frac{\sqrt{x(1+x)}}{\sqrt{x}+\sqrt{1+x}}dx = \int (\sqrt{1+x}-\sqrt{x})\sqrt{x(1+x)}dx = \int [(1+x)\sqrt{x} - x\sqrt{1+x}]dx$

$= \frac{2}{3}x\sqrt{x} + \frac{2}{5}x^2\sqrt{x} + \frac{2}{3}(1+x)^{\frac{3}{2}} - \frac{2}{5}(1+x)^{\frac{5}{2}} + C$

20. $\int \frac{\sqrt{2x^2+3}}{x}dx \xrightarrow{x=\sqrt{\frac{3}{2}}\tan t} \int \sqrt{2}\frac{\sqrt{\frac{3}{2}}\sec t}{\sqrt{\frac{3}{2}}\tan t}\sqrt{\frac{3}{2}}\sec^2 t\,dt = \sqrt{3}\int \frac{dt}{\sin t\cos^2 t}$

$= \sqrt{3}\int\left[\frac{\sin t}{\cos^2 t}+\frac{1}{\sin t}\right]dt = \sqrt{3}\left[\sec t - \ln|\csc t+\cot t|\right]+C$

$= \sqrt{2x^2+3}-\sqrt{3}\ln\frac{\sqrt{2x^2+3}+\sqrt{3}}{\sqrt{2}x}+C$

21. $\int \frac{dx}{\sqrt{(x-2)(x-3)}} = \int \frac{dx}{\sqrt{(x-\frac{5}{2})^2-\frac{1}{4}}} = \ln\left(x-\frac{5}{2}+\sqrt{x^2-5x+6}\right)+C$

22. $\int \frac{e^{\arctan x}}{(1+x^2)\sqrt{1+x^2}}dx \xrightarrow{\arctan x=t} \int e^t\cos t\,dt = \frac{e^t}{2}(\cos t+\sin t)+C = \frac{(1+x)e^{\arctan x}}{\sqrt{1+x^2}}+C$

23. $\int \frac{\cos x-\sin x}{1+\sin x\cos x}dx = \int \frac{\cos x-\sin x}{\frac{1}{2}+\frac{1}{2}+\sin x\cos x}dx = \int \frac{2(\cos x-\sin x)}{1+(\cos x+\sin x)^2}dx$

$= 2\int \frac{d(\cos x+\sin x)}{1+(\cos x+\sin x)^2} = 2\arctan(\cos x+\sin x)+C$

24. $\int \frac{\sin x+\cos x}{1+\sin x\cos x}dx = \int \frac{\cos x+\sin x}{\frac{3}{2}-\frac{1}{2}+\sin x\cos x}dx = \int -\frac{2(\cos x+\sin x)}{3-(\sin x-\cos x)^2}dx$

$= 2\int \frac{d(\sin x-\cos x)}{3-(\sin x-\cos x)^2} = \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}+C$

25. 由 $\int \frac{\cos x-\sin x}{\sqrt{2+\sin 2x}}dx = \int \frac{d(\sin x+\cos x)}{\sqrt{1+(\sin x+\cos x)^2}} = \ln\left(\sin x+\cos x+\sqrt{2+\sin 2x}\right)+C$

$\int \frac{\cos x+\sin x}{\sqrt{2+\sin 2x}}dx = \int \frac{d(\sin x-\cos x)}{\sqrt{3-(\sin x-\cos x)^2}} = \arcsin\left(\frac{\sin x-\cos x}{\sqrt{3}}\right)+C$

故原式 $=\frac{1}{2}\left[\arcsin\left(\frac{\sin x-\cos x}{\sqrt{3}}\right)-\ln\left(\sin x+\cos x+\sqrt{2+\sin 2x}\right)\right]+C$

26. 当 $a\neq 0$ 时，$\int\frac{\tan x}{a^2\sin^2 x+b^2\cos^2 x}dx=\int\frac{\tan x}{b^2+a^2\tan^2 x}d(\tan x)=\frac{1}{2a^2}\ln(b^2+a^2\tan^2 x)+C$

当 $a=0$ 时，$\int\frac{\tan x}{a^2\sin^2 x+b^2\cos^2 x}dx=\frac{1}{b^2}\int\frac{\sin x}{\cos^3 x}dx=\frac{1}{2b^2}\sec^2 x+C$

27. $\int\frac{1+x}{x(1+xe^x)}dx=\int\frac{(1+x)e^x}{xe^x(1+xe^x)}dx=\int\frac{d(xe^x)}{xe^x(1+xe^x)}$ 令 $xe^x=t$

$=\int\frac{dt}{t(1+t)}=\int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt=\ln\left|\frac{t}{1+t}\right|+C=\ln\left|\frac{xe^x}{1+xe^x}\right|+C$

28. $\int\sqrt{\frac{e^x-1}{e^x+1}}dx=\int\frac{e^x-1}{\sqrt{e^{2x}-1}}dx=\int\frac{e^x}{\sqrt{e^{2x}-1}}dx-\int\frac{1}{\sqrt{e^{2x}-1}}dx$

$=\int\frac{de^x}{\sqrt{e^{2x}-1}}+\int\frac{de^{-x}}{\sqrt{1-e^{-2x}}}=\ln\left(e^x+\sqrt{e^{2x}-1}\right)+\arcsin e^{-x}+C$

29. $\int\frac{\ln\tan x}{\sin 2x}dx=\int\frac{\ln\tan x}{2\frac{\sin x}{\cos x}\cos^2 x}dx=\frac{1}{2}\int\frac{\ln\tan x}{\tan x}d(\tan x)$

$=\frac{1}{2}\int\ln\tan x\,d(\ln\tan x)=\frac{1}{4}(\ln\tan x)^2+C$

30. $\int\frac{dx}{x^4\sqrt{1+x^2}}\xrightarrow{x=\tan t}\int\frac{\cos^3 t}{\sin^4 t}dt=\int\frac{1-\sin^2 t}{\sin^4 t}d(\sin t)$

$=-\frac{1}{3}\frac{1}{\sin^3 t}+\frac{1}{\sin t}+C=-\frac{1}{3}\frac{\sqrt{(1+x^2)^3}}{x^3}+\frac{\sqrt{1+x^2}}{x}+C$

31. $\int\frac{\sqrt{a^2-x^2}}{x^4}dx\xrightarrow{x=a\sin t}\frac{1}{a^2}\int(\csc^4 t-\csc^2 t)dt=\frac{\cot t}{a^2}+\frac{1}{a^2}\int\csc^4 t\,dt$

$\int \csc^4 t\,dt = -\cot t\cdot\csc^2 t - 2\int \csc^2 t\cdot\cot^2 t\,dt = -\cot t\cdot\csc^2 t - 2\int \csc^4 t\,dt + 2\int \csc^2 t\,dt$

$= -(2+\csc^2 t)\cot t - 2\int \csc^4 t\,dt$

得 $\int \csc^4 t\,dt = -\frac{1}{3}(2+\csc^2 t)\cot t + C$

故 $\int \frac{\sqrt{a^2-x^2}}{x^4}dx = \frac{\cot t}{a^2} - \frac{\cot t}{3a^2}(2+\csc^2 t) + C = \frac{\sqrt{a^2-x^2}}{3a^2 x}\left(1-\frac{a^2}{x^2}\right) + C$

另解 $\int \frac{\sqrt{a^2-x^2}}{x^4}dx \xrightarrow{x=\frac{1}{t}} \int t\sqrt{a^2t^2-1}\,dt = -\frac{1}{3a^2}(a^2t^2-1)\sqrt{a^2t^2-1} + C = -\frac{\sqrt{a^2-x^2}}{3a^2 x}\left(1-\frac{a^2}{x^2}\right) + C$

32. $\int x e^x \sin x\,dx = x\frac{e^x(\sin x-\cos x)}{2} - \frac{1}{2}\int e^x(\sin x-\cos x)dx = x\frac{e^x(\sin x-\cos x)}{2} + \frac{e^x}{2}\cos x + C$

33. 设 $\frac{x^3}{(x+1)^2(x^2+x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+x+1}$

则 $A(x+1)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x+1)^2 = x^3$

令 $x=-1$，得 $B=-1$，令 $x=0$，得 $A+D=1$

又 x^3 的系数得 $A+C=1$，比较 x 的系数且 $B=-1$，得 $2A+C+2D=1$

解得 $A=2$，$C=D=-1$

从而 $\int \frac{x^3}{(x+1)^2(x^2+x+1)}dx = 2\ln|x+1| + \frac{1}{x+1} - \int \frac{x+1}{x^2+x+1}dx$

而 $\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int \frac{dx}{x^2+x+1}$

$= \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\int \frac{d\frac{2x+1}{\sqrt{3}}}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}$

$= \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + C$

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故得 $\int \frac{x^3}{(x+1)^2(x^2+x+1)}dx = 2\ln|x+1| + \frac{1}{x+1} - \frac{1}{2}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + C$

34. $\int_0^{\pi}\sqrt{\sin x - \sin^3 x}\,dx = \int_0^{\frac{\pi}{2}} |\cos x|\sqrt{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \cos x\sqrt{\sin x}\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos x\sqrt{\sin x}\,dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,d(\sin x) - \int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin x}\,d(\sin x) = \frac{2}{3}(\sin x)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}} - \frac{2}{3}(\sin x)^{\frac{3}{2}}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{4}{3}$

35. $\int_0^{k\pi}\sqrt{1-\sin^2 x}\,dx = \int_0^{k\pi}\sqrt{1-\sin^2 x}\,dx = k\int_0^{\pi}\sqrt{1-\sin^2 x}\,dx = k\pi\int_0^{\pi}|\cos x|\,dx = 2k$

36. $\int_0^{\pi}\frac{\sin nx}{\sin x}dx = I_n = \int_0^{\pi}\frac{\sin[(n-1)+1]x}{\sin x}dx = \int_0^{\pi}\frac{\sin(n-1)x\cos x}{\sin x}dx + \int_0^{\pi}\frac{\cos(n-1)x\sin x}{\sin x}dx$

$= \int_0^{\pi}\frac{\sin(n-1)x\cos x}{\sin x}dx = \int_0^{\pi}\frac{\sin(n-2)x\cos^2 x}{\sin x}dx + \int_0^{\pi}\frac{\cos(n-2)x\sin x\cos x}{\sin x}dx$

$= \int_0^{\pi}\frac{\sin(n-2)x}{\sin x} + \int_0^{\pi}[\cos(n-2)x\cos x - \sin(n-2)x\sin x]dx$

$= I_{n-2} + \int_0^{\pi}\cos(n-1)x\,dx = I_{n-2}$

$\therefore I_{2n} = I_2 = 0,\quad I_{2n+1} = I_1 = \pi$

37. 原式 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x\sec x\,d\left(\frac{-1}{x\sin x + \cos x}\right) = \frac{-x\sec x}{x\sin x+\cos x}\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(x\sin x+\cos x)\sec^2 x}{x\sin x+\cos x}dx$

$= \frac{4\pi}{\sqrt{3}\pi+18} - \frac{4\pi}{6\sqrt{3}\pi+3} + \frac{2\sqrt{3}}{3}$

38. $\int_0^{\frac{\pi}{4}}\ln(1+\tan x)dx = \int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos x+\sin x}{\cos x}\right)dx = \int_0^{\frac{\pi}{4}}\ln(\cos x+\sin x)dx - \int_0^{\frac{\pi}{4}}\ln\cos x\,dx$

$$= \int_0^{\frac{\pi}{4}} \ln\sqrt{2}\,dx + \int_0^{\frac{\pi}{4}} \ln\cos\left(x-\frac{\pi}{4}\right)dx - \int_0^{\frac{\pi}{4}} \ln\cos x\,dx = \frac{\pi}{8}\ln 2$$

39. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{1+e^{-x}}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+e^x-1)\sin^4 x}{1+e^x}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\,dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{1+e^x}dx$$

令 $x=-t$，则 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{1+e^x}dx = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\sin^4 t}{1+e^{-t}}dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{1+e^{-x}}dx = I$

$\therefore I = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\,dx = \int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{16}\pi$

40. $$\int_0^3 \arcsin\sqrt{\frac{x}{x+1}}dx = x\arcsin\sqrt{\frac{x}{x+1}}\Big|_0^3 - \int_0^3 x\left[\arcsin\sqrt{\frac{x}{x+1}}\right]'dx$$

$$= x\arcsin\sqrt{\frac{x}{x+1}}\Big|_0^3 - \frac{1}{2}\int_0^3 \frac{\sqrt{x}}{x+1}dx = 3\arcsin\frac{\sqrt{3}}{2} - \frac{1}{2}\int_0^3 \frac{\sqrt{x}}{1+x}dx$$

令 $\sqrt{x}=t$，$\int_0^3 \frac{\sqrt{x}}{x+1}dx = \int_0^{\sqrt{3}} \frac{2t^2}{1+t^2}dt = 2\int_0^{\sqrt{3}} \frac{t^2+1-1}{t^2+1}dt$

$$= 2(t-\operatorname{arctan} t)\Big|_0^{\sqrt{3}} = 2(\sqrt{3}-\arctan\sqrt{3}) = 2\sqrt{3}-\frac{2}{3}\pi$$

$\therefore \int_0^3 \arcsin\sqrt{\frac{x}{x+1}}dx = 3\cdot\frac{\pi}{3} - \sqrt{3} + \frac{\pi}{3} = \frac{4}{3}\pi - \sqrt{3}$