

微积分A期末讲座

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1. 换元积分法 $\frac{d}{dx} f(\varphi(x)) = f'(\varphi(x)) \cdot \varphi'(x)$

$$\Rightarrow \int f'(\varphi(x))\varphi'(x)dx = f(\varphi(x)) + C.$$

与 $\int f'(u)du = f(u) + C$ 比较, 得

$$\int f'(\varphi(x))\varphi'(x)dx = \int f'(\varphi(x))d\varphi(x) = f(\varphi(x)) + C$$

$$\underline{\underline{u = \varphi(x)}} \int f'(u)du = f(u) + C = f(\varphi(x)) + C$$

——第一换元法 (凑微分法)

$$\int f'(u)du$$

$$\underline{\underline{u = \varphi(x)}} \int f'(\varphi(x))\varphi'(x)dx = g(x) + C = g(\varphi^{-1}(u)) + C$$

——第二换元法

08 / 不定积分

2. 分部积分法

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$$

$$\Rightarrow \int u'(x)v(x)dx + \int u(x)v'(x)dx = \int (u(x)v(x))' dx$$

$$\Rightarrow \int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

也记作 $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x).$

——分部积分法

08 / 不定积分

3. 有理函数 $\frac{p(x)}{q(x)}$ (p, q 为多项式)

Thm. 有理真分式 $\frac{p(x)}{q(x)}$ 可以唯一地分解成最简分式之和:

(1) $q(x)$ 的一次 k 重因式 $(ax + b)^k$ 对应 k 项

$$\frac{A_1}{ax + b}, \frac{A_2}{(ax + b)^2}, \dots, \frac{A_k}{(ax + b)^k};$$

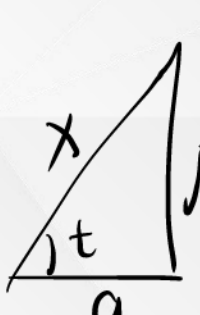
(2) $q(x)$ 的二次 k 重因式 $(px^2 + qx + r)^k$ 对应 k 项

$$\frac{B_1x + C_1}{px^2 + qx + r}, \frac{B_2x + C_2}{(px^2 + qx + r)^2}, \dots, \frac{B_kx + C_k}{(px^2 + qx + r)^k}.$$

$$\int \frac{dx}{x-a} = \ln|x-a| + C, \quad \int \frac{dx}{\sqrt{x^2-a^2}} \quad \text{令 } x = a \sec t$$

$$\int \frac{dx}{(x-a)^k} = \frac{-1}{(k-1)(x-a)^{k-1}} + C, \quad \text{则 } dx = a \sec t \cdot \tan t \cdot dt$$

$$\int \frac{x+a}{((x+a)^2+b^2)^k} dx = \frac{-1}{2(k-1)((x+a)^2+b^2)^{k-1}},$$



$$\star J_k = \int \frac{1}{((x+a)^2+b^2)^k} dx \quad \int \frac{a \sec t \tan t}{a t \tan t} dt = \int \sec t dt$$

$$J_1 = \frac{1}{b} \arctan \frac{x+a}{b} + C, \quad = \ln|\sec t + \tan t| + C$$

$$J_k = \int \frac{d(x+a)^{k+1}}{((x+a)^2+b^2)^k} \quad J_{k+1} = \int \frac{d(x+a)}{((x+a)^2+b^2)^k \cdot ((x+a)^2+b^2)}$$

$$= \frac{1}{2kb^2} ((x+a)((x+a)^2+b^2)^{-k} + (2k-1)J_k) \quad \ln|x+\sqrt{x^2-a^2}| + C$$

令 $x+a=t$.

$$J_k = \int \frac{dt}{(t^2+b^2)^k} \quad J_{k+1} = \int \frac{dt}{(t^2+b^2)^{k+1}}$$

$$= \frac{t}{(t^2+b^2)^{k+1}} - \int t \cdot (-k-1) \cdot \frac{2t \cdot dt}{(t^2+b^2)^{k+2}}$$

不~~换元~~你积个寂寞?

$$= \frac{t}{(t^2+b^2)^{k+1}} + (2k+2) \int \frac{t^2}{(t^2+b^2)^{k+2}} dt$$

$$= \frac{t}{(t^2+b^2)^{k+1}} + (2k+2) \int \left(\frac{1}{(t^2+b^2)^{k+1}} - \frac{b^2}{(t^2+b^2)^{k+2}} \right) dt$$

$$0 = \frac{t}{(t^2+b^2)^{k+1}} + (2k+1) J_{k+1} - 2b^2(k+1) \int \frac{dt}{(t^2+b^2)^{k+2}}$$

$$\frac{t}{(t^2+b^2)^{k+1}} + (2k+1) J_{k+1} - 2b^2(k+1) J_{k+2}$$

$$J_{k+2} = \frac{1}{2b^2(k+1)} \left(\frac{t}{(t^2+b^2)^{k+1}} + (2k+1) J_{k+1} \right)$$

$$J_{k+1} = \frac{1}{2b^2 k} \left(\frac{t}{(t^2+b^2)^k} + (2k-1) J_k \right)$$

$$J_{k+1} = \frac{1}{2b^2 k} \left((x+a) ((x+a)^2+b^2)^{-k} + (2k-1) J_k \right)$$

08 / 不定积分

4. 三角有理式 $R(\sin x, \cos x)$: $\sin x, \cos x$ 有限次四则运算

万能变换 $t = \tan \frac{x}{2}$, $x = 2 \arctan t$, $dx = \frac{2}{1+t^2} dt$,

$$\sin x = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{2 \tan \frac{x}{2}}{\tan^2 \frac{x}{2} + 1} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

$$\int R(\sin x, \cos x) dx = R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{dt}{1+t^2}.$$

$$\begin{aligned} \int 4x \cdot e^{-2x} dx &= \int 2x \cdot e^{-2x} \cdot d2x \\ &= \int t \cdot e^{-t} dt = - \int t de^{-t} = -t \cdot e^{-t} + \int e^{-t} dt \\ &= -t \cdot e^{-t} - e^{-t} + C \\ &= -2x \cdot e^{-2x} - e^{-2x} + C \end{aligned}$$

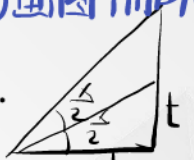
$$\text{Ex. } I = \int \frac{1 + \sin x}{1 + \cos x} dx.$$

解法一: 令 $t = \tan \frac{x}{2}$, 则 $x = 2 \arctan t$, $dx = \frac{2}{1+t^2} dt$,

$$\int \frac{1 + \sin x}{1 + \cos x} dx = \int \frac{1 + \frac{2t}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{(1+t)^2}{2} \cdot \frac{2}{1+t^2} dt$$

$$= \int \left(1 + \frac{2t}{1+t^2} \right) dt = t + \ln(1+t^2) + C = \tan \frac{x}{2} - 2 \ln \left| \cos \frac{x}{2} \right| + C.$$

① 画图辅助



解法二: $I = \int \frac{1}{2 \cos^2 \frac{x}{2}} dx + \int \frac{\sin x}{1 + \cos x} dx$

$$= \int \frac{1}{2 \cos^2 \frac{x}{2}} dx + \ln(1 + \tan^2 \frac{x}{2}) = \int \frac{1}{2 \cos^2 \frac{x}{2}} dx + \ln \sec^2 \frac{x}{2}$$

$$= \tan \frac{x}{2} - \ln(1 + \cos x) + C = \tan \frac{x}{2} + 2 \ln |\sec \frac{x}{2}|$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2}$$

① 共轭算法 (涉及 $\cos x$; 不熟悉)

② 万能公式 (其实并不复杂)

③ 折项 降幂升角公式

Remark. 初等函数的原函数不一定是初等函数, 如

$$e^{-x^2}, \sin x^2, \cos x^2, \frac{\sin x}{x}, \frac{\cos x}{x},$$

$$\frac{1}{\ln x}, \sqrt{1-k^2 \sin^2 x} (0 < k < 1).$$

如果被积函数是分段函数，需要注意得出的原函数的连续性

Ex. 求 $\int e^{|x|} dx$.

解: $e^{|x|}$ 在 \mathbb{R} 上连续, 因而可积, 设 $F(x)$ 为一个原函数, 则

$$F'(x) = e^{|x|} = \begin{cases} e^x, & x \geq 0 \\ e^{-x}, & x < 0 \end{cases}, \quad F(x) = \begin{cases} e^x + C_1, & x \geq 0 \\ -e^{-x} + C_2, & x < 0 \end{cases}.$$

$F(x)$ 在 $x = 0$ 处连续, 则 $1 + C_1 = -1 + C_2$,

$$\int e^{|x|} dx = F(x) = \begin{cases} e^x + C, & x \geq 0 \\ -e^{-x} + 2 + C, & x < 0 \end{cases}. \quad \square$$

如果被积函数是分段函数，需要注意得出的原函数的连续性

Ex. $\int |(x-1)(3x-2)| dx.$

$$|(x-1)(3x-2)| = \begin{cases} 3x^2 - 5x + 2, & x \geq \frac{2}{3} \\ -3x^2 + 5x - 2, & \frac{2}{3} < x < 1. \\ 3x^2 - 5x + 2, & x \geq 1 \end{cases}$$

$$\int |(x-1)(3x-2)| dx = \begin{cases} x^3 - \frac{5}{2}x^2 + 2x + C_1, & x \geq \frac{2}{3} \\ -x^3 + \frac{5}{2}x^2 - 2x + C_2, & \frac{2}{3} < x < 1. \\ x^3 - \frac{5}{2}x^2 + 2x + C_3, & x \geq 1 \end{cases}$$

原函数要在2/3和1处连续.

$$\therefore \frac{14}{27} + C_1 = -\frac{14}{27} + C_2, -\frac{1}{2} + C_2 = \frac{1}{2} + C_3, \text{即 } C_1 = C_2 - \frac{28}{27}, C_3 = C_2 - 1$$

0. 常见例子.

$$\int \sin^{2n} x dx, \int \sin^{2n-1} x dx, \int \cos^{2n} x dx, \int \cos^{2n-1} x dx,$$

- ① 奇次项则提出一次项并换元
- ② 偶次项则降幂升角并换元

1.敏锐地观察,是否已经可以凑微分

$$\int x e^{-x^2} dx = \frac{1}{2} \int e^{-x^2} dx^2 = -\frac{1}{2} e^{-x^2} + C.$$

$$\int \sin^n x \cos x dx = \int \sin^n x d \sin x = \frac{1}{n+1} \sin^{n+1} x + C$$

Ex. $\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{d \sin x}{1 - \sin^2 x} = \frac{1}{2} \cdot \ln \frac{1 + \sin x}{1 - \sin x}$

$$= \frac{1}{2} \int \left(\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} \right) d \sin x \quad \int \sec x dx = \ln |\sec x + \tan x| + C.$$

$$= \frac{1}{2} \ln(1 + \sin x) - \frac{1}{2} \ln(1 - \sin x) + C$$

$$= \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + C$$

2. 三角代换用于换元法:

$$"\sqrt{x^2-1}" \leftrightarrow x = \sec t, \sqrt{x^2-1} = \tan t$$

$$"\sqrt{x^2+1}" \leftrightarrow x = \tan t, \sqrt{x^2+1} = \sec t$$

$$"\sqrt{1-x^2}" \leftrightarrow x = \sin t, \sqrt{1-x^2} = \cos t$$

注意: $\sqrt{x^2-1}$ 用 $\sec t$ 代换, 需要引起高度关注!

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

$$Q: "\sqrt{ax^2+bx+c}"?$$



$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

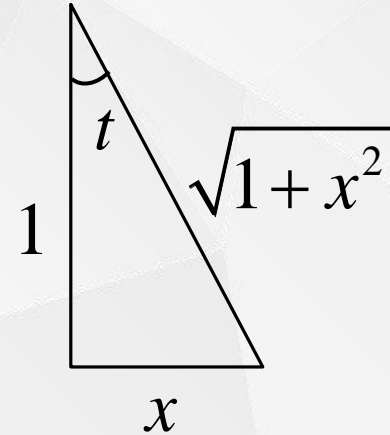
$$\hookrightarrow \sqrt{t^2 + d}$$

$$t = \sqrt{d} \cos \alpha, \tan \alpha, \sec \alpha$$

$$\forall t, |t| < \frac{\pi}{2}, \sqrt{1 + \tan^2 t} = |\sec t| = \frac{1}{|\cos t|} = \frac{1}{\cos t}$$

Ex. $\int \frac{dx}{x^2 \sqrt{1+x^2}}$

解: 令 $x = \tan t, |t| < \frac{\pi}{2},$



$$\int \frac{dx}{x^2 \sqrt{1+x^2}} = \int \frac{\sec^2 t dt}{\tan^2 t \sec t}$$

$$= \int \frac{\cos t dt}{\sin^2 t} = \int \frac{d \sin t}{\sin^2 t} = -\frac{1}{\sin t} + C = -\frac{\sqrt{1+x^2}}{x} + C.$$

$$\text{Ex.} \int \frac{x+1}{\sqrt{-x^2+2x+3}} dx$$

解:

$$\text{原式} = \int \frac{x+1}{\sqrt{4-(x-1)^2}} dx = \frac{1}{2} \int \frac{x+1}{\sqrt{1-(\frac{x-1}{2})^2}} dx = \int \frac{x+1}{\sqrt{1-(\frac{x-1}{2})^2}} d \frac{x-1}{2}$$

$$\text{令 } x-1=2t$$

$$\text{令 } t = \sin y$$

$$= \int \frac{2t+2}{\sqrt{1-t^2}} dt = \int \frac{2 \sin y + 2}{\cos y} d \sin y = 2(y - \cos y) + C$$

$$= 2(\arcsin(\frac{x-1}{2}) - \sqrt{1-(\frac{x-1}{2})^2}) + C$$

$$2 \int \frac{t+1}{\sqrt{1-t^2}} dt = \int \frac{t}{\sqrt{1-t^2}} dt + \int \frac{dt}{\sqrt{1-t^2}}$$

$$= -\sqrt{1-t^2} + \arcsin t + C$$

$$\text{Ex. } \int \frac{\sqrt{x^2 - 4}}{x} dx \quad \forall t, 0 \leq t \leq \pi, \sqrt{4\sec^2 t - 4} = 2|\tan t| = \begin{cases} 2 \tan t, 0 \leq t \leq \frac{\pi}{2} \\ -2 \tan t, \frac{\pi}{2} \leq t \leq \pi \end{cases}$$

解： 令 $x = 2\sec t, 0 \leq t \leq \pi$

$$\frac{\sqrt{x^2 - 4}}{x} = \frac{1}{\sec t} |\tan t|, dx = 2 \sec t \tan t dt$$

$$\int \frac{\sqrt{x^2 - 4}}{x} dx = \int \frac{|\tan t|}{\sec t} 2 \sec t \tan t dt = 2 \int |\tan t| \tan t dt$$

1°. $0 \leq t \leq \pi/2$, 即 $x \geq 2$.

$$\text{原式} = 2 \int \tan^2 t dt = 2 \int \sec^2 t - 1 dt = 2 \tan t - 2t + C_1 \quad \because \tan t = \frac{1}{2} \sqrt{x^2 - 4}, t = \arccos(2/x)$$

$$\therefore \text{原式} = \sqrt{x^2 - 4} - 2 \arccos(2/x) + C_1$$

$$\text{Ex. } \int \frac{\sqrt{x^2 - 4}}{x} dx \quad \forall t, 0 \leq t \leq \pi, \sqrt{4\sec^2 t - 4} = 2 |\tan t| = \begin{cases} 2 \tan t, 0 \leq t \leq \frac{\pi}{2} \\ -2 \tan t, \frac{\pi}{2} \leq t \leq \pi \end{cases}$$

解： 令 $x = 2\sec t, 0 \leq t \leq \pi$

$$\frac{\sqrt{x^2 - 4}}{x} = \frac{1}{\sec t} |\tan t|, dx = 2 \sec t \tan t dt$$

$$\int \frac{\sqrt{x^2 - 4}}{x} dx = \int \frac{|\tan t|}{\sec t} 2 \sec t \tan t dt = 2 \int |\tan t| \tan t dt$$

$$\because \tan t = -\frac{1}{2} \sqrt{x^2 - 4}, t = \arccos(2/x)$$

2°. $\pi/2 \leq t \leq \pi$ 即 $x \leq -2$

$$\text{原式} = -2 \int \tan^2 t dt = -2 \int \sec^2 t - 1 dt = -2 \tan t + 2t + C_2$$

$$\therefore \text{原式} = \sqrt{x^2 - 4} + 2 \arccos(2/x) + C_2$$

$$\text{Ex. } \int \frac{\sqrt{x^2-4}}{x} dx$$

$$\frac{dx}{x} = \frac{dz}{z^2}$$

$$\text{解: 原式} = \int \frac{\sqrt{x^2-4}}{x^2} x dx = \frac{1}{2} \int \frac{\sqrt{x^2-4}}{x^2} dx^2$$

$$= \frac{1}{2} \int \frac{\sqrt{y-4}}{y} dy \quad \begin{matrix} y=x^2 \\ \sqrt{y-4}=z, z^2+4=y, dy=2zdz \end{matrix} = \int \frac{z^2}{z^2+4} dz = \int \frac{z^2+4-4}{z^2+4} dz = \int 1 dz - 4 \int \frac{1}{z^2+4} dz$$

$$= z - 2 \int \frac{1}{(z/2)^2 + 1} dz / 2 = z - 2 \arctan(z/2) + C$$

$$= \sqrt{x^2-4} - 2 \arctan(\sqrt{x^2-4}/2) + C$$

1.敏锐地观察,是否已经可以凑微分

$$\text{Ex.} \int \frac{x-1}{\sqrt{2-2x-x^2}} dx = \int \frac{x-1}{\sqrt{3-(x+1)^2}} dx = \int \frac{x+1-2}{\sqrt{3-(x+1)^2}} dx$$

$$= \int \frac{x+1}{\sqrt{3-(x+1)^2}} dx - 2 \int \frac{1}{\sqrt{3-(x+1)^2}} dx$$
$$\int \frac{x+1}{\sqrt{3-(x+1)^2}} dx = -\sqrt{3-(x+1)^2}$$
$$\int \frac{1}{\sqrt{3-(x+1)^2}} dx = \arcsin \frac{x+1}{\sqrt{3}} + C$$
$$= \frac{1}{2} \int \frac{d(x+1)^2}{\sqrt{3-(x+1)^2}} - \frac{2}{\sqrt{3}} \int \frac{1}{\sqrt{1-(\frac{x+1}{\sqrt{3}})^2}} dx = \frac{1}{2} \int \frac{d(x+1)^2}{\sqrt{3-(x+1)^2}} - 2 \int \frac{1}{\sqrt{1-(\frac{x+1}{\sqrt{3}})^2}} d \frac{x+1}{\sqrt{3}}$$

$$= -\sqrt{3-(x+1)^2} - 2 \arcsin\left(\frac{x+1}{\sqrt{3}}\right) + C$$

3. 一次根式, 直接令其整体等于一个新变量:

$$\text{Ex. } \int \frac{1}{1+\sqrt{1+x}} dx$$

$$\text{令 } \sqrt{1+x} = t, x = t^2 - 1,$$

$$\begin{aligned} \text{原式} &= \int \frac{1}{1+t} d(t^2 - 1) = \int \frac{2t}{1+t} dt = \int \frac{2t+2-2}{1+t} dt = \int 2 - \frac{2}{1+t} dt = 2t - 2\ln(1+t) \\ &= 2\sqrt{1+x} - 2\ln(1+\sqrt{1+x}) + C \end{aligned}$$

$$\text{Ex. } \int \frac{\ln x}{x\sqrt{1+\ln x}} dx \quad \text{令 } \sqrt{1+\ln x} = t, \ln x = t^2 - 1$$

$$\frac{1}{x} dx = 2t dt, \quad dx = 2tx dt$$

$$\int \frac{\ln x}{x\sqrt{1+\ln x}} dx = \int \frac{t^2 - 1}{xt} 2tx dt = 2 \int t^2 - 1 dt = \frac{2}{3} t^3 - 2t + C = \frac{2}{3} (\ln x + 1)^{3/2} - 2(\ln x + 1)^{1/2} + C$$

$$\int \frac{\ln x}{\sqrt{1+\ln x}} d\ln x = \int \frac{t}{\sqrt{1+t}} dt$$

• 例题（含有一次根式的函数）

$$\int \frac{1}{\sqrt[3]{t} + \sqrt{t}} dt = \int \frac{1}{x^2 + x^3} dx^6 (t = x^6) = \int \frac{6x^5}{x^2 + x^3} dx = \int \frac{6x^3}{1+x} dx = \int \frac{6(t-1)^3}{t} dt$$

4. 高次根式, 考虑设法化为1次 (通过还原)

$$\begin{aligned}\int \frac{1}{x \sqrt{1+x^5}} &= \int \frac{x^4 dx}{x^5 \sqrt{1+x^5}} = \frac{1}{5} \int \frac{dy}{y \sqrt{1+y}} = \frac{1}{5} \int \frac{d(t^2-1)}{t(t^2-1)} \\ &= \frac{2}{5} \int \frac{1}{t^2-1} = \frac{1}{5} \int \frac{1}{t-1} - \frac{1}{5} \int \frac{1}{t+1} = \frac{1}{5} \ln\left(\frac{t-1}{t+1}\right) + C = \frac{1}{5} \ln\left(\frac{\sqrt{1+x^5}-1}{\sqrt{1+x^5}+1}\right) + C\end{aligned}$$

5.感到无所适从, 不知道怎么办的时候, 往往可以通过分部积分打开局面

$$\begin{aligned}\text{Ex. } \int \ln x dx &= x \ln x - \int x d \ln x \\ &= x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C\end{aligned}$$

$$x(\ln x - 1)$$

$$\ln x - 1 + 1 = \ln x$$

$$\text{Ex. } \int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \arcsin x - \frac{1}{2} \int \frac{dx^2}{\sqrt{1-x^2}} = x \arcsin x + \frac{1}{2} \sqrt{1-x^2} + C.$$

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2}$$

$$\begin{aligned}\frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot -2x \\ = -\frac{x}{\sqrt{1-x^2}}\end{aligned}$$

- 例题5 (实在不会, 就先分部积分, 创造条件)

$$\int (3x^2 + 2x) \arctan x dx$$

$$\int \arctan x d(x^3 + x^2) = (x^3 + x^2) \arctan x - \int \frac{x^3 + x^2}{x^2 + 1} dx$$

$$\int \frac{x^3 + x^2}{x^2 + 1} dx = \int \left(x + \frac{x^2 - x}{x^2 + 1} \right) dx = \int \left(x + 1 - \frac{x + 1}{x^2 + 1} \right) dx$$

$$= \frac{1}{2}x^2 + x - \int \frac{x}{x^2 + 1} dx - \int \frac{dx}{x^2 + 1}$$

$$= \frac{1}{2}x^2 + x - \arctan x - \frac{1}{2} \ln x^2 + 1 + C$$

$x = a \tan t \quad (t \in (-\frac{\pi}{2}, \frac{\pi}{2})) \quad dx = a \sec^2 t dt$

$\int \frac{x^2}{\sqrt{x^2+a^2}} dx$

$\int \frac{a^2 \tan^2 t \cdot a \sec^2 t dt}{a \sec t}$

$= \int (\sec^3 t - \sec t) dt$

$= \int \sec t dt \tan t - \ln |\sec t + \tan t| + C$

$= \sec t \tan t - \int \tan^2 t \sec t dt - \ln |\sec t + \tan t| + C$

$\therefore \frac{1}{2} \sec t \tan t - \frac{1}{2} \ln |\sec t + \tan t| + C$

$= \frac{1}{2} \cdot \frac{\sqrt{x^2+a^2}}{a} \cdot \frac{x}{a} - \frac{1}{2} \ln \left| \frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a} \right| + C$

再补上 $a^2!$

$\int \frac{x^2}{\sqrt{x^2+a^2}} dx = \frac{x}{2} \sqrt{x^2+a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2+a^2}| + C$

$\int \frac{x^2}{\sqrt{x^2-a^2}} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2-a^2}| + C$

分类讨论原则:

① $x = a \sin t \quad (t \in [-\frac{\pi}{2}, \frac{\pi}{2}])$
 不用讨论: 因为 $\cos t$ 恒正

② $x = a \tan t \quad (t \in (-\frac{\pi}{2}, \frac{\pi}{2}))$
 不用讨论: 因为 $\sec t$ 恒正

③ 仅需讨论 $x = \sec t$.

$\int \frac{x^2}{\sqrt{x^2+a^2}} dx \quad x = a \tan t \quad (t \in [0, \frac{\pi}{2}))$

$dx = a \sec^2 t dt$

- 例题5 (实在不会, 就先分部积分, 创造条件)

$$\begin{aligned} & \int \frac{1 + \sin x}{1 + \cos x} e^x dx \\ &= \int \frac{(\cos(x/2) + \sin(x/2))^2}{2 \cos^2(x/2)} e^x dx = \int \frac{(\cos(x/2) + \sin(x/2))^2}{\cos^2(x/2)} e^x dx / 2 \\ &= \int \frac{(\cos(t) + \sin(t))^2}{\cos^2 t} e^{2t} dt \\ &= \int (1 + \tan t)^2 e^{2t} dt \\ &= \int \sec^2 t e^{2t} dt + 2 \int \tan t e^{2t} dt = \int e^{2t} d \tan t + 2 \int \tan t e^{2t} dt \\ &= \tan t e^{2t} - 2 \int \tan t e^{2t} d + 2 \int \tan t e^{2t} dt = \tan t e^{2t} + C \end{aligned}$$

• 例题 (分部积分-设而不求, 消元)

$$\int \left(1 - \frac{2}{x}\right)^2 e^x dx$$

$$\int \left(1 - \frac{2}{x}\right)^2 e^x dx = \int \left(1 - \frac{4}{x} + \frac{4}{x^2}\right) e^x dx = \int e^x dx + 4 \int \left(-\frac{1}{x} + \frac{1}{x^2}\right) e^x dx$$

$$\int \left(-\frac{1}{x} + \frac{1}{x^2}\right) e^x dx = \int \frac{1}{x^2} e^x dx - \int \frac{1}{x} e^x dx$$

$$\because -\int \frac{1}{x} de^x = -\frac{1}{x} e^x - \int \frac{1}{x^2} e^x dx \therefore \int \left(-\frac{1}{x} + \frac{1}{x^2}\right) e^x dx = -\frac{1}{x} e^x + C$$

$$\int (f(x) + f'(x)) e^x dx$$

$$= f(x) \cdot e^x - \int f'(x) \cdot e^x dx + \int f'(x) \cdot e^x dx$$

$$= f(x) \cdot e^x + C$$

- 例题（分部积分-导出递推式）

$$\int \ln^n(x) dx$$

$$\int \ln^n(x) dx = x \ln^n(x) - n \int \ln^{n-1}(x) dx$$

$$f_n(x) := \int \ln^n(x) dx$$

$$f_n(x) = x \ln^n(x) - n f_{n-1}(x)$$

- 例题（实在不会，就先分部积分，创造条件）

$$\int \frac{x^2}{(x \sin x + \cos x)^2} dx$$

$$\int \frac{x^2}{(x \sin x + \cos x)^2} dx = - \int \frac{x}{\cos x} d \frac{1}{x \sin x + \cos x}$$

$$= - \frac{x}{\cos x} \frac{1}{x \sin x + \cos x} + \int \frac{1}{x \sin x + \cos x} d \frac{x}{\cos x}$$

$$= - \frac{x}{\cos x} \frac{1}{x \sin x + \cos x} + \int \frac{1}{x \sin x + \cos x} \frac{\cos x - (-\sin x)x}{\cos^2 x} dx$$

$$= - \frac{x}{\cos x} \frac{1}{x \sin x + \cos x} + \int \sec^2 x dx = - \frac{x}{\cos x} \frac{1}{x \sin x + \cos x} + \tan x$$

$$\left(\frac{1}{x \sin x + \cos x} \right)' = - \frac{1}{(x \sin x + \cos x)^2} (\sin x + \cos x \cdot x - \sin x)$$

6.含有三角的情况：虽然万能变换是通法，但是万能变换不一定简单.

$$\begin{aligned} \text{Ex. } & \int \frac{\tan x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \\ &= \int \frac{\tan x}{a^2 + b^2 \tan^2 x} \cdot \frac{dx}{\cos^2 x} = \int \frac{\tan x}{a^2 + b^2 \tan^2 x} d \tan x \\ &= \frac{1}{2} \int \frac{d \tan^2 x}{a^2 + b^2 \tan^2 x} = \frac{1}{2b^2} \ln(a^2 + b^2 \tan^2 x) + C. \end{aligned}$$

6.含有三角的情况：虽然万能变换是通法，但是万能变换不一定简单。

$$\text{Ex. } \int \frac{\sin x}{\cos x + \sin x} dx$$

$$\text{令 } t = x + \pi/4$$

$$= \int \frac{\sin x}{\sqrt{2} \sin(x + \frac{\pi}{4})} dx$$

$$= \int \frac{\sin(t - \frac{\pi}{4})}{\sqrt{2} \sin t} dt$$

$$= \int \frac{\frac{\sqrt{2}}{2} \sin t - \frac{\sqrt{2}}{2} \cos t}{\sqrt{2} \sin t} dt = \frac{1}{2} \int 1 - \cot t dt = \frac{1}{2} x - \frac{1}{2} \ln |\sin x + \cos x| + C$$

$$\begin{aligned} \sin x + \cos x &= \sqrt{2} \left(\cos x \cdot \frac{\sqrt{2}}{2} + \sin x \cdot \frac{\sqrt{2}}{2} \right) \\ &= \sqrt{2} \cos(x - \frac{\pi}{4}) \end{aligned}$$

$$\int \frac{\cos x}{\sqrt{2} \cos(x - \frac{\pi}{4})} dx = \frac{\sqrt{2}}{2} \int \frac{\cos(t + \frac{\pi}{4})}{\cos t} dt = \frac{\sqrt{2}}{2} \int \frac{\cos t \cdot \frac{\sqrt{2}}{2} - \sin t \cdot \frac{\sqrt{2}}{2}}{\cos t} dt$$

$$= \frac{1}{2} t - \frac{1}{2} \int \tan t dt = \frac{1}{2} t + \frac{1}{2} \ln |\cos t| + C$$

$$= \frac{1}{2} x + \frac{1}{2} \ln |\sin x + \cos x| + C$$

7. 如果被积函数只和 e^x 有关, 令 $e^x = t$ 是一个不错的选择.

Ex. $\int \frac{\arcsin e^x}{e^x} dx$

$$= \int \frac{\arcsin t}{t} d(\ln t) = \int \arcsin t / t^2 dt = \int \arcsin t d(-1/t) = -\frac{\arcsin t}{t} + \int \frac{1}{t\sqrt{1-t^2}} dt$$

$$= \int \frac{1}{t\sqrt{1-t^2}} dt = \int \frac{1}{\sin y \cos y} d \sin y = \int \frac{1}{\sin y} dy = \int \frac{\sin y}{\sin^2 y} dy = -\int \frac{1}{1-\cos^2 y} d \cos y$$

$$\int \frac{1}{\cos^2 y - 1} d \cos y = \frac{1}{2} \ln \left(\frac{\cos y - 1}{\cos y + 1} \right) + C = \frac{1}{2} \ln \left(\frac{\sqrt{1-e^{2x}} - 1}{\sqrt{1-e^{2x}} + 1} \right) + C$$

↗ 负的

$$\therefore \int \frac{\arcsin e^x}{e^x} dx = -\frac{\arcsin e^x}{e^x} + \frac{1}{2} \ln \left(\frac{\sqrt{1-e^{2x}} - 1}{\sqrt{1-e^{2x}} + 1} \right) + C = \frac{-e^{2x}}{(\sqrt{1-e^{2x}} + 1)^2}$$

$t = e^x \quad x = \ln t \quad dx = \frac{1}{t} dt$

$$\int \frac{\arcsin t}{t} \cdot \frac{1}{t} dt = \int \frac{\arcsin t}{t^2} dt = -\int \arcsin t d \frac{1}{t} = -\frac{\arcsin t}{t} + \int \frac{1}{t} \cdot \frac{1}{\sqrt{1-t^2}} dt$$

$$\int \frac{dt}{t\sqrt{1-t^2}} \quad \text{令 } t = \sin m, m \in (-\frac{\pi}{2}, \frac{\pi}{2}) = \int \frac{\cos m dm}{\sin m \cdot \cos m} = \int \csc m dm$$

$$= -\ln |\csc m + \cot m| + C \quad \begin{array}{c} \triangle \\ \hline m \end{array} \quad t = -\ln \left| \frac{1}{t} + \frac{\sqrt{1-t^2}}{t} \right| + C$$

$$= -\frac{\arcsin t}{t} - \ln \left| \frac{1 + \sqrt{1-e^{2x}}}{e^x} \right| + C \quad \rightarrow \text{明确对应关系}$$

$$= -\frac{\arcsin t}{t} + \ln \left(\frac{e^x}{1 + \sqrt{1-e^{2x}}} \right)$$

09 / 定积分

考点：

- (1) 利用黎曼和求极限；
- (2) 变上限定积分-变上限定积分用于不等式证明
- (3) 定积分计算的三大法宝：**N-L公式，换元积分法，分部积分法；**
- (4) 定积分相关不等式证明（较难）

Step1.分割

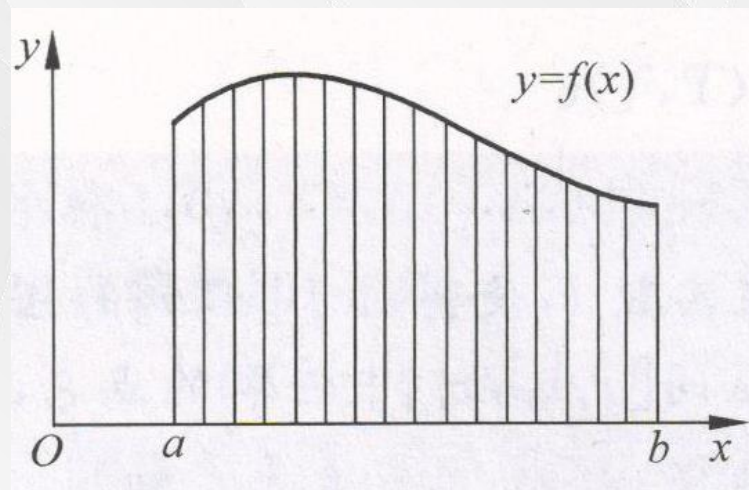
$$T : a = x_0 < x_1 < \cdots < x_n = b.$$

$$\Delta x_i \triangleq x_i - x_{i-1}, \quad |T| = \max_{1 \leq i \leq n} \{\Delta x_i\}.$$

Step2.取标志点 $\xi_i \in [x_{i-1}, x_i]$.

$$\text{Step3.近似求和. } S \approx \sum_{i=1}^n f(\xi_i) \Delta x_i,$$

$$\text{Step4.取极限. } \lim_{|T| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = S.$$



Def. 设 f 为闭区间 $[a, b]$ 上的有界函数, 若存在实数 I , s.t. 对 $[a, b]$ 的任何一个分割 $T: a = x_0 < x_1 < \cdots < x_n = b$, 对任意 $\{\xi_i\}, \xi_i \in [x_{i-1}, x_i], 1 \leq i \leq n$, 只要 $|T| = \max_{1 \leq i \leq n} \{\Delta x_i\} \rightarrow 0$, 就有

$$\lim_{|T| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = I,$$

即 $\forall \varepsilon > 0, \exists \delta > 0$, 当 $|T| < \delta$ 时, 无论 $\xi_i \in [x_{i-1}, x_i]$ 如何取, 都有

$$\left| \sum_{i=1}^n f(\xi_i) \Delta x_i - I \right| < \varepsilon,$$

则称 f 在 $[a, b]$ 上Riemann可积, 称 I 为 f 在 $[a, b]$ 上的Riemann

积分, 记为 $\int_a^b f(x) dx = I$.

a, b, f, x 分别称为积分上、下限, 被积函数和积分变量.

09 / 定积分——利用黎曼和求极限

Ex. $\lim_{n \rightarrow \infty} \sum_{k=n+1}^{2n} \frac{1}{k} = \underline{\hspace{2cm}}$.

$\rightarrow \Delta x_i \cdot f(\xi_i)$

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \dots + \frac{1}{n+n} = \frac{1}{n} \left(\frac{1}{1+1/n} + \frac{1}{1+2/n} + \dots + \frac{1}{1+n/n} \right)$$

$$f(x) := \frac{1}{1+x} = \frac{1}{n} \left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right)$$

$$\lim_{n \rightarrow \infty} \sum_{k=n+1}^{2n} \frac{1}{k} = \lim_{n \rightarrow \infty} \frac{1}{n} \left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right) = \int_0^1 f(x) dx = \int_0^1 \frac{1}{1+x} dx = \ln 2$$

$$\int_0^1 \frac{1}{1+x} dx = \ln(1+x) \Big|_0^1 = \ln 2$$

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n\sqrt{n}} (1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n}) &= \frac{1}{n} \cdot (\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \dots + \sqrt{\frac{n}{n}}) \\ &= \sum \Delta x_i \cdot f(\xi_i) = \int_0^1 \sqrt{x} dx = \frac{2}{3} \cdot x^{\frac{3}{2}} \Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

09 / 定积分——利用黎曼和求极限

Ex. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{(k+n)^2} = \underline{\hspace{2cm}}$. $\frac{n}{k^2 + 2nk + n^2} = \frac{1}{n} \cdot \frac{1}{(\frac{k}{n})^2 + (\frac{k}{n}) + 1}$

$2? = \frac{1}{n} \cdot \sum \frac{1}{(\frac{k}{n})^2 + (\frac{k}{n}) + 1}$

$$\sum_{k=1}^n \frac{n}{(k+n)^2} = \frac{n}{(1+n)^2} + \dots + \frac{n}{(n+n)^2} = \frac{1}{n} \left(\frac{n^2}{(1+n)^2} + \dots + \frac{n^2}{(n+n)^2} \right) = \int_0^1 \frac{1}{x^2+x+1} dx$$

$$\frac{1}{n} \left(\frac{1}{(1/n+1)^2} + \dots + \frac{1}{(n/n+1)^2} \right) = \frac{1}{n} \left(f\left(\frac{1}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right) \rightarrow f(x) = \frac{1}{(1+x)^2}$$

$$\int_0^1 f(x) dx = \int_0^1 \frac{1}{(1+x)^2} dx = \frac{1}{2}$$

~~$$\int \frac{1}{x^2+x+1} dx = \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} d(x+\frac{1}{2}) = \int \frac{1}{t^2 + \frac{3}{4}} dt = \frac{2}{\sqrt{3}} \arctan \frac{2t}{\sqrt{3}} \Big|_{\frac{1}{2}}$$~~

~~$$\left(\frac{\pi}{3} - \frac{\pi}{6} \right) \frac{2}{\sqrt{3}}$$~~

~~$$\frac{\pi}{6} \cdot \frac{2\sqrt{3}}{3} = \frac{\sqrt{3}\pi}{9}$$~~

Thm.(微积分基本定理)

$f \in R[a, b], F(x) = \int_a^x f(t)dt \ (a \leq x \leq b)$, 则

(1) $F \in C[a, b]$;

(2) 若 f 在 $x_0 \in [a, b]$ 连续, 则 F 在 x_0 可导, 且 $F'(x_0) = f(x_0)$;

(3) 若 $f \in C[a, b]$, 则 F 是 f 在 $[a, b]$ 上的一个原函数. 若 G 为 f 的任一个原函数, 则

$$\int_a^b f(t)dt = G(b) - G(a) \triangleq G(x) \Big|_a^b. \text{ (Newton-Leibniz)}$$

09 / 定积分——变上限积分

Ex. f 连续, u, v 可导, $G(x) = \int_{v(x)}^{u(x)} f(t) dt$, 求 $G'(x)$.

解: 令 $F(u) = \int_a^u f(t) dt$, 则 $F'(u) = f(u)$.

$$\begin{aligned} G(x) &= \int_a^{u(x)} f(t) dt - \int_a^{v(x)} f(t) dt \\ &= F(u(x)) - F(v(x)) \end{aligned}$$

$$\begin{aligned} G'(x) &= F'(u(x)) \cdot u'(x) - F'(v(x)) \cdot v'(x) \\ &= f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x). \square \end{aligned}$$

$$\begin{aligned} G(x) &= \int_0^{u(x)} f(t) dt - \int_0^{v(x)} f(t) dt \\ G'(x) &= f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x) \end{aligned}$$

Ex. (2019) 求 $\int_{x^2}^{x^3} \frac{\sin t}{t} dt$ 的一阶导数 _____

$$\begin{aligned} \text{解: } 3x^2 \frac{\sin(x^3)}{x^3} - 2x \frac{\sin(x^2)}{x^2} &= \frac{3\sin(x^3) - 2\sin(x^2)}{x} \\ &= \frac{\sin x^3}{x^3} \cdot 3x^2 - \frac{\sin x^2}{x^2} \cdot 2x \end{aligned}$$

Ex. (未知年份) 求 $\int_{x^2}^{2x} \ln(1 + \sin t) dt$ 的一阶导数 _____

$$\begin{aligned} &\int_0^{2x} \ln(1 + \sin t) dt - \int_0^{x^2} \ln(1 + \sin t) dt \\ &= \ln(1 + \sin 2x) \cdot 2 - \ln(1 + \sin x^2) \cdot 2x \end{aligned}$$

Note. 被积函数中, 如果含有被积变量以外的其他变量, 必须设法分离出来

Ex. f 连续, $F(x) = \int_a^x (x-t)f(t)dt$, 求 $F''(x)$.

解: $F(x) = x \int_a^x f(t)dt - \int_a^x tf(t)dt$, x 在积分过程中是个常数

$$F'(x) = \int_a^x f(t)dt + xf(x) - xf(x) = \int_a^x f(t)dt, F''(x) = f(x). \square$$

Ex. (变上限积分, 适合于使用洛必达法则) $\lim_{x \rightarrow +\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt} = \lim_{x \rightarrow +\infty} \frac{2e^{x^2} \int_0^x e^{t^2} dt}{e^{2x^2}}$

$$= \lim_{x \rightarrow +\infty} \frac{2 \int_0^x e^{t^2} dt}{e^{x^2}} = \lim_{x \rightarrow +\infty} \frac{2e^{x^2}}{2xe^{x^2}} = 0. \square$$

Ex. (2019) $f \in C[0, +\infty)$, $\int_a^{ab} f(x)dx$ 和 a 无关, $a, b > 0$. 求证: $f(x) = c/x$.

解: $\int_a^{ab} f(x)dx$ 和 a 无关, $\therefore \frac{d \int_a^{ab} f(x)dx}{da} = 0 \quad \therefore bf(ab) - f(a) = 0, \forall a, b > 0$

$$\therefore f(ab) = \frac{1}{b} f(a), \forall a, b > 0 \quad \therefore \text{取 } a = 1, \therefore f(b) = \frac{f(1)}{b}$$

$$\therefore c = f(1) \quad \int_0^{ab} f(x)dx - \int_0^a f(x)dx$$

$$f(ab) \cdot b - f(a) = 0$$

$$f(a) = f(ab) \cdot b$$

$$f(1) = f(b) \cdot b$$

$$f(x) = \frac{f(1)}{x} = \frac{c}{x}$$

Note. 被积函数中, 如果含有被积变量以外的其他变量, 必须设法分离出来

Ex. f 连续, $\lim_{x \rightarrow 0} \frac{f(x)}{x} = a$, 定义 $\phi(x) = \int_0^1 f(xt) dt$, 计算 $\phi'(x)$,

并考察 $\phi'(x)$ 在 0 处的连续性. $\because f$ 连续, $\therefore f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x \frac{f(x)}{x} = 0 \times a = 0$

解: $\phi(x) = \int_0^1 f(xt) dt = \frac{1}{x} \int_0^1 f(xt) dx = \frac{1}{x} \int_0^x f(y) dy \quad \therefore \phi(x) = \frac{1}{x} \int_0^x f(y) dy$

$$\therefore \phi'(x) = \frac{f(x)x - \int_0^x f(y) dy}{x^2}$$

$$\therefore \lim_{x \rightarrow 0} \phi'(x) = \lim_{x \rightarrow 0} \frac{f(x)x - \int_0^x f(y) dy}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{x} - \frac{\int_0^x f(y) dy}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{f(x)}{x} - \lim_{x \rightarrow 0} \frac{\int_0^x f(y) dy}{x^2} = a - \lim_{x \rightarrow 0} \frac{\int_0^x f(y) dy}{x^2} \stackrel{L-H}{=} a - \lim_{x \rightarrow 0} \frac{f(x)}{2x} = \frac{a}{2}$$

$$\phi'(0) = \lim_{x \rightarrow 0} \frac{\phi(x) - \phi(0)}{x} \xrightarrow{\text{上定义}} = \lim_{x \rightarrow 0} \frac{\phi(x)}{x} = \lim_{x \rightarrow 0} \frac{\int_0^x f(y) dy}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{2x} = a/2$$

x 不是变量, x 是变化的常数

核心构造: 当 x 不是积分因子时, x 可以移出 \int , 也可以抓进 dt

Thm: 若 f, g 在 x_0 可导, 则 $f \cdot g$ 在 x_0 可导.

$y = f(x) \cdot g(x)$. 若 $g(x)$ 或 $f(x)$ 在 $x = x_0$ 处不可导, 则 y 不一定不可导.

09 / 定积分——利用变上限积分完成不等式证明

Note. 有一些看起来很难的定积分不等式证明, 可以利用这个方法

Ex. f 在 $[a, b]$ 上二阶导函数连续, f 上凸, 求证: $\int_a^b f(x) dx \geq \frac{f(a)+f(b)}{2}(b-a)$

证明: 考察函数 $g(x) = \int_a^x f(t) dt - \frac{f(a)+f(x)}{2}(x-a)$ 目标: $g(b) \geq 0 \Rightarrow$ 构造变上限积分法

$$g'(x) = f(x) - \frac{f'(x)}{2}(x-a) - \frac{f(a)+f(x)}{2} = \frac{1}{2}(f(x) - f(a) - f'(x)(x-a))$$

$$= \frac{1}{2}(f'(\xi)(x-a) - f'(x)(x-a)), a < \xi < x = \frac{1}{2}(x-a)(f'(\xi) - f'(x)), a < \xi < x$$

利用中值定理

$$= \frac{1}{2}(x-a)(\xi-x)f''(\zeta) \geq 0, a < \xi < \zeta < x \quad g'(x) \geq 0 \quad g(a) = 0, \therefore g(x) \geq 0, \forall x > a$$

09 / 定积分——利用变上限积分完成不等式证明

Note. 有一些看起来很难的定积分不等式证明, 可以利用这个方法

Hw. f 在 $[a, b]$ 上二阶导函数连续, f 上凸, 求证: $\int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right) \frac{(b-a)}{2}$

$$\int_a^x f(t) dt \leq f\left(\frac{x+a}{2}\right) \cdot (x-a)$$

$$G(x) = \int_a^x f(t) dt - f\left(\frac{x+a}{2}\right) \cdot (x-a)$$

$$G'(x) = f(x) - f'\left(\frac{x+a}{2}\right) \cdot \frac{x-a}{2} - f\left(\frac{x+a}{2}\right)$$

$$= f(x) - f\left(\frac{x+a}{2}\right) - f'\left(\frac{x+a}{2}\right) \cdot \frac{x-a}{2}$$

$$= [f'(\xi) - f'\left(\frac{x+a}{2}\right)] \cdot \frac{x-a}{2}$$

$$\xi \in \left(\frac{x+a}{2}, x\right) = f'(\xi) \cdot \left(\xi - \frac{x+a}{2}\right) \cdot \frac{x-a}{2}$$

$$\xi \in \left(\frac{x+a}{2}, \xi\right)$$

09 / 定积分——利用变上限积分完成不等式证明

Ex. f 在 $[0,1]$ 上有一阶导数, $f(0) = 0, 0 \leq f'(x) \leq 1$, 求证: $\int_0^1 f^3(x)dx \leq (\int_0^1 f(x)dx)^2$

$$G(t) = \int_0^t f^3(x)dx - (\int_0^t f(x)dx)^2 \quad G(1) = \int_0^1 f^3(x)dx - (\int_0^1 f(x)dx)^2 \quad G(0) = 0$$

$$G'(t) = f^3(t) - 2f(t)(\int_0^t f(x)dx) = f(t)(f^2(t) - 2\int_0^t f(x)dx)$$

希望说明 $G'(t) \leq 0, \because f(t) \geq 0, \therefore$ 只需证明 $f^2(t) - 2\int_0^t f(x)dx \leq 0$

$$Q(t) = f^2(t) - 2\int_0^t f(x)dx, Q(0) = 0, \text{下证 } Q'(t) \leq 0$$

$$Q'(t) = 2f(t)f'(t) - 2f(t) = 2f(t)(f'(t) - 1)$$

$\because 0 \leq f'(x) \leq 1, \therefore f'(x) - 1 \leq 0$, 又 $f(x) \geq 0$

$$\therefore Q'(t) \leq 0 \quad G(x) = \int_0^x f^3(t)dt - (\int_0^x f(t)dt)^2 \leq 0$$

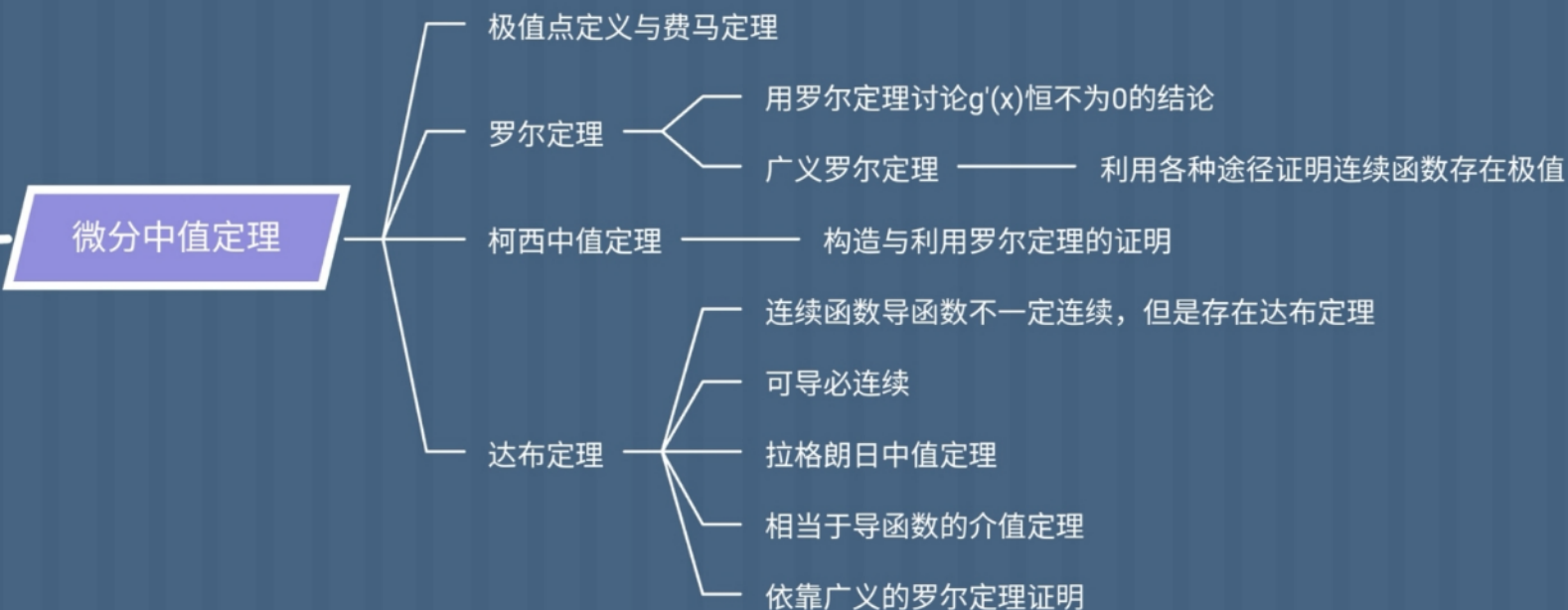
$$G'(x) = f^3(x) - 2\int_0^x f(t)dt \cdot f(x) \\ = f(x) \cdot (f^2(x) - 2\int_0^x f(t)dt) \leq 0$$

$$H(x) = f^2(x) - 2\int_0^x f(t)dt \leq 0$$

$$H(0) = 0 \quad H'(x) = 2f(x) \cdot f'(x) - 2f(x) \\ = 2f(x) \cdot (f'(x) - 1) < 0$$

09 / 定积分——利用变上限积分完成不等式证明

Note. 变上限积分函数本身是一个有很好性质的函数.
期中之前学过的中值定理和泰勒公式可以用上



$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-x_0)^{n+1}$$

Note. 结合泰勒公式 \ 中值定理

Ex. f 在 $[0,1]$ 可导, $f(1) = 4 \int_0^{1/4} e^{1-x^3} f(x) dx$. 则 $\exists \xi \in (0,1)$, s.t.

$f'(\xi) = 3\xi^2 f(\xi)$. 回溯

$g(x) = e^{1-x^3} \cdot f(x)$ $g'(x) = e^{1-x^3} \cdot (-3x^2 \cdot f(x) + f'(x) \cdot e^{1-x^3}) = 0$
 $f(\xi) = 3\xi^2 \cdot f(\xi)$

Proof. 由积分第一中值定理, $\exists \eta \in (0,1)$, s.t.

$f(1) = 4 \int_0^{1/4} e^{1-x^3} f(x) dx = e^{1-\eta^3} f(\eta)$.

令 $g(x) = e^{1-x^3} f(x)$, 则

$h(x) = e^{1-x^3} \cdot f(x)$ $T(x) = 1$
 $4 \cdot h(\eta) \cdot (\frac{1}{4} - 0) = h(\eta)$

$g'(x) = e^{1-x^3} (f'(x) - 3x^2 f(x))$, $g(\eta) = g(1)$.

由 Rolle 定理, $\exists \xi \in (\eta, 1) \subset (0,1)$, s.t. $g'(\xi) = 0$, 即

$f'(\xi) = 3\xi^2 f(\xi)$. \square

$\int_a^b f(x)g(x) dx = f(\xi) \cdot \int_a^b g(x) dx$

① $g(x)$ 不变号 ② $\xi \in [a,b]$ 且可能与
其他参数有关

$\int_a^b f(x) dx = f(\xi) \cdot (b-a)$

Note. 结合泰勒公式 \ 中值定理

Ex. f 在 $[0,1]$ 可导, $f(1) = 4 \int_0^{1/4} e^{1-x^3} f(x) dx$. 则 $\exists \xi \in (0,1)$, s.t.

$$f'(\xi) = 3\xi^2 f(\xi).$$

Proof. 由积分第一中值定理, $\exists \eta \in (0,1)$, s.t.

$$f(1) = 4 \int_0^{1/4} e^{1-x^3} f(x) dx = e^{1-\eta^3} f(\eta).$$

令 $g(x) = e^{1-x^3} f(x)$, 则

$$g'(x) = e^{1-x^3} (f'(x) - 3x^2 f(x)), \quad g(\eta) = g(1).$$

由 Rolle 定理, $\exists \xi \in (\eta, 1) \subset (0,1)$, s.t. $g'(\xi) = 0$, 即

$$f'(\xi) = 3\xi^2 f(\xi). \square$$

Note. 结合泰勒公式 \ 中值定理 $g(y) = \int_0^y \frac{1}{1+x} dx$, $g'(y) = \frac{1}{1+y}$, $g''(y) = -(\frac{1}{1+y})^2$

Ex. (A Hard Problem!!) 计算 $\lim_{n \rightarrow \infty} n^2 (\ln 2 - \sum_{i=1}^n \frac{2n+2i+1}{2(n+i)^2})$

分析. 原式 = $n^2 (\ln 2 - \sum_{i=1}^n \frac{2(n+i)+1}{2(n+i)^2}) = n^2 (\ln 2 - \sum_{i=1}^n \frac{1}{n+i} - \frac{1}{2} \sum_{i=1}^n \frac{1}{(n+i)^2})$

$$= n^2 (\ln 2 - \frac{1}{n} \sum_{i=1}^n \frac{1}{1+i/n} - \frac{1}{2n^2} \sum_{i=1}^n \frac{1}{(1+i/n)^2})$$

$$\ln 2 - \frac{1}{n} \sum_{i=1}^n \frac{1}{1+i/n} - \frac{1}{2n^2} \sum_{i=1}^n \frac{1}{(1+i/n)^2} = \int_0^1 \frac{1}{1+x} dx - \frac{1}{n} \sum_{i=1}^n \frac{1}{1+i/n} - \frac{1}{2n^2} \sum_{i=1}^n \frac{1}{(1+i/n)^2}$$

$$\int_0^1 \frac{1}{1+x} dx - \frac{1}{n} \sum_{i=1}^n \frac{1}{1+i/n} - \frac{1}{2n^2} \sum_{i=1}^n \frac{1}{(1+i/n)^2}$$

$$= \sum_{i=1}^n (\int_{(i-1)/n}^{i/n} \frac{1}{1+x} dx - \frac{1}{n} \frac{1}{1+i/n} - \frac{1}{2n^2} \frac{1}{(1+i/n)^2})$$

$$= \sum_{i=1}^n (g(\frac{i}{n}) - g(\frac{i-1}{n}) - \frac{1}{n} g'(\frac{i}{n}) + \frac{1}{2n^2} g''(\frac{i}{n}))$$

Note. 结合泰勒公式 \ 中值定理 $g(y) = \int_0^y \frac{1}{1+x} dx, g'(y) = \frac{1}{1+y}, g''(y) = -(\frac{1}{1+y})^2$

Ex. 计算 $\lim_{n \rightarrow \infty} n^2 (\ln 2 - \sum_{i=1}^n \frac{2n+2i+1}{2(n+i)^2})$

分析. ∴ 原式 = $n^2 \sum_{i=1}^n (g(\frac{i}{n}) - g(\frac{i-1}{n}) - \frac{1}{n} g'(\frac{i}{n}) + \frac{1}{2n^2} g''(\frac{i}{n}))$

$$\because g(\frac{i-1}{n}) = g(\frac{i}{n}) - \frac{1}{n} g'(\frac{i}{n}) + \frac{1}{2n^2} g''(\frac{i}{n}) - \frac{1}{6n^3} g'''(\xi)$$

$$\therefore g(\frac{i}{n}) - g(\frac{i-1}{n}) = \frac{1}{n} g'(\frac{i}{n}) - \frac{1}{2n^2} g''(\frac{i}{n}) + \frac{1}{6n^3} g'''(\xi_i), \xi \in [\frac{i-1}{n}, \frac{i}{n}]$$

$$\therefore n^2 \sum_{i=1}^n (g(\frac{i}{n}) - g(\frac{i-1}{n}) - \frac{1}{n} g'(\frac{i}{n}) + \frac{1}{2n^2} g''(\frac{i}{n})) = n^2 \sum_{i=1}^n \frac{1}{6n^3} g'''(\xi_i)$$

$$= \frac{1}{6} \frac{1}{n} \sum_{i=1}^n g'''(\xi_i) \rightarrow \frac{1}{6} \int_0^1 g'''(x) dx = \frac{g''(1) - g''(0)}{6} = \frac{1}{8}$$

∴ $\xi \in (\frac{i-1}{n}, \frac{i}{n}) \quad i=1 \sim n$. 故积分区间为 (0,1)

09 / 定积分——利用变上限积分完成不等式证明

Note. 结合泰勒公式 \ 中值定理

$$\text{Ex. } \sum_{k=1}^{n-1} \frac{\ln k + \ln(k+1)}{2} \leq \int_1^n \ln x dx \leq \sum_{k=1}^{n-1} \frac{\ln k + \ln(k+1)}{2} + \frac{1}{8}$$

$$\text{即 } \frac{n^{(n+\frac{1}{2})}}{e^n} e^{\frac{7}{8}} \leq n! \leq \frac{n^{(n+\frac{1}{2})}}{e^n} e, \forall n \in \mathbb{N}^+$$

$$\text{注: } n! \sim \sqrt{2\pi} \frac{n^{(n+\frac{1}{2})}}{e^n},$$

$$\sqrt{2\pi} \approx 2.5066, e^{7/8} \approx 2.3988, e \approx 2.7183$$

$$\frac{\ln 1 + \ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \dots + \frac{\ln^{n-2} + \ln^n}{2}$$

$$= \frac{\ln^n}{2} + \ln^{(n-1)}! \Rightarrow \sqrt{n} \cdot (n-1)!$$

$$\times (\ln^{x-1}) \Big|_1^n = n(\ln^n - 1) + 1 = n \ln^n - n + 1$$

$$\Rightarrow \frac{n^n}{e^n} \cdot e$$

$$\sqrt{n} \cdot (n-1)! \cdot e^{\frac{1}{8}}$$

$$\sqrt{n} \cdot (n-1)! \leq \frac{n^n}{e^n} \cdot e \leq \sqrt{n} \cdot (n-1)! \cdot e^{\frac{1}{8}}$$

$$\frac{\sqrt{n}}{n!} \cdot e^{-\frac{1}{8}} \leq \frac{e^{n-1}}{n^n} \leq \frac{\sqrt{n}}{n!}$$

$$\frac{n^{(n+\frac{1}{2})}}{e^n} \cdot e^{\frac{7}{8}} \leq n! \leq \frac{n^{(n+\frac{1}{2})}}{e^n}$$

09 / 定积分——N-L公式计算定积分

通过找原函数求定积分,但是要注意符号.

Ex. $\int_0^{\pi} \sqrt{\sin x - \sin^3 x} dx = \underline{\hspace{2cm}}$

绝对值万分重要

$$\int_0^{\pi} \sqrt{\sin x - \sin^3 x} dx = \int_0^{\pi} \sqrt{\sin x (1 - \sin^2 x)} dx = \int_0^{\pi} \sqrt{\sin x \cos^2 x} dx = \int_0^{\pi} \sqrt{\sin x} |\cos x| dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin x} (-\cos x) dx = \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cos x dx - \int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin x} (\cos x) dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin x} d \sin x - \int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin x} d \sin x = \frac{2}{3} \sin^{3/2} x \Big|_0^{\frac{\pi}{2}} - \frac{2}{3} \sin^{3/2} x \Big|_{\frac{\pi}{2}}^{\pi} = 4/3$$

09 / 定积分——换元、分部积分

Thm.(定积分的换元法) $f \in C[a, b]$, $\varphi \in C^1[\alpha, \beta]$, $\varphi(\alpha) = a$,
 $\varphi(\beta) = b$, $a \leq \varphi(t) \leq b$, 则 $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$.

Thm.(定积分的分部积分法) $u, v \in C[a, b]$, 则

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx.$$

Ex.(1) $f \in C[a, b]$, 则 $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$; \Rightarrow 基理

$$(2) I = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x}{x(\pi-2x)} dx = \frac{1}{\pi} \ln 2. \quad \left(\frac{\pi}{2} - x \right) \left(\pi - \left(\frac{\pi}{2} - x \right) \cdot 2 \right) \\ = 2x \cdot \left(\frac{1}{2}\pi - x \right) = x(\pi - 2x)$$

Proof.(1) $\int_a^b f(a+b-x) dx$

$$\underline{t = a+b-x} \quad - \int_b^a f(t) dt = \int_a^b f(t) dt = \int_a^b f(x) dx.$$

$$(2) \text{利用(1), } I = \int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{x(\pi-2x)} dx = \frac{1}{2} \int_{\pi/6}^{\pi/3} \frac{1}{x(\pi-2x)} dx \\ = \frac{1}{\pi} \int_{\pi/6}^{\pi/3} \left(\frac{1}{2x} + \frac{1}{\pi-2x} \right) dx = \frac{1}{2\pi} \ln \frac{2x}{\pi-2x} \Big|_{\pi/6}^{\pi/3} = \frac{1}{\pi} \ln 2. \square$$

$$\frac{1}{\pi} \int_{\pi/6}^{\pi/3} \left(\frac{1}{2x} + \frac{1}{\pi-2x} \right) dx = \frac{1}{\pi} \cdot \left(\frac{1}{2} \ln 2x - \frac{1}{2} \ln \pi - 2x \right) \Big|_{\pi/6}^{\pi/3} \\ = \frac{1}{2\pi} \ln \frac{2x}{\pi-2x} \Big|_{\pi/6}^{\pi/3} = \frac{1}{2\pi} (\ln 2 - \ln \frac{1}{2}) = \frac{1}{\pi} \ln 2$$

Ex. $f(x) = \int_1^{x^2} \exp(-t^2) dt$, 计算 $I = \int_0^1 xf(x) dx$

解. $f(x) = \int_1^{x^2} e^{-t^2} dt$

怎么可能让你两次积分!

利用 $f(0) = 0$ 来昏布鸡昏

$$\int_0^1 xf(x) dx = \int_0^1 x \left(\int_1^{x^2} \exp(-t^2) dt \right) dx = \frac{1}{2} \int_0^1 \left(\int_1^{x^2} \exp(-t^2) dt \right) dx^2$$

$$= \frac{1}{2} x^2 \left(\int_1^{x^2} \exp(-t^2) dt \right) \Big|_0^1 - \frac{1}{2} \int_0^1 x^2 \times 2x \exp(-x^4) dx$$

分部积分

$$G(0) = G(1) = 0 = - \int_0^1 x^3 \exp(-x^4) dx = - \frac{1}{4} \int_0^1 \exp(-x^4) dx^4$$

后定积分部分

为0是基本操作

$$= \frac{1}{4} \left(\frac{1}{e} - 1 \right)$$

$$- \int_0^1 x^3 \cdot e^{-x^4} dx$$

$$= - \frac{1}{4} \int_0^1 e^{-x^4} dx^4$$

$$= - \frac{1}{4} \int_0^1 e^{-t} dt$$

$$= + \frac{1}{4} e^{-t} \Big|_0^1 = \frac{1}{4e} - \frac{1}{4}$$

Ex. 证明 $I_n \triangleq \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$, 并求 I_n .

Proof. 令 $t = \frac{\pi}{2} - x$, 则

$$\int_0^{\pi/2} \sin^n x dx = -\int_{\pi/2}^0 \sin^n \left(\frac{\pi}{2} - t\right) dt = \int_0^{\pi/2} \cos^n t dt.$$

$$\begin{aligned} I_n &= -\int_0^{\pi/2} \sin^{n-1} x d \cos x \\ &= -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^2 x \sin^{n-2} x dx \\ &= (n-1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^{n-2} x dx \\ &= (n-1) I_{n-2} - (n-1) I_n. \end{aligned}$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

$$I_1 = \int_0^{\pi/2} \sin x dx = -\cos x \Big|_0^{\pi/2} = 1,$$

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2},$$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} \cdot 1 = \frac{(2n-2)!!}{(2n-1)!!}. \square$$

09 / 定积分——设法消除未知项

Ex. $\int_{-\pi}^{\pi} \frac{\cos x}{1+e^x} dx = \underline{\hspace{2cm}}$ $\frac{\cos x}{1+e^x}$ 的原函数不好求!

$$\text{原式} = \int_0^{\pi} \frac{\cos x}{1+e^x} dx + \int_{-\pi}^0 \frac{\cos x}{1+e^x} dx$$

$$\int_{-\pi}^0 \frac{\cos x}{1+e^x} dx \stackrel{y=-x}{=} \int_{\pi}^0 \frac{\cos(-y)}{1+e^{-y}} d-y = -\int_{\pi}^0 \frac{\cos(y)}{1+e^{-y}} dy = \int_0^{\pi} \frac{\cos(y)}{1+e^{-y}} dy = \int_0^{\pi} \frac{e^y \cos(y)}{1+e^y} dy$$

$$\therefore \text{原式} = \int_0^{\pi} \frac{\cos x}{1+e^x} dx + \int_0^{\pi} \frac{e^y \cos(y)}{1+e^y} dy = \int_0^{\pi} \frac{\cos x}{1+e^x} + \frac{e^x \cos(x)}{1+e^x} dx = \int_0^{\pi} \frac{\cos x (1+e^x)}{1+e^x} dx$$

$$= \int_0^{\pi} \cos x dx = \sin \pi - \sin 0 = 0$$

带有三角函数的定积分与对称区间定积分格外有技巧

变换积分区间

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\int_{-a}^a f(x) dx = \int_0^a f(x) dx + \int_0^a f(x) dx = \int_0^a (f(x) + f(-x)) dx$$

$$\int_{-\pi}^{\pi} \frac{\cos x}{1+e^x} dx = \int_0^{\pi} \left(\frac{\cos x}{1+e^x} + \frac{\cos x}{1+e^{-x}} \right) dx = \int_0^{\pi} \left(\frac{\cos x (e^x + 1)}{e^x + 1} \right) dx = 0$$

$$\int_{-\pi}^{\pi} \frac{\cos x}{1+e^x} dx = \int_{-\pi}^{\pi} \frac{\cos x}{1+e^{-x}} dx = \int_{-\pi}^{\pi} \frac{e^x \cdot \cos x}{e^x + 1} dx$$

$$\therefore \int_{-\pi}^{\pi} \frac{\cos x}{1+e^x} dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos x dx = \frac{1}{2} \cdot \sin x \Big|_{-\pi}^{\pi} = 0$$

09 / 定积分——奇函数

Ex. $\int_{-1/2}^{1/2} \frac{\arcsin x}{\sqrt{1-3x^2}} dx = \underline{\quad 0 \quad}$ $\frac{\arcsin x}{\sqrt{1-3x^2}}$ 的原函数不好求!

$$f(x) = \frac{\arcsin x}{\sqrt{1-3x^2}}, f(-x) = \frac{\arcsin(-x)}{\sqrt{1-3x^2}} = -f(x)$$

Hint. 奇函数在关于原点对称的区间上积分为0

$$\int_0^{\frac{1}{2}} \left(\frac{\arcsin x}{\sqrt{1-3x^2}} + \frac{\arcsin(-x)}{\sqrt{1-3x^2}} \right) dx = 0$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-3x^2}} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\arcsin(-x)}{\sqrt{1-3x^2}} dx = - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\arcsin(x)}{\sqrt{1-3x^2}} dx$$

如果定积分仅仅是在不定积分基础上加上N-L公式，
不利用定积分特殊性，那也太没意思了

09 / 定积分——偶函数

思路: $\frac{x^2}{4}$, $0 \sim x$, $xt - t^2$

Ex. $\int_0^x e^{xt-t^2} dt = e^{x^2/4} \int_0^x e^{-t^2/4} dt$

\Rightarrow 构造关于 $t = \frac{x}{2}$ 的广义对称区间

$\int_0^x e^{xt-t^2} dt = e^{\frac{x^2}{4}} \int_0^x e^{-x^2/4 + xt - t^2} dt = e^{\frac{x^2}{4}} \int_0^x e^{-(t-\frac{x}{2})^2} dt$

x 为参数
 $e^{x^2/4}$ 是手动构造而来

$2 \int_0^{\frac{x}{2}} e^{-(t-\frac{x}{2})^2} dt = 2 \int_{-\frac{x}{2}}^0 e^{-m^2} dm$

$\int_0^x e^{-(t-\frac{x}{2})^2} dt = \int_{-\frac{x}{2}}^{\frac{x}{2}} e^{-s^2} ds = 2 \int_0^{\frac{x}{2}} e^{-s^2} ds$ (偶函数)

$= -2 \int_0^{\frac{x}{2}} e^{-m^2} dm$
 $= 2 \int_0^{\frac{x}{2}} e^{-m^2} dm$

$\stackrel{s=q/2}{=} 2 \int_0^x e^{-q^2/4} d(q/2) = \int_0^x e^{-q^2/4} dq$

① $\int_0^x e^{x(x-t) - (x-t)^2} dt = \int_0^x e^{x^2 - xt - x^2 + 2xt - t^2} dt = \int_0^x e^{xt - t^2} dt$

$f(x) = f(a+b-x) \Rightarrow$ 偶函数

② $f(t)$ 为关于 $t=a$ 的偶函数

$\int_{a-x}^{a+x} f(t) dt = 2 \int_{a-x}^a f(t) dt = 2 \int_a^{a+x} f(t) dt$

$\int_{-x}^x f(t) dt = 2 \int_0^x f(t) dt = 2 \int_{-x}^0 f(t) dx$ 关键点是 $\int_{-x}^0 f(t) dt = - \int_0^{-x} f(t) dt$

切记不 \leftarrow 交换一次 $\leftarrow = \int_0^x f(t) dt$
要既换上下界又换上下界符号

09/ 定积分相关不等式证明

Prop3. (单调性) $f, g \in R[a, b]$, 且 $f(x) \leq g(x)$, 则

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Prop4.(积分估值) $f \in R[a, b] \Rightarrow |f| \in R[a, b]$, 且

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Thm. 设 $f \in C[a, b]$, $f(x) \geq 0$, $\int_a^b f(x)dx = 0$. 求证: $f(x) \equiv 0$.

Proof. 反证. 设 $f(x)$ 在不恒为0, 则 $\exists x_0 \in [a, b]$, s.t. $f(x_0) > 0$.

不妨设 $x_0 \in (a, b)$. $f \in C[a, b]$, 则 $\exists \delta > 0$, s.t.

$$f(x) > f(x_0)/2 > 0, \quad \forall x \in [x_0 - \delta, x_0 + \delta].$$

而 $f(x) \geq 0$, 于是

$$\begin{aligned} 0 &= \int_a^b f(x)dx = \int_a^{x_0-\delta} f(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)dx + \int_{x_0+\delta}^b f(x)dx \\ &\geq 0 + \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2} dx + 0 \geq f(x_0)\delta > 0, \text{矛盾.} \square \end{aligned}$$

Thm.(Cauchy不等式) $f, g \in R[a, b]$, 则

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx.$$

Proof. 令 $A = \int_a^b f^2(x)dx, B = \int_a^b f(x)g(x)dx, C = \int_a^b g^2(x)dx.$

则 $0 \leq \int_a^b [tf(x) + g(x)]^2 dx = At^2 + 2Bt + C, \forall t \in \mathbb{R}.$

故 $(2B)^2 - 4AC \leq 0. \square$

积的积分的平方 \leq 平方的积分的积

Thm.(积分第一中值定理) $f \in C[a, b], g \in R[a, b], g$ 不变号,

则 $\exists \xi \in [a, b], s.t. \int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx. (*)$

特别地, $g(x) \equiv 1$ 时, $\int_a^b f(x)dx = f(\xi)(b-a).$

Proof.不妨设 $g \geq 0$.记 f 在 $[a, b]$ 上的最大值与最小值为 $M,$

$m,$ 则 $m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx.$

若 $\int_a^b g(x)dx = 0,$ 则 $\int_a^b f(x)g(x)dx = 0, \forall \xi \in [a, b], (*)$ 成立.

若 $\int_a^b g(x)dx > 0, \exists \xi \in [a, b], s.t.$

$$f(\xi) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \in [m, M]. \square$$

Ex. $\forall x > 0$, 证明: $\int_0^x \frac{\sin t}{1+t} dt \geq 0$

分析: $\sin x$ 在 $[2k\pi, (2k+1)\pi]$ 取非负, $[(2k+1)\pi, (2k+2)\pi]$ 取非正

$\forall x > 0$, 1°. 如果 $\exists k > 0, k \in \mathbb{N}, s.t. 2k\pi \leq x \leq (2k+1)\pi$

$$\because \forall x, 2k\pi \leq x \leq (2k+1)\pi, \text{ 有 } \frac{\sin x}{1+x} \geq 0 \quad \therefore \int_0^x \frac{\sin t}{1+t} dt \geq \int_0^{2k\pi} \frac{\sin t}{1+t} dt$$

2°. 如果 $\exists k > 0, k \in \mathbb{N}, s.t. (2k+1)\pi \leq x \leq (2k+2)\pi$

$$\because \int_x^{(2k+2)\pi} \frac{\sin t}{1+t} dt \leq 0, \therefore \int_0^x \frac{\sin t}{1+t} dt \geq \int_0^x \frac{\sin t}{1+t} dt + \int_x^{(2k+2)\pi} \frac{\sin t}{1+t} dt = \int_0^{(2k+2)\pi} \frac{\sin t}{1+t} dt$$

如果可以证明: $\int_0^{2k\pi} \frac{\sin t}{1+t} dt \geq 0$, 问题即告解决.

Ex. $\forall x > 0$, 证明: $\int_0^x \frac{\sin t}{1+t} dt \geq 0$

现在证明: $\int_0^{2k\pi} \frac{\sin t}{1+t} dt \geq 0$.

$$\int_0^{2k\pi} \frac{\sin t}{1+t} dt = \int_0^{2\pi} \frac{\sin t}{1+t} dt + \int_{2\pi}^{4\pi} \frac{\sin t}{1+t} dt + \dots + \int_{(2k-2)\pi}^{2k\pi} \frac{\sin t}{1+t} dt$$

$$\therefore \int_{(2m-2)\pi}^{2m\pi} \frac{\sin t}{1+t} dt = \int_{(2m-2)\pi}^{(2m-1)\pi} \frac{\sin t}{1+t} dt + \int_{(2m-1)\pi}^{2m\pi} \frac{\sin t}{1+t} dt$$

$$= \int_{(2m-2)\pi}^{(2m-1)\pi} \frac{\sin t}{1+t} dt + \int_{(2m-2)\pi}^{(2m-1)\pi} \frac{\sin(t+\pi)}{1+t+\pi} d(t+\pi)$$

$$= \int_{(2m-2)\pi}^{(2m-1)\pi} \sin t \left(\frac{1}{1+t} - \frac{1}{1+t+\pi} \right) dt = \int_{(2m-2)\pi}^{(2m-1)\pi} \sin t \frac{\pi}{(1+t)(\pi+1+t)} dt \geq 0 \quad \therefore \int_0^{2k\pi} \frac{\sin t}{1+t} dt \geq 0$$

} trig 以 π 为单位的换元

Ex. $f \in C^1[0,1], f(0) = 0, f(1) = 1$

证明: $\int_0^1 |f(x) - f'(x)| dx \geq 1/e$

$$\because 0 \leq e^{-x} \leq 1, \therefore e^{-x} |f(x) - f'(x)| \leq |f(x) - f'(x)|$$

$$\begin{aligned} \therefore \int_0^1 |f(x) - f'(x)| dx &\geq \int_0^1 |e^{-x}(f(x) - f'(x))| dx \geq \left| \int_0^1 e^{-x}(f(x) - f'(x)) dx \right| \\ &= |e^{-x}f(x)|_0^1 = \frac{1}{e} \end{aligned}$$

与积分第一中值定理无关

$$[e^x \cdot f(x)]' = e^x \cdot (f(x) + f'(x))$$

$$[e^{-x} \cdot f(x)]' = e^{-x} \cdot (f'(x) - f(x))$$

} 思路来源

Ex. $f \in C^1[a, b]$, 证明: $\max_{a \leq x \leq b} |f(x)| \leq \frac{1}{b-a} \left| \int_a^b f(x) dx \right| + \int_a^b |f'(x)| dx$

证. $\because f \in C^1[a, b], \therefore \exists \xi \in [a, b], s.t. \frac{1}{b-a} \left| \int_a^b f(x) dx \right| = \left| \frac{1}{b-a} \int_a^b f(x) dx \right| = |f(\xi)|$

即证. $\max_{a \leq x \leq b} |f(x)| \leq |f(\xi)| + \int_a^b |f'(x)| dx$

$\forall x \in [a, b]. |f(x)| \leq |f(\xi)| + |f(x) - f(\xi)| \leq |f(\xi)| + \left| \int_{\xi}^x f'(x) dx \right|$

$\leq |f(\xi)| + \left| \int_{\xi}^x |f'(x)| dx \right| \leq |f(\xi)| + \int_a^b |f'(x)| dx$

由 x 的任意性. $\max_{a \leq x \leq b} |f(x)| \leq |f(\xi)| + \int_a^b |f'(x)| dx$

Ex. $f \in C^1[0,1], f(0) = 0$, 证明: $\int_0^1 f^2(x)dx \leq \int_0^1 f'^2(x)dx$

证. $\because f(x) = f(0) + \int_0^x f'(t)dt = \int_0^x f'(t)dt$

$f^2(x) \leq (\int_0^x f'(x)dx)^2 = (\int_0^x 1 \times f'(x)dx)^2$ 由柯西-施瓦茨不等式

$$\leq \int_0^x 1^2 dx \int_0^x f'^2(x)dx = x \int_0^x f'^2(x)dx \leq \int_0^1 f'^2(x)dx, \forall x \in [0,1]$$

$$\therefore \int_0^1 f^2(x)dx \leq \int_0^1 (\int_0^1 f'^2(x)dx)dx = \int_0^1 f'^2(x)dx$$

\searrow 常数

Ex. f 在 $[a, b]$ 上连续可导, 则 $\lim_{\lambda \rightarrow +\infty} \int_a^b f(x) \cos \lambda x dx = 0$

证. $\because \int_a^b f(x) \cos \lambda x dx = \frac{1}{\lambda} \int_a^b f(x) d \sin \lambda x \quad | f'(x) | \leq M, \forall x \in [a, b]$

$$= \frac{1}{\lambda} (f(b) \sin(\lambda b) - f(a) \sin(\lambda a)) - \frac{1}{\lambda} \int_a^b f'(x) \sin \lambda x dx$$

$$\left| \frac{1}{\lambda} \int_a^b f'(x) \sin \lambda x dx \right| \leq \frac{1}{\lambda} \int_a^b |f'(x) \sin \lambda x| dx \leq \frac{1}{\lambda} \int_a^b |f'(x)| dx \leq \frac{1}{\lambda} (b-a)M$$

$$\therefore \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} \int_a^b f'(x) \sin \lambda x dx = 0 \quad \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} (f(b) \sin(\lambda b) - f(a) \sin(\lambda a)) = 0$$

Ex.(A Hard Problem!) $\forall x > 0$, 证明: $\lim_{x \rightarrow +\infty} \int_0^x \frac{\sin t}{1+t} dt$ 存在

取 $k = [x/2\pi]$

$$\because \max\left(\int_0^{2k\pi} \frac{\sin t}{1+t} dt, \int_0^{(2k+2)\pi} \frac{\sin t}{1+t} dt\right) \leq \int_0^x \frac{\sin t}{1+t} dt \leq \int_0^{(2k+1)\pi} \frac{\sin t}{1+t} dt$$

转而考虑数列极限 $\lim_{n \rightarrow \infty} \int_0^{2n\pi} \frac{\sin t}{1+t} dt$ 和 $\lim_{n \rightarrow \infty} \int_0^{(2n+1)\pi} \frac{\sin t}{1+t} dt$

$$\left| \int_0^{2n\pi} \frac{\sin t}{1+t} dt \right| = \sum_{m=1}^n \int_{(2m-2)\pi}^{(2m-1)\pi} \sin t \frac{\pi}{(1+t)(\pi+1+t)} dt = \left| \sum_{m=1}^n \int_{(2m-2)\pi}^{2m\pi} \frac{\sin t}{1+t} dt \right|$$

$$\leq \pi \sum_{m=1}^n \int_{(2m-2)\pi}^{(2m-1)\pi} \frac{1}{(1+t)^2} dt \leq \pi \int_0^{(2n-1)\pi} \frac{1}{(1+t)^2} dt \leq \pi \int_0^{+\infty} \frac{1}{(1+t)^2} dt \Rightarrow \text{反常积分收敛}$$

$$\int_0^{2n\pi} \frac{\sin t}{1+t} dt \uparrow \therefore \int_0^{2n\pi} \frac{\sin t}{1+t} dt \text{ 有极限, 又 } \lim_{n \rightarrow \infty} \int_{(2n-1)\pi}^{2n\pi} \frac{\sin t}{1+t} dt = 0.$$

$$\int_{2n\pi}^{(2n+1)\pi} \frac{\sin t}{1+t} dt < \int_{2n\pi}^{(2n+1)\pi} \frac{1}{1+t} dt = \ln(t+1) \Big|_{2n\pi}^{(2n+1)\pi} = \ln \frac{2n\pi + \pi + 1}{2n\pi + 1} = 0$$

10 / 反常积分

• 无穷限积分

Def. 若 $\lim_{A \rightarrow +\infty} \int_a^A f(x) dx = I$, 则称 f 在 $[a, +\infty)$ 上的广义积分收敛, 称 I 为 f 在 $[a, +\infty)$ 上的广义积分(值), 记作

$$\int_a^{+\infty} f(x) dx = \lim_{A \rightarrow +\infty} \int_a^A f(x) dx.$$

若 $\lim_{A \rightarrow +\infty} \int_a^A f(x) dx$ 不存在, 则称广义积分 $\int_a^{+\infty} f(x) dx$ 发散.

Remark. $\int_{-\infty}^a f(x) dx \triangleq \lim_{A \rightarrow -\infty} \int_A^a f(x) dx.$

•瑕积分(无界函数积分)

Def. f 在 $[a, b)$ 上定义, 在 b 点附近无界(此时称 $x = b$ 为 f 的一个瑕点), 若 $\forall \delta \in (0, b - a), f \in R[a, b - \delta]$, 且

$$\lim_{\delta \rightarrow 0^+} \int_a^{b-\delta} f(x) dx = I,$$

则称 f 在 $[a, b)$ 上的瑕积分收敛, 称 I 为 f 在 $[a, b)$ 上的瑕积分(值), 记作

$$\int_a^b f(x) dx = \lim_{\delta \rightarrow 0^+} \int_a^{b-\delta} f(x) dx.$$

若 $\lim_{\delta \rightarrow 0^+} \int_a^{b-\delta} f(x) dx$ 不存在, 则称瑕积分 $\int_a^b f(x) dx$ 发散.

10 / 反常积分

考点：

- (1) 反常积分的计算；【类似于定积分，三个法宝】
- (2) 判定反常积分是否收敛；
- 重点掌握：不变号函数
- 了解：阿贝尔和狄利克雷判敛

10 反常积分的计算

$$\int_a^{+\infty} f(x)dx = \lim_{A \rightarrow +\infty} \int_a^A f(x)dx. \quad \int_a^b f(x)dx = \lim_{\delta \rightarrow 0^+} \int_a^{b-\delta} f(x)dx$$

Ex. $\int_0^{+\infty} x^{n-1} e^{-x} dx = \underline{\hspace{2cm}}$.

$$\int_0^{+\infty} x^{n-1} e^{-x} dx = \int_0^{+\infty} -x^{n-1} de^{-x} = -x^{n-1} e^{-x} - \int_0^{+\infty} e^{-x} d-x^{n-1} =$$

$$-x^{n-1} e^{-x} \Big|_0^{+\infty} + (n-1) \int_0^{+\infty} e^{-x} x^{n-2} dx$$

$$\int_0^{+\infty} x^{n-1} e^{-x} dx = (n-1) \int_0^{+\infty} x^{n-2} e^{-x} dx = (n-1)(n-2) \int_0^{+\infty} x^{n-3} e^{-x} dx = \dots = (n-1)! \int_0^{+\infty} e^{-x} dx = -e^{-x} \Big|_0^{+\infty} = 1$$

(Handwritten blue annotations: A bracket under the first integral term points to $\frac{x^{n-1}}{e^{-x}} \rightarrow 0$ as $x \rightarrow \infty$.)

10 反常积分的计算

Ex. 已知 $\int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2$, 求

(1). $\int_0^{\pi/2} x \cot x dx = \underline{\hspace{2cm}}$;

(1) $\int_a^{\pi/2} x \cot x dx = \int_a^{\pi/2} \frac{x \cos x}{\sin x} dx = \int_a^{\pi/2} \frac{x d(\sin x)}{\sin x}$

$= \int_a^{\pi/2} x d(\ln \sin x) = x \ln \sin x \Big|_a^{\pi/2} - \int_a^{\pi/2} \ln \sin x dx$

$\therefore \int_a^{\pi/2} x \cot x dx = -a \ln \sin a - \int_a^{\pi/2} \ln \sin x dx$

令 $a \rightarrow 0^+$, 有 $\int_0^{\pi/2} x \cot x dx = \lim_{a \rightarrow 0^+} \int_a^{\pi/2} x \cot x dx =$

$= \lim_{a \rightarrow 0^+} -a \ln \sin a - \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \ln \sin x dx = -\int_0^{\pi/2} \ln \sin x dx = \frac{\pi}{2} \ln 2$

一眼望穿, $\int \cot x dt = +\ln|\sin x| \rightarrow$ 记忆关键处不妨多记一记

$\int_0^{\pi/2} x \cot x dx = -\int_0^{\pi/2} x d \ln \sin x = -x \cdot \ln \sin x \Big|_0^{\pi/2} + \int_0^{\pi/2} \ln \sin x dx$

$\lim_{x \rightarrow 0^+} x \cdot \ln \sin x = x \cdot \ln x = 0$

\rightarrow 不要L-H: 等价无穷小一波带走

10 反常积分的计算

Ex. 已知 $\int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2$, 求

(2). $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx = \underline{\hspace{2cm}}$;

令 $x = \sin t$. $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} \frac{\ln \sin t}{\cos t} d \sin t = \int_0^{\pi/2} \ln \sin t dt$

(3). $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$.

令 $x = \tan t$. 则 $dx = \sec^2 t dt$

$\int_0^{\pi/4} \ln(1+\tan t) dt = \int_0^{\pi/4} \ln\left(\frac{\sin t + \cos t}{\cos t}\right) dt$

$\int_0^{\pi/4} \ln(1+\tan(\frac{\pi}{4}-t)) dt$?

不要犹豫, 这不是反常积分

$\ln(1+\tan t) \Big|_0^{\pi/4} - \int_0^{\pi/4} t d \ln(1+\tan t) \rightarrow \frac{\sec^2 t}{1+\tan t} \cdot t dt$

$\frac{1-\tan t}{1+\tan t} + 1 = \frac{2}{1+\tan t}$

$$\text{Ex. } I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx.$$

辅助角公式的应用

$$\text{解: } I = \int_0^{\pi/4} \ln(1 + \tan t) dt \quad (t = \arctan x)$$

$$= \int_0^{\pi/4} \ln(\sin t + \cos t) dt - \int_0^{\pi/4} \ln(\cos t) dt = I_1 - I_2.$$

$$I_1 = \int_0^{\pi/4} \left(\ln \sqrt{2} + \ln \sin\left(t + \frac{\pi}{4}\right) \right) dt$$

$$= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln \sin\left(t + \frac{\pi}{4}\right) dt$$

$$= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln \cos\left(\frac{\pi}{4} - t\right) dt = \frac{\pi}{8} \ln 2 + I_2. \quad I = \frac{\pi}{8} \ln 2. \square$$

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

10 反常积分的判敛

Hint. $\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx$, $\int_a^t f(x)dx$ 记成 $\phi(t)$

$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \phi(t)$ 如果 $\phi(t)$ 是单调增加的, 那么

$\lim_{t \rightarrow +\infty} \phi(t) = a$ 或是 $+\infty$ $\lim_{t \rightarrow +\infty} \phi(t) = a \Leftrightarrow \phi(t)$ 有上界

f 连续, $\phi(t)$ 是单调增加 $\Leftrightarrow f(x) \geq 0$

\therefore 优先考虑不变号函数 $f(x)$ 的积分 $\int_a^{+\infty} f(x)dx$

10 反常积分的判敛

Thm. (比较判敛法) 设 $0 \leq f(x) \leq Cg(x), \forall x > K$, 则

(1) $\int_a^{+\infty} g(x)dx$ 收敛 $\Rightarrow \int_a^{+\infty} f(x)dx$ 绝对收敛;

(2) $\int_a^{+\infty} f(x)dx$ 发散 $\Rightarrow \int_a^{+\infty} g(x)dx$ 发散.

Thm. (比较判敛法-极限形式) 设 f, g 非负, $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = C$.

(1) 若 $C > 0$, 则 $\int_a^{+\infty} f(x)dx$ 与 $\int_a^{+\infty} g(x)dx$ 同敛散;

(2) 若 $C = 0$, 且 $\int_a^{+\infty} g(x)dx$ 收敛, 则 $\int_a^{+\infty} f(x)dx$ 收敛;

(3) 若 $C = +\infty$, 且 $\int_a^{+\infty} g(x)dx$ 发散, 则 $\int_a^{+\infty} f(x)dx$ 发散.

} 不一定只用于 C 为正常数

10 反常积分的判敛

Ex. 判别广义积分的收敛性

$$\frac{C \cdot x^n}{x^{\frac{3}{2}}}$$

$$\int_0^{+\infty} \frac{\sin^2 x}{1+x^2} dx, \int_1^{+\infty} \frac{x^2 dx}{e^x + x}, \int_1^{+\infty} \frac{\ln x dx}{\sqrt{x^3 + 2x + 1}}, \int_2^{+\infty} \frac{\ln x dx}{x(\ln x + 9)}$$

解: (1) $\frac{\sin^2 x}{1+x^2} \leq \frac{1}{1+x^2}$, $\int_0^{+\infty} \frac{1}{1+x^2} dx$ 收敛, 故 $\int_0^{+\infty} \frac{\sin^2 x}{1+x^2} dx$ 绝对收敛.

(2) $\lim_{x \rightarrow +\infty} \frac{x^2}{e^x + x} / \frac{1}{x^2} = \lim_{x \rightarrow +\infty} \frac{x^4}{e^x + x} = 0$. $\int_1^{+\infty} \frac{dx}{x^2}$ 收敛, $\int_1^{+\infty} \frac{x^2 dx}{e^x + x}$ 收敛.

(3) $\lim_{x \rightarrow +\infty} \frac{\ln x}{\sqrt{x^3 + 2x + 1}} / \frac{1}{x^{5/4}} = 0$, $\int_1^{+\infty} \frac{dx}{x^{5/4}}$ 收敛, $\int_1^{+\infty} \frac{\ln x dx}{\sqrt{x^3 + 2x + 1}}$ 收敛.

(4) $\lim_{x \rightarrow +\infty} \frac{\ln x}{x(\ln x + 9)} / \frac{1}{x} = 1$, $\int_1^{+\infty} \frac{dx}{x}$ 发散, $\int_1^{+\infty} \frac{\ln x}{x(\ln x + 9)}$ 发散.

(2) 步骤聚 \Rightarrow 人为猜测最后等价的式子. 比如

(3) $< C \cdot x^{n-\frac{3}{2}}$. 令 $n=0.1$ 则 (3) $< C \cdot x^{-1.4}$. 收敛. 写为 $\lim_{x \rightarrow \infty} f(x) \cdot x^{1.4} = 0$
 $\therefore f(x) = O(x^{-1.4})$ $f(x)$ 收敛

高阶无穷小写法

(4) $\sim \frac{\ln x}{x \ln x} \sim \frac{1}{x}$. $f(x) / \frac{1}{x} = 1$

等价写法.

10 反常积分的判敛

如果 $f(x)$ 变号【不是恒为正或负】,考虑 $|f(x)|$ 的无穷限积分 $\int_a^{+\infty} |f(x)| dx$ 收敛

Def. 若 $\int_a^{+\infty} |f(x)| dx$ 收敛,则称 $\int_a^{+\infty} f(x) dx$ 绝对收敛;

$\int_a^{+\infty} |f(x)| dx$ 发散, $\int_a^{+\infty} f(x) dx$ 收敛,则称 $\int_a^{+\infty} f(x) dx$ 条件收敛.

Ex. $\int_a^{+\infty} |f(x)| dx$ 收敛,则 $\int_a^{+\infty} f(x) dx$ 收敛.

Ex. $\int_a^{+\infty} \frac{\sin x}{x^2} dx$ 收敛 解. $\because \int_a^{+\infty} \frac{|\sin x|}{x^2} dx$ 收敛

瑕积分是类似的！

Thm. (比较判敛法) 设 b 为瑕积分 $\int_a^b f(x)dx$ 的唯一瑕点,

$|f(x)| \leq Cg(x), \forall x \in (b-\delta, b)$, 则

(1) $\int_a^b g(x)dx$ 收敛 $\Rightarrow \int_a^b f(x)dx$ 绝对收敛;

(2) $\int_a^b |f(x)|dx$ 发散 $\Rightarrow \int_a^b g(x)dx$ 发散.

Thm. (比较判敛法-极限形式) 设 b 为瑕积分 $\int_a^b f(x)dx$ 的唯

一瑕点, f, g 非负, $\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = C$.

(1) 若 $C > 0$, 则 $\int_a^b f(x)dx$ 与 $\int_a^b g(x)dx$ 同敛散; 找 $f(x)$ 的等价无穷大.

(2) 若 $C = 0$, 且 $\int_a^b g(x)dx$ 收敛, 则 $\int_a^b f(x)dx$ 收敛;

(3) 若 $C = +\infty$, 且 $\int_a^b g(x)dx$ 发散, 则 $\int_a^b f(x)dx$ 发散.

Ex. $\int_0^1 \sqrt{\cot x} dx$ 的收敛性.

找 $f(x)$ 的等价无穷大.

解: $x=0$ 是瑕点.

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{\cot x}}{1/\sqrt{x}} = \lim_{x \rightarrow 0^+} \sqrt{\cos x \cdot \frac{x}{\sin x}} = 1,$$

$$\int_0^1 \frac{1}{\sqrt{x}} dx \text{ 收敛,}$$

故 $\int_0^1 \sqrt{\cot x} dx$ 收敛. \square

$$\begin{aligned} \int_0^1 \sqrt{\cot x} dx &\approx \int_0^1 \sqrt{\frac{1}{\tan x}} dx = \int_0^1 \sqrt{\frac{1}{x}} dx \\ &= \int_0^1 x^{-\frac{1}{2}} dx \\ &= \int_0^1 \frac{1}{\sqrt{x}} dx \\ &\quad \text{422} \end{aligned}$$

Ex. $p, q > 0$, 讨论 $\int_0^{\pi/2} \frac{1}{\sin^p x \cos^q x} dx$ 的收敛性.

解: $\frac{1}{\sin^p x \cos^q x} \geq 0$ $\frac{1}{\sin^p x \cos^q x}$ 在 0 和 $\frac{\pi}{2}$ 处存在瑕点.

$$\frac{1}{\sin^p x \cos^q x} \sim \frac{1}{x^p}, x \rightarrow 0$$

$$\frac{1}{\sin^p x \cos^q x} \sim \frac{1}{x^q}, x \rightarrow \frac{\pi}{2} \quad \therefore p, q < 1 \text{ 时收敛.}$$

$$\cos x (x \rightarrow \frac{\pi}{2}) = \sin(\frac{\pi}{2} - x) \sim \frac{\pi}{2} - x$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{(\frac{\pi}{2} - x)^q} dx \quad t = \frac{\pi}{2} - x. = \int_{\frac{\pi}{2}}^0 \frac{1}{t^q} d-t = \int_0^{\frac{\pi}{2}} \frac{1}{t^q} dt$$

Ex. $p > 0$, 讨论 $\int_0^{+\infty} \frac{\ln(1+x)}{x^p} dx$ 的收敛性. $\int_0^1 \frac{\ln(1+x)}{x^p} dx$

解: $\int_0^{+\infty} \frac{\ln(1+x)}{x^p} dx$ 收敛

$\Leftrightarrow \int_0^1 \frac{\ln(1+x)}{x^p} dx$ 与 $\int_1^{+\infty} \frac{\ln(1+x)}{x^p} dx$ 同时收敛. $\lim(\ln^{x+1}) = x$
 读本上都在处理 \ln^x 为 $+\infty, -\infty$ 的情况, 等于 0 何必如此

$\lim_{x \rightarrow 0^+} x^{p-1} \cdot \frac{\ln(1+x)}{x^p} = 1$, $\int_0^1 \frac{1}{x^{p-1}} dx$ 收敛 $\Leftrightarrow p-1 < 1$

故 $\int_0^1 \frac{\ln(1+x)}{x^p} dx$ 收敛 $\Leftrightarrow p < 2$.

$$\int_1^{+\infty} \frac{\ln(1+x)}{x^p} dx \leq \int_1^{+\infty} \frac{C \cdot x^n}{x^p} dx$$

$$\frac{C}{x^{p-n}} dx$$

$p-n > 1$
 $p > 1+n$
 $p < 1+n$
 $p = 1+n$

当 $0 < p \leq 1$ 时, $\int_1^{+\infty} \frac{1}{x^p} dx$ 发散, 从而 $\int_1^{+\infty} \frac{\ln(1+x)}{x^p} dx$ 发散.

当 $p > 1$ 时, $\forall q \in (1, p)$, $\lim_{x \rightarrow +\infty} x^q \cdot \frac{\ln(1+x)}{x^p} = 0$, $\int_1^{+\infty} \frac{1}{x^q} dx$ 收敛,

从而 $\int_1^{+\infty} \frac{\ln(1+x)}{x^p} dx$ 收敛.

综上, $\int_0^{+\infty} \frac{\ln(1+x)}{x^p} dx$ 收敛 $\Leftrightarrow 1 < p < 2$. \square

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