微积分A期末讲座

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目 录

contents

1.换元积分法 $\frac{\mathrm{d}}{\mathrm{d}x} f(\varphi(x)) = f'(\varphi(x)) \cdot \varphi'(x)$

$$\Rightarrow \int f'(\varphi(x))\varphi'(x)\mathrm{d}x = f(\varphi(x)) + C.$$

与 $\int f'(u)\mathrm{d}u = f(u) + C$ 比较, 得

$$\int f'(\varphi(x))\varphi'(x)\mathrm{d}x = \int f'(\varphi(x))\mathrm{d}\varphi(x) = f(\varphi(x)) + C$$

$$\xlongequal{u=\varphi(x)} \int f'(u)\mathrm{d}u = f(u) + C = f(\varphi(x)) + C$$

——第一换元法(凑微分法)

$$\int f'(u)\mathrm{d}u$$

$$\xlongequal{u=\varphi(x)} \int f'(\varphi(x))\varphi'(x)\mathrm{d}x = g(x) + C = g(\varphi^{-1}(u)) + C$$

——第二换元法

2.分部积分法

$$\left(u(x)v(x)\right)' = u'(x)v(x) + u(x)v'(x)$$

$$\Rightarrow \int u'(x)v(x)\mathrm{d}x + \int u(x)v'(x)\mathrm{d}x = \int \left(u(x)v(x)\right)' \mathrm{d}x$$

$$\Rightarrow \int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

也记作 $\int u(x)\mathrm{d}v(x) = u(x)v(x) - \int v(x)\mathrm{d}u(x).$

——分部积分法

3.有理函数$\dfrac{p(x)}{q(x)}$(p,q为多项式)

Thm.有理真分式$\dfrac{p(x)}{q(x)}$可以唯一地分解成最简分式之和:

(1)$q(x)$的一次k重因式$(ax+b)^k$对应k项

$$\frac{A_1}{ax+b},\frac{A_2}{(ax+b)^2},\cdots,\frac{A_k}{(ax+b)^k};$$

(2)$q(x)$的二次k重因式$(px^2+qx+r)^k$对应k项

$$\frac{B_1x+C_1}{px^2+qx+r},\frac{B_2x+C_2}{(px^2+qx+r)^2},\cdots,\frac{B_kx+C_k}{(px^2+qx+r)^k}.$$

- $$\int \frac{\mathrm{d}x}{x-a} = \ln|x-a| + C,$$
- $$\int \frac{\mathrm{d}x}{(x-a)^k} = \frac{-1}{(k-1)(x-a)^{k-1}} + C,$$
- $$\int \frac{x+a}{((x+a)^2+b^2)^k}\,\mathrm{d}x = \frac{-1}{2(k-1)((x+a)^2+b^2)^{k-1}},$$
- $$J_k = \int \frac{1}{((x+a)^2+b^2)^k}\,\mathrm{d}x$$

$$J_1 = \frac{1}{b}\arctan\frac{x+a}{b} + C,$$

$$J_{k+1} = \frac{1}{2kb^2}\left((x+a)((x+a)^2+b^2)^{-k} + (2k-1)J_k\right).$$

4.三角有理式$R(\sin x,\cos x)$:$\sin x,\cos x$有限次四则运算

万能变换 $t=\tan\frac{x}{2}$, $x=2\arctan t$, $\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$,

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}.$$

$$\int R(\sin x,\cos x)\mathrm{d}x=R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{\mathrm{d}t}{1+t^2}.$$

Ex. $I=\int\frac{1+\sin x}{1+\cos x}\mathrm{d}x.$

解法一：令$t=\tan\frac{x}{2}$，则 $x=2\arctan t,\ \mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t,$

$$\int\frac{1+\sin x}{1+\cos x}\mathrm{d}x=\int\frac{1+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t=\int\frac{(1+t)^2}{2}\cdot\frac{2}{1+t^2}\mathrm{d}t$$

$$=\int\left(1+\frac{2t}{1+t^2}\right)\mathrm{d}t=t+\ln(1+t^2)+C=\tan\frac{x}{2}-2\ln\left|\cos\frac{x}{2}\right|+C.$$

解法二：$I=\int\frac{1}{2\cos^2\frac{x}{2}}\mathrm{d}x+\int\frac{\sin x}{1+\cos x}\mathrm{d}x$

$$=\tan\frac{x}{2}-\ln(1+\cos x)+C.$$

Remark.初等函数的原函数不一定是初等函数,如

$$e^{x^2}, \sin x^2, \cos x^2, \frac{\sin x}{x}, \frac{\cos x}{x},$$

$$\frac{1}{\ln x}, \sqrt{1-k^2\sin^2 x}\,(0<k<1).$$

如果被积函数是分段函数，需要注意得出的原函数的连续性

Ex.求$\int e^{|x|}dx.$

解:$e^{|x|}$在$\mathbb{R}$上连续,因而可积, 设$F(x)$为一个原函数, 则

$$F'(x)=e^{|x|}=\begin{cases} e^{x}, & x\geq 0 \\ e^{-x}, & x<0 \end{cases},\quad F(x)=\begin{cases} e^{x}+C_1, & x\geq 0 \\ -e^{-x}+C_2, & x<0 \end{cases}.$$

$F(x)$在$x=0$处连续,则 $1+C_1=-1+C_2$,

$$\int e^{|x|}dx=F(x)=\begin{cases} e^{x}+C, & x\geq 0 \\ -e^{-x}+2+C, & x<0 \end{cases}.\ \square$$

如果被积函数是分段函数，需要注意得出的原函数的连续性

Ex. $\int |(x-1)(3x-2)|\,dx.$

$$|(x-1)(3x-2)| = \begin{cases} 3x^2-5x+2, & x \ge \dfrac{2}{3} \\ -3x^2+5x-2, & \dfrac{2}{3} < x < 1. \\ 3x^2-5x+2, & x \ge 1 \end{cases}$$

$$\int |(x-1)(3x-2)|\,dx = \begin{cases} x^3-\dfrac{5}{2}x^2+2x+C_1, & x \ge \dfrac{2}{3} \\ -x^3+\dfrac{5}{2}x^2-2x+C_2, & \dfrac{2}{3} < x < 1. \\ x^3-\dfrac{5}{2}x^2+2x+C_3, & x \ge 1 \end{cases}$$

原函数要在2/3和1处连续.

$$\therefore \frac{14}{27}+C_1=-\frac{14}{27}+C_2, -\frac{1}{2}+C_2=\frac{1}{2}+C_3, \text{即} C_1=C_2-\frac{28}{27}, C_3=C_2-1$$

0.常见例子.

$$\int \sin^{2n} x dx, \int \sin^{2n-1} x dx, \int \cos^{2n} x dx, \int \cos^{2n-1} x dx,$$

1.敏锐地观察, 是否已经可以凑微分

$$\int xe^{-x^2}dx = \frac{1}{2}\int e^{-x^2}dx^2 = -\frac{1}{2}e^{-x^2} + C.$$

$$\int \sin^n x\cos x dx = \int \sin^n x d\sin x = \frac{1}{n+1}\sin^{n+1}x + C$$

Ex. $\int \sec x\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$

$$= \frac{1}{2}\int\left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right)\mathrm{d}\sin x$$

$$= \frac{1}{2}\ln(1+\sin x) - \frac{1}{2}\ln(1-\sin x) + C$$

$$= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C$$

2.三角代换用于换元法:

$"\sqrt{x^2-1}" \leftrightarrow x=\sec t, \sqrt{x^2-1}=\tan t$

$"\sqrt{x^2+1}" \leftrightarrow x=\tan t, \sqrt{x^2+1}=\sec t$

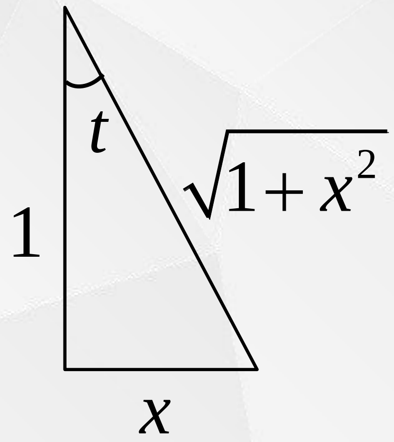
$"\sqrt{1-x^2}" \leftrightarrow x=\sin t, \sqrt{1-x^2}=\cos t$

Q:$"\sqrt{ax^2+bx+c}"$?

注意:$\sqrt{x^2-1}$用$\sec t$代换, 需要引起高度关注!

$$\forall t, |t| < \frac{\pi}{2}, \sqrt{1+\tan^2 t} = |\sec t| = \left|\frac{1}{\cos t}\right| = \frac{1}{\cos t}$$

Ex. $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}$



解: 令 $x = \tan t, |t| < \frac{\pi}{2}$,

$$\int \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}} = \int \frac{\sec^2 t\mathrm{d}t}{\tan^2 t \sec t}$$

$$= \int \frac{\cos t\mathrm{d}t}{\sin^2 t} = \int \frac{\mathrm{d}\sin t}{\sin^2 t} = -\frac{1}{\sin t} + C = -\frac{\sqrt{1+x^2}}{x} + C.$$

Ex. $\int \frac{x+1}{\sqrt{-x^2+2x+3}}dx$

解:

原式 $=\int \frac{x+1}{\sqrt{4-(x-1)^2}}dx = \frac{1}{2}\int \frac{x+1}{\sqrt{1-(\frac{x-1}{2})^2}}dx = \int \frac{x+1}{\sqrt{1-(\frac{x-1}{2})^2}}d\frac{x-1}{2}$

令 $x-1=2t$ 令 $t=\sin y$

$= \int \frac{2t+2}{\sqrt{1-t^2}}dt$ $= \int \frac{2\sin y+2}{\cos y}d\sin y = 2(y-\cos y)+C$

$= 2(\arcsin(\frac{x-1}{2}) - \sqrt{1-(\frac{x-1}{2})^2}) + C$

Ex. $\int \frac{\sqrt{x^2-4}}{x}dx$

$$\forall t, 0 \le t \le \pi, \sqrt{4\sec^2 t - 4} = 2|\tan t| = \begin{cases} 2\tan t, 0 \le t \le \frac{\pi}{2} \\ -2\tan t, \frac{\pi}{2} \le t \le \pi \end{cases}$$

解： 令$x = 2\sec t, 0 \le t \le \pi$

$$\frac{\sqrt{x^2-4}}{x} = \frac{1}{\sec t}|\tan t|, dx = 2\sec t \tan t dt$$

$$\int \frac{\sqrt{x^2-4}}{x}dx = \int \frac{|\tan t|}{\sec t} 2\sec t \tan t dt = 2\int |\tan t| \tan t dt$$

$1°. 0 \le t \le \pi/2$, 即$x \ge 2$.

原式$= 2\int \tan^2 t dt = 2\int \sec^2 t - 1 dt = 2\tan t - 2t + C_1$ $\because \tan t = \frac{1}{2}\sqrt{x^2-4}, t = \arccos(2/x)$

$\therefore$ 原式$= \sqrt{x^2-4} - 2\arccos(2/x) + C_1$

$\text{Ex.}\int\frac{\sqrt{x^2-4}}{x}dx$

$$\forall t, 0\le t\le\pi, \sqrt{4\sec^2 t-4}=2|\tan t|=\begin{cases}2\tan t, 0\le t\le\dfrac{\pi}{2}\\ -2\tan t, \dfrac{\pi}{2}\le t\le\pi\end{cases}$$

解：令$x=2\sec t, 0\le t\le\pi$

$$\frac{\sqrt{x^2-4}}{x}=\frac{1}{\sec t}|\tan t|, dx=2\sec t\tan t dt$$

$$\int\frac{\sqrt{x^2-4}}{x}dx=\int\frac{|\tan t|}{\sec t}2\sec t\tan t dt=2\int|\tan t|\tan t dt$$

$2°.\pi/2\le t\le\pi.$即$x\le-2$

$$\because \tan t=-\frac{1}{2}\sqrt{x^2-4}, t=\arccos(2/x)$$

$$原式=-2\int\tan^2 t dt=-2\int\sec^2 t-1dt=-2\tan t+2t+C_2$$

$$\therefore 原式=\sqrt{x^2-4}+2\arccos(2/x)+C_2$$

Ex. $\int \frac{\sqrt{x^2-4}}{x}dx$

解： 原式 $= \int \frac{\sqrt{x^2-4}}{x^2}xdx = \frac{1}{2}\int \frac{\sqrt{x^2-4}}{x^2}dx^2$

$$\overset{y=x^2}{=} \frac{1}{2}\int \frac{\sqrt{y-4}}{y}dy \overset{\sqrt{y-4}=z,z^2+4=y,dy=2zdz}{=} \int \frac{z^2}{z^2+4}dz = \int \frac{z^2+4-4}{z^2+4}dz = \int 1dz - 4\int \frac{1}{z^2+4}dz$$

$$= z - 2\int \frac{1}{(z/2)^2+1}dz/2 = z - 2\arctan(z/2) + C$$

$$= \sqrt{x^2-4} - 2\arctan(\sqrt{x^2-4}/2) + C$$

1.敏锐地观察, 是否已经可以凑微分

Ex. $$\int \frac{x-1}{\sqrt{2-2x-x^2}}\mathrm{d}x = \int \frac{x-1}{\sqrt{3-(x+1)^2}}dx = \int \frac{x+1-2}{\sqrt{3-(x+1)^2}}dx$$

$$= \int \frac{x+1}{\sqrt{3-(x+1)^2}}dx - 2\int \frac{1}{\sqrt{3-(x+1)^2}}dx$$

$$= \frac{1}{2}\int \frac{d(x+1)^2}{\sqrt{3-(x+1)^2}} - \frac{2}{\sqrt{3}}\int \frac{1}{\sqrt{1-(\frac{x+1}{\sqrt{3}})^2}}dx = \frac{1}{2}\int \frac{d(x+1)^2}{\sqrt{3-(x+1)^2}} - 2\int \frac{1}{\sqrt{1-(\frac{x+1}{\sqrt{3}})^2}}d\frac{x+1}{\sqrt{3}}$$

$$= -\sqrt{3-(x+1)^2} - 2\arcsin(\frac{x+1}{\sqrt{3}}) + C$$

3.一次根式, 直接令其整体等于一个新变量:

Ex. $\int \frac{1}{1+\sqrt{1+x}}\mathrm{d}x$

令 $\sqrt{1+x}=t, x=t^2-1,$

原式 $=\int \frac{1}{1+t}\mathrm{d}(t^2-1) \quad =\int \frac{2t}{1+t}\mathrm{d}t \ =\int \frac{2t+2-2}{1+t}\mathrm{d}t \ =\int 2-\frac{2}{1+t}\mathrm{d}t \ =2t-2\ln(1+t)$

$=2\sqrt{1+x}-2\ln(1+\sqrt{1+x})+C$

Ex. $\int \frac{\ln x}{x\sqrt{1+\ln x}}\mathrm{d}x$ 令 $\sqrt{1+\ln x}=t, \ln x=t^2-1$

$\frac{1}{x}dx=2tdt, \quad dx=2txdt$

$$\int \frac{\ln x}{x\sqrt{1+\ln x}}\mathrm{d}x=\int \frac{t^2-1}{xt}2tx\mathrm{d}t=2\int t^2-1\mathrm{d}t=\frac{2}{3}t^3-2t+C=\frac{2}{3}(\ln x+1)^{3/2}-2(\ln x+1)^{1/2}+C$$

- 例题（含有一次根式的函数）

$$\int \frac{1}{\sqrt[3]{t}+\sqrt[2]{t}}dt = \int \frac{1}{x^2+x^3}dx^6 \, (t = x^6) = \int \frac{6x^5}{x^2+x^3}dx = \int \frac{6x^3}{1+x}dx = \int \frac{6(t-1)^3}{t}dt$$

4.高次根式,考虑设法化为1次（通过还原）

$$\int \frac{1}{x\sqrt{1+x^5}} = \int \frac{x^4dx}{x^5\sqrt{1+x^5}} = \frac{1}{5}\int \frac{dy}{y\sqrt{1+y}} = \frac{1}{5}\int \frac{d(t^2-1)}{t(t^2-1)}$$

$$= \frac{2}{5}\int \frac{1}{t^2-1} = \frac{1}{5}\int \frac{1}{t-1} - \frac{1}{5}\int \frac{1}{t+1} = \frac{1}{5}\ln(\frac{t-1}{t+1}) + C = \frac{1}{5}\ln(\frac{\sqrt{1+x^5}-1}{\sqrt{1+x^5}+1}) + C$$

5.感到无所适从, 不知道怎么办的时候, 往往可以通过分部积分打开局面

Ex. $\int \ln x\mathrm{d}x = x\ln x - \int x\mathrm{d}\ln x$

$$= x\ln x - \int x\cdot\frac{1}{x}\mathrm{d}x = x\ln x - x + C$$

Ex. $\int \arcsin x\mathrm{d}x = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$

$$= x\arcsin x - \frac{1}{2}\int\frac{\mathrm{d}x^2}{\sqrt{1-x^2}} = x\arcsin x + \frac{1}{2}\sqrt{1-x^2} + C.$$

- 例题5 （实在不会，就先分部积分，创造条件）

$$\int (3x^2+2x)\arctan x dx$$

$$\int \arctan x d(x^3+x^2) = (x^3+x^2)\arctan x - \int \frac{x^3+x^2}{x^2+1} dx$$

Ex. $\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int \frac{x\cdot 2x}{2\sqrt{x^2+a^2}}dx$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + \int \frac{a^2}{\sqrt{x^2+a^2}}dx$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)+C.$$

Ex. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)+C.$

- 例题5 （实在不会，就先分部积分，创造条件）

$$\int \frac{1+\sin x}{1+\cos x} e^x dx$$

$$=\int \frac{(\cos(x/2)+\sin(x/2))^2}{2\cos^2(x/2)} e^x dx = \int \frac{(\cos(x/2)+\sin(x/2))^2}{\cos^2(x/2)} e^x dx/2$$

$$=\int \frac{(\cos(t)+\sin(t))^2}{\cos^2 t} e^{2t} dt$$

$$=\int (1+\tan t)^2 e^{2t} dt$$

$$=\int \sec^2 t e^{2t} dt + 2\int \tan t e^{2t} dt = \int e^{2t} d\tan t + 2\int \tan t e^{2t} dt$$

$$=\tan t e^{2t} - 2\int \tan t e^{2t} d + 2\int \tan t e^{2t} dt = \tan t e^{2t} + C$$

- 例题（分部积分-设而不求，消元）

$$\int (1-\frac{2}{x})^2 e^x dx$$

$$\int (1-\frac{2}{x})^2 e^x dx = \int (1-\frac{4}{x}+\frac{4}{x^2})e^x dx = \int e^x dx + 4\int (-\frac{1}{x}+\frac{1}{x^2})e^x dx$$

$$\int (-\frac{1}{x}+\frac{1}{x^2})e^x dx = \int \frac{1}{x^2}e^x dx - \int \frac{1}{x}e^x dx$$

$$\because -\int \frac{1}{x} de^x = -\frac{1}{x}e^x - \int \frac{1}{x^2}e^x dx \therefore \int (-\frac{1}{x}+\frac{1}{x^2})e^x dx = -\frac{1}{x}e^x + C$$

- 例题（分部积分-导出递推式）

$$\int \ln^n(x)dx$$

$$\int \ln^n(x)dx = x\ln^n(x) - n\int \ln^{n-1}(x)dx$$

$$f_n(x) := \int \ln^n(x)dx$$

$$f_n(x) = x\ln^n(x) - nf_{n-1}(x)$$

- 例题（实在不会，就先分部积分，创造条件）

$$\int \frac{x^2}{(x\sin x+\cos x)^2}dx$$

$$\int \frac{x^2}{(x\sin x+\cos x)^2}dx=-\int \frac{x}{\cos x}d\frac{1}{x\sin x+\cos x}$$

$$=-\frac{x}{\cos x}\frac{1}{x\sin x+\cos x}+\int \frac{1}{x\sin x+\cos x}d\frac{x}{\cos x}$$

$$=-\frac{x}{\cos x}\frac{1}{x\sin x+\cos x}+\int \frac{1}{x\sin x+\cos x}\frac{\cos x-(-\sin x)x}{\cos^2 x}dx$$

$$=-\frac{x}{\cos x}\frac{1}{x\sin x+\cos x}+\int \sec^2 xdx=-\frac{x}{\cos x}\frac{1}{x\sin x+\cos x}+\tan x$$

6.含有三角的情况：虽然万能变换是通法，但是万能变换不一定简单.

Ex. $\int \frac{\tan x}{a^2\cos^2 x + b^2\sin^2 x}\mathrm{d}x$

$$= \int \frac{\tan x}{a^2 + b^2\tan^2 x}\cdot\frac{\mathrm{d}x}{\cos^2 x} = \int \frac{\tan x}{a^2 + b^2\tan^2 x}\mathrm{d}\tan x$$

$$= \frac{1}{2}\int \frac{\mathrm{d}\tan^2 x}{a^2 + b^2\tan^2 x} = \frac{1}{2b^2}\ln(a^2 + b^2\tan^2 x) + C.$$

6.含有三角的情况：虽然万能变换是通法，但是万能变换不一定简单.

Ex. $\int \frac{\sin x}{\cos x+\sin x}\mathrm{d}x$

令$t=x+\pi/4$

$$=\int \frac{\sin x}{\sqrt{2}\sin(x+\frac{\pi}{4})}\mathrm{d}x \qquad =\int \frac{\sin(t-\frac{\pi}{4})}{\sqrt{2}\sin t}\mathrm{d}t$$

$$=\int \frac{\frac{\sqrt{2}}{2}\sin t-\frac{\sqrt{2}}{2}\cos t}{\sqrt{2}\sin t}\mathrm{d}t =\frac{1}{2}\int 1-\cot t dt=\frac{1}{2}x-\frac{1}{2}\ln|\sin x+\cos x|+C$$

7.如果被积函数只和e^x有关,令$e^x=t$是一个不错的选择.

Ex. $\int \frac{\arcsin e^x}{e^x}\mathrm{d}x$

$$=\int \frac{\arcsin t}{t}d(\ln t)=\int \arcsin t \big/ t^2 dt=\int \arcsin t\, d(-1/t)=-\frac{\arcsin t}{t}+\int \frac{1}{t\sqrt{1-t^2}}dt$$

$$=\int \frac{1}{t\sqrt{1-t^2}}dt=\int \frac{1}{\sin y\cos y}d\sin y=\int \frac{1}{\sin y}dy=\int \frac{\sin y}{\sin^2 y}dy=-\int \frac{1}{1-\cos^2 y}d\cos y$$

$$\int \frac{1}{\cos^2 y-1}d\cos y=\frac{1}{2}\ln(\frac{\cos y-1}{\cos y+1})+C=\frac{1}{2}\ln(\frac{\sqrt{1-e^{2x}}-1}{\sqrt{1-e^{2x}}+1})+C$$

$$\therefore \int \frac{\arcsin e^x}{e^x}\mathrm{d}x=-\frac{\arcsin e^x}{e^x}+\frac{1}{2}\ln(\frac{\sqrt{1-e^{2x}}-1}{\sqrt{1-e^{2x}}+1})+C$$

09/ 定积分

考点：

（1） 利用黎曼和求极限；

（2）变上限定积分-变上限定积分用于不等式证明

（3）定积分计算的三大法宝：N-L公式，换元积分法，分部积分法；

（4）定积分相关不等式证明（较难）

Step1.分割

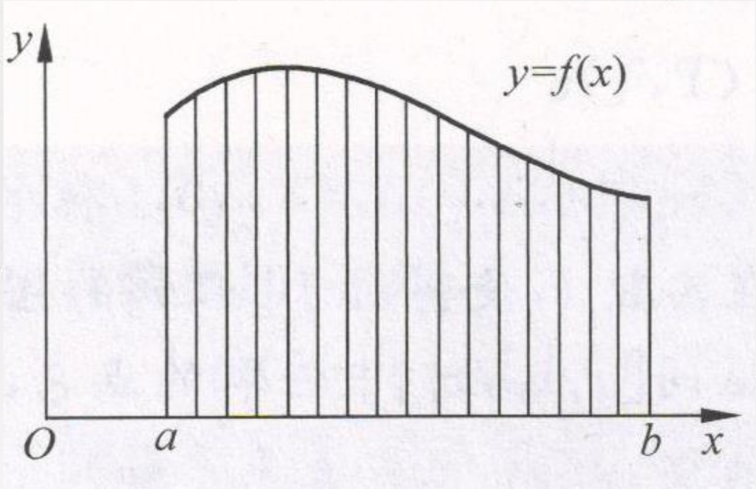
$$T: a = x_0 < x_1 < \cdots < x_n = b.$$

$$\Delta x_i \triangleq x_i - x_{i-1},\ |T| = \max_{1\le i\le n}\{\Delta x_i\}.$$

Step2.取标志点$\xi_i \in [x_{i-1}, x_i]$.

Step3.近似求和. $S \approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i,$

Step4.取极限. $\lim_{|T|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = S.$



Def. 设f为闭区间$[a,b]$上的有界函数,若存在实数I,s.t.对$[a,b]$的任何一个分割$T: a = x_0 < x_1 < \cdots < x_n = b$,对任意$\{\xi_i\}, \xi_i \in [x_{i-1}, x_i], 1 \le i \le n$,只要$|T| = \max\limits_{1\le i\le n}\{\Delta x_i\} \to 0$,就有

$$\lim_{|T|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \mathrm{I},$$

即$\forall \varepsilon > 0, \exists \delta > 0$,当$|T| < \delta$时,无论$\xi_i \in [x_{i-1}, x_i]$如何取,都有

$$\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \mathrm{I}\right| < \varepsilon,$$

则称f在$[a,b]$上Riemann可积,称I为f在$[a,b]$上的Riemann积分,记为

$$\int_a^b f(x)dx = \mathrm{I}.$$

a, b, f, x分别称为积分上、下限,被积函数和积分变量.

09 定积分——利用黎曼和求极限

Ex. $\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k}=$ ________.

$$\sum_{k=n+1}^{2n}\frac{1}{k}=\frac{1}{n+1}+\ldots+\frac{1}{n+n}=\frac{1}{n}\left(\frac{1}{1+1/n}+\frac{1}{1+2/n}+\ldots+\frac{1}{1+n/n}\right)$$

$$f(x):=\frac{1}{1+x} \qquad =\frac{1}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right)$$

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k}=\lim_{n\to\infty}\frac{1}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right)=\int_0^1 f(x)dx=\int_0^1\frac{1}{1+x}dx=\ln 2$$

09 定积分——利用黎曼和求极限

Ex. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(k+n)^2}=$ ________.

$$\sum_{k=1}^{n}\frac{n}{(k+n)^2}=\frac{n}{(1+n)^2}+\dots+\frac{n}{(n+n)^2}=\frac{1}{n}\left(\frac{n^2}{(1+n)^2}+\dots+\frac{n^2}{(n+n)^2}\right)=$$

$$\frac{1}{n}\left(\frac{1}{(1/n+1)^2}+\dots+\frac{1}{(n/n+1)^2}\right)=\frac{1}{n}\left(f\left(\frac{1}{n}\right)+\dots+f\left(\frac{n}{n}\right)\right)\to \qquad f(x)=\frac{1}{(1+x)^2}$$

$$\int_0^1 f(x)dx=\int_0^1\frac{1}{(1+x)^2}dx=\frac{1}{2}$$

Thm.(微积分基本定理)

$$f\in R[a,b], F(x)=\int_a^x f(t)dt\ \ (a\le x\le b),\text{则}$$

(1)$F\in C[a,b]$;

(2)若f在$x_0\in[a,b]$连续,则F在x_0可导,且$F'(x_0)=f(x_0)$;

(3)若$f\in C[a,b]$,则F是f在$[a,b]$上的一个原函数. 若G为f的任一个原函数,则

$$\int_a^b f(t)dt=G(b)-G(a)\triangleq G(x)\Big|_a^b.\ (\text{Newton-Leibniz})$$

09 定积分——变上限积分

Ex. f连续,u,v可导,$G(x)=\int_{v(x)}^{u(x)} f(t)dt$,求$G'(x)$.

解: 令$F(u)=\int_a^u f(t)dt$,则$F'(u)=f(u)$.

$$G(x)=\int_a^{u(x)} f(t)dt-\int_a^{v(x)} f(t)dt$$

$$=F(u(x))-F(v(x))$$

$$G'(x)=F'(u(x))\cdot u'(x)-F'(v(x))\cdot v'(x)$$

$$=f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x).\square$$

Ex. (2019)求$\int_{x^2}^{x^3}\frac{\sin t}{t}dt$的一阶导数______________

解:$3x^2\frac{\sin(x^3)}{x^3}-2x\frac{\sin(x^2)}{x^2}=\frac{3\sin(x^3)-2\sin(x^2)}{x}$

Ex. (未知年份)求$\int_{x^2}^{2x}\ln(1+\sin t)dt$的一阶导数______________

Note.被积函数中,如果含有被积变量以外的其他变量，必须设法分离出来

Ex. f连续, $F(x)=\int_a^x (x-t)f(t)dt$, 求$F''(x)$.

解: $F(x)=x\int_a^x f(t)dt-\int_a^x tf(t)dt$,

$$F'(x)=\int_a^x f(t)dt+xf(x)-xf(x)=\int_a^x f(t)dt,\ F''(x)=f(x).\square$$

Ex. (变上限积分, 适合于使用洛必达法则) $\lim\limits_{x\to+\infty}\dfrac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}=\lim\limits_{x\to+\infty}\dfrac{2e^{x^2}\int_0^x e^{t^2}dt}{e^{2x^2}}$

$$=\lim_{x\to+\infty}\frac{2\int_0^x e^{t^2}dt}{e^{x^2}}=\lim_{x\to+\infty}\frac{2e^{x^2}}{2xe^{x^2}}=0.\square$$

Ex. (2019)$f\in C[0,+\infty),\int_a^{ab}f(x)dx$和$a$无关$,a,b>0.$求证$:f(x)=c/x.$

解: $\int_a^{ab}f(x)dx$和a无关$,\therefore\dfrac{d\int_a^{ab}f(x)dx}{da}=0\qquad\therefore bf(ab)-f(a)=0,\forall a,b>0$

$\therefore f(ab)=\dfrac{1}{b}f(a),\forall a,b>0\ \therefore$ 取$a=1,\therefore f(b)=\dfrac{f(1)}{b}$

$\therefore c=f(1)$

Note.被积函数中, 如果含有被积变量以外的其他变量，必须设法分离出来

Ex. f连续, $\lim_{x\to 0}\dfrac{f(x)}{x}=a$, 定义$\phi(x)=\int_0^1 f(xt)dt$, 计算$\phi'(x)$,

并考察$\phi'(x)$在0处的连续性. $\because f$连续, $\therefore f(0)=\lim_{x\to 0} f(x)=\lim_{x\to 0} x\dfrac{f(x)}{x}=0\times a=0$

解: $\phi(x)=\int_0^1 f(xt)dt=\dfrac{1}{x}\int_0^1 f(xt)dxt=\dfrac{1}{x}\int_0^x f(y)dy \quad \therefore \phi(x)=\dfrac{1}{x}\int_0^x f(y)dy$

$$\therefore \phi'(x)=\frac{f(x)x-\int_0^x f(y)dy}{x^2}$$

$$\therefore \lim_{x\to 0}\phi'(x)=\lim_{x\to 0}\frac{f(x)x-\int_0^x f(y)dy}{x^2}=\lim_{x\to 0}\frac{f(x)}{x}-\frac{\int_0^x f(y)dy}{x^2}$$

$$=\lim_{x\to 0}\frac{f(x)}{x}-\lim_{x\to 0}\frac{\int_0^x f(y)dy}{x^2}=a-\lim_{x\to 0}\frac{\int_0^x f(y)dy}{x^2}\overset{L-H}{=}a-\lim_{x\to 0}\frac{f(x)}{2x}=\frac{a}{2}$$

$$\phi'(0)=\lim_{x\to 0}\frac{\phi(x)-\phi(0)}{x}=\lim_{x\to 0}\frac{\phi(x)}{x}=\lim_{x\to 0}\frac{\int_0^x f(y)dy}{x^2}=\lim_{x\to 0}\frac{f(x)}{2x}=a/2$$

09/ 定积分——利用变上限积分完成不等式证明

Note.有一些看起来很难的定积分不等式证明,可以利用这个方法

Ex.f在$[a,b]$上二阶导函数连续, f上凸, 求证:$\int_a^b f(x)dx \geq \frac{f(a)+f(b)}{2}(b-a)$

证明:考察函数$g(x)=\int_a^x f(t)dt-\frac{f(a)+f(x)}{2}(x-a)$ 目标:$g(b)\geq 0$

$$g'(x)=f(x)-\frac{f'(x)}{2}(x-a)-\frac{f(a)+f(x)}{2}=\frac{1}{2}(f(x)-f(a)-f'(x)(x-a))$$

$$=\frac{1}{2}(f'(\xi)(x-a)-f'(x)(x-a)),a<\xi<x \quad =\frac{1}{2}(x-a)(f'(\xi)-f'(x)),a<\xi<x$$

$$=\frac{1}{2}(x-a)(\xi-x)f''(\zeta)\geq 0,a<\xi<\zeta<x \qquad g'(x)\geq 0 \quad g(a)=0,\therefore g(x)\geq 0,\forall x>a$$

09 定积分——利用变上限积分完成不等式证明

Note.有一些看起来很难的定积分不等式证明,可以利用这个方法

Hw.f在$[a,b]$上二阶导函数连续, f上凸, 求证:$\int_a^b f(x)dx \le f(\frac{a+b}{2})\frac{(b-a)}{2}$

09/ 定积分——利用变上限积分完成不等式证明

Ex. f在[0,1]上有一阶导数, $f(0)=0, 0\le f'(x)\le 1$, 求证: $\int_0^1 f^3(x)dx \le (\int_0^1 f(x)dx)^2$

$G(t)=\int_0^t f^3(x)dx-(\int_0^t f(x)dx)^2 \quad G(1)=\int_0^1 f^3(x)dx-(\int_0^1 f(x)dx)^2 \quad G(0)=0$

$G'(t)=f^3(t)-2f(t)(\int_0^t f(x)dx)=f(t)(f^2(t)-2\int_0^t f(x)dx)$

希望说明$G'(t)\le 0$, $\because f(t)\ge 0 \therefore$ 只需证明$f^2(t)-2\int_0^t f(x)dx\le 0$

$Q(t)=f^2(t)-2\int_0^t f(x)dx, Q(0)=0$, 下证$Q'(t)\le 0$

$Q'(t)=2f(t)f'(t)-2f(t)=2f(t)(f'(t)-1)$

$\because 0\le f'(x)\le 1, \therefore f'(x)-1\le 0$, 又$f(x)\ge 0$

$\therefore Q'(t)\le 0$

09 定积分——利用变上限积分完成不等式证明

Note.变上限积分函数本身是一个有很好性质的函数.

期中之前学过的中值定理和泰勒公式可以用上

Note.结合泰勒公式\中值定理

Ex. f在[0,1]可导, $f(1)=4\int_0^{1/4}e^{1-x^3}f(x)dx$.则$\exists\xi\in(0,1),s.t.$

$$f'(\xi)=3\xi^2f(\xi).$$

Proof.由积分第一中值定理,$\exists\eta\in(0,1),s.t.$

$$f(1)=4\int_0^{1/4}e^{1-x^3}f(x)dx=e^{1-\eta^3}f(\eta).$$

令$g(x)=e^{1-x^3}f(x)$,则

$$g'(x)=e^{1-x^3}(f'(x)-3x^2f(x)),\ \ g(\eta)=g(1).$$

由Rolle定理, $\exists\xi\in(\eta,1)\subset(0,1),s.t.g'(\xi)=0$,即

$$f'(\xi)=3\xi^2f(\xi).\square$$

Note.结合泰勒公式\中值定理

Ex. f在[0,1]可导, $f(1)=4\int_0^{1/4} e^{1-x^3} f(x)dx$.则$\exists \xi \in (0,1), s.t.$

$f'(\xi)=3\xi^2 f(\xi)$.

Proof.由积分第一中值定理,$\exists \eta \in (0,1), s.t.$

$$f(1)=4\int_0^{1/4} e^{1-x^3} f(x)dx = e^{1-\eta^3} f(\eta).$$

令$g(x)=e^{1-x^3} f(x)$,则

$$g'(x)=e^{1-x^3}(f'(x)-3x^2 f(x)),\ \ g(\eta)=g(1).$$

由Rolle定理, $\exists \xi \in (\eta,1) \subset (0,1), s.t. g'(\xi)=0$,即

$$f'(\xi)=3\xi^2 f(\xi).\square$$

Note.结合泰勒公式\中值定理 $g(y)=\int_0^y \frac{1}{1+x}dx, g'(y)=\frac{1}{1+y}, g''(y)=-(\frac{1}{1+y})^2$

Ex. (A Hard Problem!!)计算 $\lim_{n\to\infty} n^2(\ln 2-\sum_{i=1}^{n}\frac{2n+2i+1}{2(n+i)^2})$

分析.原式 $=n^2(\ln 2-\sum_{i=1}^{n}\frac{2(n+i)+1}{2(n+i)^2})=n^2(\ln 2-\sum_{i=1}^{n}\frac{1}{n+i}-\frac{1}{2}\sum_{i=1}^{n}\frac{1}{(n+i)^2})$

$$=n^2(\ln 2-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n}-\frac{1}{2n^2}\sum_{i=1}^{n}\frac{1}{(1+i/n)^2})$$

$$\ln 2-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n}-\frac{1}{2n^2}\sum_{i=1}^{n}\frac{1}{(1+i/n)^2}=\int_0^1\frac{1}{1+x}dx-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n}-\frac{1}{2n^2}\sum_{i=1}^{n}\frac{1}{(1+i/n)^2}$$

$$\int_0^1\frac{1}{1+x}dx-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n}-\frac{1}{2n^2}\sum_{i=1}^{n}\frac{1}{(1+i/n)^2}$$

$$=\sum_{i=1}^{n}(\int_{(i-1)/n}^{i/n}\frac{1}{1+x}dx-\frac{1}{n}\frac{1}{1+i/n}-\frac{1}{2n^2}\frac{1}{(1+i/n)^2})$$

$$=\sum_{i=1}^{n}(g(\frac{i}{n})-g(\frac{i-1}{n})-\frac{1}{n}g'(\frac{i}{n})+\frac{1}{2n^2}g''(\frac{i}{n}))$$

Note.结合泰勒公式\中值定理 $g(y)=\int_0^y \frac{1}{1+x}dx, g'(y)=\frac{1}{1+y}, g''(y)=-(\frac{1}{1+y})^2$

Ex. 计算 $\lim_{n\to\infty} n^2(\ln 2-\sum_{i=1}^{n}\frac{2n+2i+1}{2(n+i)^2})$

分析.∴ 原式 $=n^2\sum_{i=1}^{n}(g(\frac{i}{n})-g(\frac{i-1}{n})-\frac{1}{n}g'(\frac{i}{n})+\frac{1}{2n^2}g''(\frac{i}{n}))$

$$\because g(\frac{i-1}{n})=g(\frac{i}{n})-\frac{1}{n}g'(\frac{i}{n})+\frac{1}{2n^2}g''(\frac{i}{n})-\frac{1}{6n^3}g'''(\xi)$$

$$\therefore g(\frac{i}{n})-g(\frac{i-1}{n})=\frac{1}{n}g'(\frac{i}{n})-\frac{1}{2n^2}g''(\frac{i}{n})+\frac{1}{6n^3}g'''(\xi_i),\ \xi\in[\frac{i-1}{n},\frac{i}{n}]$$

$$\therefore n^2\sum_{i=1}^{n}(g(\frac{i}{n})-g(\frac{i-1}{n})-\frac{1}{n}g'(\frac{i}{n})+\frac{1}{2n^2}g''(\frac{i}{n}))=n^2\sum_{i=1}^{n}\frac{1}{6n^3}g'''(\xi_i)$$

$$=\frac{1}{6}\frac{1}{n}\sum_{i=1}^{n}g'''(\xi_i)\to\frac{1}{6}\int_0^1 g'''(x)dx=\frac{g''(1)-g''(0)}{6}=\frac{1}{8}$$

09 定积分——利用变上限积分完成不等式证明

Note.结合泰勒公式\中值定理

$$\text{Ex.}\sum_{k=1}^{n-1}\frac{\ln k+\ln(k+1)}{2}\le\int_1^n \ln x dx\le\sum_{k=1}^{n-1}\frac{\ln k+\ln(k+1)}{2}+\frac{1}{8}$$

$$\text{即}\ \frac{n^{(n+\frac{1}{2})}}{e^n}e^{\frac{7}{8}}\le n!\le\frac{n^{(n+\frac{1}{2})}}{e^n}e,\forall n\in\mathbb{N}^+$$

$$\text{注}:n!\sim\sqrt{2\pi}\frac{n^{(n+\frac{1}{2})}}{e^n},$$

$$\sqrt{2\pi}\approx 2.5066,e^{7/8}\approx 2.3988,e\approx 2.7183$$

09 定积分——N-L公式计算定积分

通过找原函数求定积分，但是要注意符号.

Ex. $\int_0^{\pi}\sqrt{\sin x-\sin^3 x}dx=$__________

$$\int_0^{\pi}\sqrt{\sin x-\sin^3 x}dx=\int_0^{\pi}\sqrt{\sin x(1-\sin^2 x)}dx=\int_0^{\pi}\sqrt{\sin x\cos^2 x}dx=\int_0^{\pi}\sqrt{\sin x}|\cos x|dx$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\cos x dx+\int_{\frac{\pi}{2}}^{\pi}\sqrt{\sin x}(-\cos x)dx=\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\cos x dx-\int_{\frac{\pi}{2}}^{\pi}\sqrt{\sin x}(\cos x)dx$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{\sin x}d\sin x-\int_{\frac{\pi}{2}}^{\pi}\sqrt{\sin x}d\sin x=\frac{2}{3}\sin^{3/2}x\Big|_0^{\frac{\pi}{2}}-\frac{2}{3}\sin^{3/2}x\Big|_{\frac{\pi}{2}}^{\pi}=4/3$$

09 定积分——换元、分部积分

Thm.(定积分的换元法) $f\in C[a,b],\varphi\in C^1[\alpha,\beta],\varphi(\alpha)=a,$ $\varphi(\beta)=b,a\le\varphi(t)\le b,$ 则 $\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$

Thm.(定积分的分部积分法) $u,v\in C[a,b],$ 则

$$\int_a^b u(x)v'(x)dx=u(x)v(x)\Big|_a^b-\int_a^b v(x)u'(x)dx.$$

Ex.(1) $f \in C[a,b]$, 则 $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$;

(2) $I = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x}{x(\pi-2x)}dx = \frac{1}{\pi}\ln 2.$

Proof.(1) $\int_a^b f(a+b-x)dx$

$$\overset{t=a+b-x}{=\!=\!=\!=} -\int_b^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx.$$

(2)利用(1), $I = \int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{x(\pi-2x)}dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} \frac{1}{x(\pi-2x)}dx$

$$= \frac{1}{\pi}\int_{\pi/6}^{\pi/3}\left(\frac{1}{2x}+\frac{1}{\pi-2x}\right)dx = \frac{1}{2\pi}\ln\frac{2x}{\pi-2x}\Bigg|_{\pi/6}^{\pi/3} = \frac{1}{\pi}\ln 2. \square$$

Ex. $f(x)=\int_1^{x^2}\exp(-t^2)dt$, 计算 $I=\int_0^1 xf(x)dx$

解.

$$\int_0^1 xf(x)dx=\int_0^1 x\left(\int_1^{x^2}\exp(-t^2)dt\right)dx=\frac{1}{2}\int_0^1\left(\int_1^{x^2}\exp(-t^2)dt\right)dx^2$$

$$=\frac{1}{2}x^2\left(\int_1^{x^2}\exp(-t^2)dt\right)\Big|_0^1-\frac{1}{2}\int_0^1 x^2\times 2x\exp(-x^4)dx$$

$$=-\int_0^1 x^3\exp(-x^4)dx \quad =-\frac{1}{4}\int_0^1\exp(-x^4)dx^4$$

$$=\frac{1}{4}\left(\frac{1}{e}-1\right)$$

Ex. 证明$I_n \triangleq \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$,并求$I_n$.

Proof. 令$t = \frac{\pi}{2} - x$,则

$$\int_0^{\pi/2} \sin^n x dx = -\int_{\pi/2}^{0} \sin^n (\frac{\pi}{2} - t) dt = \int_0^{\pi/2} \cos^n t dt.$$

$$I_n = -\int_0^{\pi/2} \sin^{n-1} x d\cos x$$

$$= -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x dx$$

$$= (n-1)\int_0^{\pi/2} (1 - \sin^2 x) \sin^{n-2} x dx$$

$$= (n-1) I_{n-2} - (n-1) I_n.$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

$$I_1 = \int_0^{\pi/2} \sin x dx = -\cos x \Big|_0^{\pi/2} = 1,$$

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \cdots \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2},$$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \cdots \cdot \frac{2}{3} \cdot 1 = \frac{(2n-2)!!}{(2n-1)!!}. \square$$

09 定积分——设法消除未知项

Ex. $\int_{-\pi}^{\pi}\frac{\cos x}{1+e^x}dx=$__________

$\frac{\cos x}{1+e^x}$的原函数不好求！

原式$=\int_{0}^{\pi}\frac{\cos x}{1+e^x}dx+\int_{-\pi}^{0}\frac{\cos x}{1+e^x}dx$

$$\int_{-\pi}^{0}\frac{\cos x}{1+e^x}dx\overset{y=-x}{=}\int_{\pi}^{0}\frac{\cos(-y)}{1+e^{-y}}d-y=-\int_{\pi}^{0}\frac{\cos(y)}{1+e^{-y}}dy=\int_{0}^{\pi}\frac{\cos(y)}{1+e^{-y}}dy=\int_{0}^{\pi}\frac{e^y\cos(y)}{1+e^y}dy$$

$\therefore$ 原式$=\int_{0}^{\pi}\frac{\cos x}{1+e^x}dx+\int_{0}^{\pi}\frac{e^y\cos(y)}{1+e^y}dy=\int_{0}^{\pi}\frac{\cos x}{1+e^x}+\frac{e^x\cos(x)}{1+e^x}dx=\int_{0}^{\pi}\frac{\cos x(1+e^x)}{1+e^x}dx$

$=\int_{0}^{\pi}\cos x dx=\sin\pi-\sin 0=0$

09 定积分——奇函数

Ex. $\int_{-1/2}^{1/2} \frac{\arcsin x}{\sqrt{1-3x^2}} dx =$ ____0____ $\frac{\arcsin x}{\sqrt{1-3x^2}}$的原函数不好求！

$$f(x) = \frac{\arcsin x}{\sqrt{1-3x^2}},\ f(-x) = \frac{\arcsin(-x)}{\sqrt{1-3x^2}} = f(x)$$

Hint. 奇函数在关于原点对称的区间上积分为0

09 定积分——偶函数

Ex. $\int_0^x e^{xt-t^2}dt = e^{x^2/4}\int_0^x e^{-t^2/4}dt$

$$\int_0^x e^{xt-t^2}dt = e^{\frac{x^2}{4}}\int_0^x e^{-x^2/4+xt-t^2}dt \ = e^{\frac{x^2}{4}}\int_0^x e^{-(t-\frac{x}{2})^2}dt$$

$$\int_0^x e^{-(t-\frac{x}{2})^2}dt \overset{s=t-\frac{x}{2}}{=} \int_{-x/2}^{x/2} e^{-s^2}ds = 2\int_0^{x/2} e^{-s^2}ds(\text{偶函数})$$

$$\overset{s=q/2}{=} 2\int_0^x e^{-q^2/4}d(q/2) = \int_0^x e^{-q^2/4}dq$$

09 定积分相关不等式证明

Prop3. (单调性) $f, g \in R[a,b]$, 且$f(x) \le g(x)$,则

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

Prop4.(积分估值) $f \in R[a,b] \Rightarrow |f| \in R[a,b]$, 且

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

Thm. 设$f\in C[a,b]$, $f(x)\ge 0$, $\int_a^b f(x)dx=0$. 求证:$f(x)\equiv 0$.

Proof. 反证. 设$f(x)$在不恒为0, 则$\exists x_0\in[a,b]$, $s.t.$ $f(x_0)>0$.

不妨设$x_0\in(a,b)$. $f\in C[a,b]$, 则$\exists\delta>0$, $s.t.$

$$f(x)>f(x_0)/2>0,\quad \forall x\in[x_0-\delta,x_0+\delta].$$

而$f(x)\ge 0$, 于是

$$0=\int_a^b f(x)dx=\int_a^{x_0-\delta} f(x)dx+\int_{x_0-\delta}^{x_0+\delta} f(x)dx+\int_{x_0+\delta}^{b} f(x)dx$$

$$\ge 0+\int_{x_0-\delta}^{x_0+\delta}\frac{f(x_0)}{2}dx+0\ \ge f(x_0)\delta>0,$$ 矛盾.□

Thm.(Cauchy不等式) $f, g \in R[a,b]$,则

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx.$$

Proof.令$A=\int_a^b f^2(x)dx, B=\int_a^b f(x)g(x)dx, C=\int_a^b g^2(x)dx.$

则 $0 \le \int_a^b [tf(x)+g(x)]^2 dx = At^2+2Bt+C, \quad \forall t \in \mathbb{R}.$

故 $(2B)^2-4AC \le 0.$□

Thm.(积分第一中值定理) $f\in C[a,b], g\in R[a,b], g$不变号，

则$\exists\xi\in[a,b]$, s.t. $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx.$ (*)

特别地，$g(x)\equiv 1$时，$\int_a^b f(x)dx = f(\xi)(b-a).$

Proof.不妨设$g\geq 0$.记f在$[a,b]$上的最大值与最小值为M,

m,则 $m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$

若$\int_a^b g(x)dx = 0$, 则$\int_a^b f(x)g(x)dx = 0, \forall\xi\in[a,b]$, (*)成立.

若$\int_a^b g(x)dx > 0, \exists\xi\in[a,b]$, s.t.

$$f(\xi) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \in [m, M].\square$$

Ex. $\forall x>0$, 证明: $\int_0^x \frac{\sin t}{1+t}dt \ge 0$

分析: $\sin x$在$[2k\pi,(2k+1)\pi]$取非负,$[(2k+1)\pi,(2k+2)\pi]$取非正

$\forall x>0$, 1°.如果$\exists k>0, k\in\mathbb{N}, s.t. 2k\pi \le x \le (2k+1)\pi$

$$\because \forall x, 2k\pi \le x \le (2k+1)\pi, \text{有} \frac{\sin x}{1+x} \ge 0 \quad \therefore \int_0^x \frac{\sin t}{1+t}dt \ge \int_0^{2k\pi} \frac{\sin t}{1+t}dt$$

2°.如果$\exists k>0, k\in\mathbb{N}, s.t.(2k+1)\pi \le x \le (2k+2)\pi$

$$\because \int_x^{(2k+2)\pi} \frac{\sin t}{1+t}dt \le 0, \therefore \int_0^x \frac{\sin t}{1+t}dt \ge \int_0^x \frac{\sin t}{1+t}dt + \int_x^{(2k+2)\pi} \frac{\sin t}{1+t}dt = \int_0^{(2k+2)\pi} \frac{\sin t}{1+t}dt$$

如果可以证明: $\int_0^{2k\pi} \frac{\sin t}{1+t}dt \ge 0$, 问题即告解决.

Ex. $\forall x > 0$, 证明：$\int_0^x \frac{\sin t}{1+t} dt \geq 0$

现在证明：$\int_0^{2k\pi} \frac{\sin t}{1+t} dt \geq 0.$

$$\int_0^{2k\pi} \frac{\sin t}{1+t} dt = \int_0^{2\pi} \frac{\sin t}{1+t} dt + \int_{2\pi}^{4\pi} \frac{\sin t}{1+t} dt + \ldots + \int_{(2k-2)\pi}^{2k\pi} \frac{\sin t}{1+t} dt$$

$$\because \int_{(2m-2)\pi}^{2m\pi} \frac{\sin t}{1+t} dt = \int_{(2m-2)\pi}^{(2m-1)\pi} \frac{\sin t}{1+t} dt + \int_{(2m-1)\pi}^{2m\pi} \frac{\sin t}{1+t} dt$$

$$= \int_{(2m-2)\pi}^{(2m-1)\pi} \frac{\sin t}{1+t} dt + \int_{(2m-2)\pi}^{(2m-1)\pi} \frac{\sin(t+\pi)}{1+t+\pi} d(t+\pi)$$

$$= \int_{(2m-2)\pi}^{(2m-1)\pi} \sin t \left(\frac{1}{1+t} - \frac{1}{1+t+\pi}\right) dt = \int_{(2m-2)\pi}^{(2m-1)\pi} \sin t \frac{\pi}{(1+t)(\pi+1+t)} dt \geq 0 \quad \therefore \int_0^{2k\pi} \frac{\sin t}{1+t} dt \geq 0$$

Ex. $f \in C^1[0,1],\ f(0)=0,\ f(1)=1$

证明：$\int_0^1 |f(x)-f'(x)|\,dx \geq 1/e$

$\because 0 \leq e^{-x} \leq 1, \therefore e^{-x}|f(x)-f'(x)| \leq |f(x)-f'(x)|$

$\therefore \int_0^1 |f(x)-f'(x)| \geq \int_0^1 |e^{-x}(f(x)-f'(x))|dx \geq \left|\int_0^1 e^{-x}(f(x)-f'(x))dx\right|$

$$= |e^{-x}f(x)|_0^1| = \frac{1}{e}$$

Ex. $f\in C^1[a,b]$, 证明：$\max_{a\le x\le b}|f(x)|\le\frac{1}{b-a}\left|\int_a^b f(x)dx\right|+\int_a^b|f'(x)|dx$

证. $\because f\in C^1[a,b]$, $\therefore\exists\xi\in[a,b]$, s.t. $\frac{1}{b-a}\left|\int_a^b f(x)dx\right|=\left|\frac{1}{b-a}\int_a^b f(x)dx\right|=|f(\xi)|$

即证. $\max_{a\le x\le b}|f(x)|\le|f(\xi)|+\int_a^b|f'(x)|dx$

$\forall x\in[a,b]$. $|f(x)|\le|f(\xi)|+|f(x)-f(\xi)|\le|f(\xi)|+\left|\int_\xi^x f'(x)dx\right|$

$\le|f(\xi)|+\left|\int_\xi^x|f'(x)|dx\right|\le|f(\xi)|+\int_a^b|f'(x)|dx$

由x的任意性. $\max_{a\le x\le b}|f(x)|\le|f(\xi)|+\int_a^b|f'(x)|dx$

Ex. $f \in C^1[0,1], f(0)=0,$ 证明：$\int_0^1 f^2(x)dx \le \int_0^1 f'^2(x)dx$

证. $\because f(x) = f(0) + \int_0^x f'(t)dt = \int_0^x f'(t)dt$

$$f^2(x) \le (\int_0^x f'(x)dx)^2 = (\int_0^x 1 \times f'(x)dx)^2$$ 由柯西-施瓦茨不等式

$$\le \int_0^x 1^2 dx \int_0^x f'^2(x)dx = x\int_0^x f'^2(x)dx \le \int_0^1 f'^2(x)dx, \forall x \in [0,1]$$

$$\therefore \int_0^1 f^2(x)dx \le \int_0^1 (\int_0^1 f'^2(x)dx)dx = \int_0^1 f'^2(x)dx$$

Ex. f在$[a,b]$上连续可导, 则$\lim_{\lambda\to+\infty}\int_a^b f(x)\cos\lambda x dx=0$

证. $\because \int_a^b f(x)\cos\lambda x dx=\frac{1}{\lambda}\int_a^b f(x) d\sin\lambda x$ $\qquad |f'(x)|\le M, \forall x\in[a,b]$

$$=\frac{1}{\lambda}(f(b)\sin(\lambda b)-f(a)\sin(\lambda a))-\frac{1}{\lambda}\int_a^b f'(x)\sin\lambda x dx$$

$$\left|\frac{1}{\lambda}\int_a^b f'(x)\sin\lambda x dx\right|\le\frac{1}{\lambda}\int_a^b |f'(x)\sin\lambda x| dx\le\frac{1}{\lambda}\int_a^b |f'(x)| dx\le\frac{1}{\lambda}(b-a)M$$

$$\therefore \lim_{\lambda\to+\infty}\frac{1}{\lambda}\int_a^b f'(x)\sin\lambda x dx=0 \quad \lim_{\lambda\to+\infty}\frac{1}{\lambda}(f(b)\sin(\lambda b)-f(a)\sin(\lambda a))=0$$

Ex.(A Hard Problem!) $\forall x>0$, 证明：$\lim_{x\to+\infty}\int_0^x \frac{\sin t}{1+t}dt$存在

取$k=[x/2\pi]$

$$\because \max\left(\int_0^{2k\pi}\frac{\sin t}{1+t}dt,\int_0^{(2k+2)\pi}\frac{\sin t}{1+t}dt\right)\le\int_0^x\frac{\sin t}{1+t}dt\le\int_0^{(2k+1)\pi}\frac{\sin t}{1+t}dt$$

转而考虑数列极限$\lim_{n\to\infty}\int_0^{2n\pi}\frac{\sin t}{1+t}dt$和$\lim_{n\to\infty}\int_0^{(2n+1)\pi}\frac{\sin t}{1+t}dt$

$$\left|\int_0^{2n\pi}\frac{\sin t}{1+t}dt\right|=\sum_{m=1}^{n}\int_{(2m-2)\pi}^{(2m-1)\pi}\sin t\frac{\pi}{(1+t)(\pi+1+t)}dt$$

$$\le\pi\sum_{m=1}^{n}\int_{(2m-2)\pi}^{(2m-1)\pi}\frac{1}{(1+t)^2}dt\le\pi\int_0^{(2n-1)\pi}\frac{1}{(1+t)^2}dt\le\pi\int_0^{+\infty}\frac{1}{(1+t)^2}dt$$

$\int_0^{2n\pi}\frac{\sin t}{1+t}dt\uparrow\ \therefore\int_0^{2n\pi}\frac{\sin t}{1+t}dt$有极限，又$\lim_{n\to\infty}\int_{(2n-1)\pi}^{2n\pi}\frac{\sin t}{1+t}dt=0.$

10/ 反常积分

- 无穷限积分

Def.若 $\lim\limits_{A\to+\infty}\int_a^A f(x)dx = I$,则称$f$在$[a,+\infty)$上的广义积分收敛,称$I$为$f$在$[a,+\infty)$上的广义积分(值),记作

$$\int_a^{+\infty} f(x)dx = \lim_{A\to+\infty}\int_a^A f(x)dx.$$

若 $\lim\limits_{A\to+\infty}\int_a^A f(x)dx$不存在,则称广义积分$\int_a^{+\infty} f(x)dx$发散.

Remark.$\int_{-\infty}^{a} f(x)dx \triangleq \lim\limits_{A\to-\infty}\int_A^a f(x)dx.$

•瑕积分(无界函数积分)

Def. f在$[a,b)$上定义,在b点附近无界(此时称$x=b$为f的一个瑕点),若$\forall\delta\in(0,b-a)$, $f\in R[a,b-\delta]$,且

$$\lim_{\delta\to0^+}\int_a^{b-\delta} f(x)dx = I,$$

则称f在$[a,b)$上的瑕积分收敛,称I为f在$[a,b)$上的瑕积分(值),记作

$$\int_a^b f(x)dx = \lim_{\delta\to0^+}\int_a^{b-\delta} f(x)dx.$$

若$\lim\limits_{\delta\to0^+}\int_a^{b-\delta} f(x)dx$不存在,则称瑕积分$\int_a^b f(x)dx$发散.

10/ 反常积分

考点：

（1） 反常积分的计算；【类似于定积分，三个法宝】

（2）判定反常积分是否收敛；

-重点掌握：不变号函数

-了解：阿贝尔和狄利克雷判敛

10/ 反常积分的计算

$$\int_a^{+\infty} f(x)dx = \lim_{A\to+\infty}\int_a^A f(x)dx. \qquad \int_a^b f(x)dx = \lim_{\delta\to 0^+}\int_a^{b-\delta} f(x)dx$$

Ex. $\int_0^{+\infty} x^{n-1}e^{-x}dx =$ ______.

$$\int_0^{+\infty} x^{n-1}e^{-x}dx = \int_0^{+\infty} -x^{n-1}de^{-x} = -x^{n-1}e^{-x} - \int_0^{+\infty} e^{-x}d\left(-x^{n-1}\right) =$$

$$-x^{n-1}e^{-x}\Big|_0^{+\infty} + (n-1)\int_0^{+\infty} e^{-x}x^{n-2}dx$$

$$\int_0^{+\infty} x^{n-1}e^{-x}dx = (n-1)\int_0^{+\infty} x^{n-2}e^{-x}dx = (n-1)(n-2)\int_0^{+\infty} x^{n-3}e^{-x}dx = \ldots = (n-1)!$$

10 反常积分的计算

Ex. 已知$\int_0^{\pi/2}\ln(\sin x)dx=-\frac{\pi}{2}\ln 2$,求

(1).$\int_0^{\pi/2}x\cot x dx=$____________;

$$(1)\int_a^{\pi/2}x\cot x dx=\int_a^{\pi/2}\frac{x\cos x}{\sin x}dx=\int_a^{\pi/2}\frac{xd(\sin x)}{\sin x}$$

$$=\int_a^{\pi/2}xd(\ln\sin x)=x\ln\sin x\Big|_a^{\pi/2}-\int_a^{\pi/2}\ln\sin x dx$$

$$\therefore\int_a^{\pi/2}x\cot x dx=-a\ln\sin a-\int_a^{\pi/2}\ln\sin x dx$$

令$a\to 0+$,有$\int_0^{\pi/2}x\cot x dx=\lim_{a\to 0+}\int_a^{\pi/2}x\cot x dx=$

$$=\lim_{a\to 0+}-a\ln\sin a-\lim_{a\to 0+}\int_a^{\pi/2}\ln\sin x dx=-\int_0^{\pi/2}\ln\sin x dx=\frac{\pi}{2}\ln 2$$

10 反常积分的计算

Ex. 已知$\int_0^{\pi/2}\ln(\sin x)dx=-\frac{\pi}{2}\ln 2$，求

(2). $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx=$ ________;

令$x=\sin t$. $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx=\int_0^{\frac{\pi}{2}}\frac{\ln\sin t}{\cos t}d\sin t=\int_0^{\frac{\pi}{2}}\ln\sin t\,dt$

(3). $I=\int_0^1 \frac{\ln(1+x)}{1+x^2}dx.$

Ex. $I=\int_0^1 \frac{\ln(1+x)}{1+x^2}dx.$

解: $I=\int_0^{\pi/4}\ln(1+\tan t)dt \qquad (t=\arctan x)$

$$=\int_0^{\pi/4}\ln(\sin t+\cos t)dt-\int_0^{\pi/4}\ln(\cos t)dt=I_1-I_2.$$

$$I_1=\int_0^{\pi/4}\left(\ln\sqrt{2}+\ln\sin(t+\frac{\pi}{4})\right)dt$$

$$=\frac{\pi}{8}\ln 2+\int_0^{\pi/4}\ln\sin(t+\frac{\pi}{4})dt$$

$$=\frac{\pi}{8}\ln 2+\int_0^{\pi/4}\ln\cos(\frac{\pi}{4}-t)dt=\frac{\pi}{8}\ln 2+I_2.\quad I=\frac{\pi}{8}\ln 2.\square$$

10、反常积分的判敛

Hint. $\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_a^t f(x)dx$, $\int_a^t f(x)dx$记成$\phi(t)$

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty}\phi(t)$　如果$\phi(t)$是单调增加的,那么

$\lim_{t\to+\infty}\phi(t) = a$或是$+\infty$　$\lim_{t\to+\infty}\phi(t) = a \Leftrightarrow \phi(t)$有上界

f连续,$\phi(t)$是单调增加$\Leftrightarrow f(x)\geq 0$

$\therefore$优先考虑不变号函数$f(x)$的积分$\int_a^{+\infty} f(x)dx$

10 反常积分的判敛

Thm.(比较判敛法) 设$0 \le f(x) \le Cg(x), \forall x > K$,则

(1)$\int_a^{+\infty} g(x)dx$ 收敛 $\Rightarrow \int_a^{+\infty} f(x)dx$ 绝对收敛;

(2)$\int_a^{+\infty} f(x)dx$发散 $\Rightarrow \int_a^{+\infty} g(x)dx$发散.

Thm.(比较判敛法-极限形式) 设f, g非负, $\lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} = C$.

(1)若$C > 0$,则$\int_a^{+\infty} f(x)dx$与$\int_a^{+\infty} g(x)dx$同敛散;

(2)若$C = 0$,且$\int_a^{+\infty} g(x)dx$收敛,则$\int_a^{+\infty} f(x)dx$收敛;

(3)若$C = +\infty$,且$\int_a^{+\infty} g(x)dx$发散,则$\int_a^{+\infty} f(x)dx$发散.

10 反常积分的判敛

Ex.判别广义积分的收敛性

$$\int_0^{+\infty}\frac{\sin^2 x}{1+x^2}dx,\int_1^{+\infty}\frac{x^2dx}{e^x+x},\int_1^{+\infty}\frac{\ln xdx}{\sqrt{x^3+2x+1}},\int_2^{+\infty}\frac{\ln xdx}{x(\ln x+9)}.$$

解:(1) $\frac{\sin^2 x}{1+x^2}\le\frac{1}{1+x^2},\int_0^{+\infty}\frac{1}{1+x^2}dx$收敛, 故$\int_0^{+\infty}\frac{\sin^2 x}{1+x^2}dx$绝对收敛.

(2) $\lim\limits_{x\to+\infty}\frac{x^2}{e^x+x}\Big/\frac{1}{x^2}=\lim\limits_{x\to+\infty}\frac{x^4}{e^x+x}=0.\int_1^{+\infty}\frac{dx}{x^2}$收敛, $\int_1^{+\infty}\frac{x^2dx}{e^x+x}$收敛.

(3) $\lim\limits_{x\to+\infty}\frac{\ln x}{\sqrt{x^3+2x+1}}\Big/\frac{1}{x^{5/4}}=0,\int_1^{+\infty}\frac{dx}{x^{5/4}}$收敛, $\int_1^{+\infty}\frac{\ln xdx}{\sqrt{x^3+2x+1}}$收敛.

(4) $\lim\limits_{x\to+\infty}\frac{\ln x}{x(\ln x+9)}\Big/\frac{1}{x}=1,\ \int_1^{+\infty}\frac{dx}{x}$发散, $\int_1^{+\infty}\frac{\ln x}{x(\ln x+9)}$发散.

10、反常积分的判敛

如果$f(x)$变号【不是恒为正或负】，考虑$|f(x)|$的无穷限积分$\int_a^{+\infty}|f(x)|dx$ 收敛

Def. 若$\int_a^{+\infty}|f(x)|dx$ 收敛，则称$\int_a^{+\infty} f(x)dx$绝对收敛；

$\int_a^{+\infty}|f(x)|dx$ 发散，$\int_a^{+\infty} f(x)dx$收敛，则称$\int_a^{+\infty} f(x)dx$条件收敛.

Ex. $\int_a^{+\infty}|f(x)|dx$ 收敛，则$\int_a^{+\infty} f(x)dx$收敛.

Ex. $\int_a^{+\infty}\frac{\sin x}{x^2}dx$ 收敛　解. $\because \int_a^{+\infty}\frac{|\sin x|}{x^2}dx$收敛

瑕积分是类似的！

Thm.(比较判敛法) 设b为瑕积分$\int_a^b f(x)dx$的唯一瑕点,

$|f(x)| \le Cg(x), \forall x \in (b-\delta, b)$, 则

(1) $\int_a^b g(x)dx$ 收敛 $\Rightarrow \int_a^b f(x)dx$ 绝对收敛;

(2) $\int_a^b |f(x)|dx$发散 $\Rightarrow \int_a^b g(x)dx$发散.

Thm.(比较判敛法-极限形式) 设b为瑕积分$\int_a^b f(x)dx$的唯一瑕点, f,g非负, $\lim\limits_{x\to b^-}\dfrac{f(x)}{g(x)}=C$.

(1)若$C>0$,则$\int_a^b f(x)dx$与$\int_a^b g(x)dx$同敛散; 找$f(x)$的等价无穷大.

(2)若$C=0$,且$\int_a^b g(x)dx$收敛,则$\int_a^b f(x)dx$收敛;

(3)若$C=+\infty$,且$\int_a^b g(x)dx$发散,则$\int_a^b f(x)dx$发散.

Ex. $\int_0^1 \sqrt{\cot x}dx$ 的收敛性.

找$f(x)$的等价无穷大.

解: $x=0$是瑕点.

$$\lim_{x\to 0^+}\frac{\sqrt{\cot x}}{1/\sqrt{x}}=\lim_{x\to 0^+}\sqrt{\cos x\cdot\frac{x}{\sin x}}=1,$$

$\int_0^1 \frac{1}{\sqrt{x}}dx$收敛,

故$\int_0^1 \sqrt{\cot x}dx$收敛.□

Ex. $p,q>0$, 讨论 $\int_0^{\pi/2}\frac{1}{\sin^p x\cos^q x}dx$ 的收敛性.

解: $\frac{1}{\sin^p x\cos^q x}\geq 0$ $\qquad \frac{1}{\sin^p x\cos^q x}$ 在 0 和 $\frac{\pi}{2}$ 处存在瑕点.

$$\frac{1}{\sin^p x\cos^q x}\sim\frac{1}{x^p}, x\to 0$$

$$\frac{1}{\sin^p x\cos^q x}\sim\frac{1}{x^q}, x\to\frac{\pi}{2}$$

$\therefore p,q<1$ 时收敛.

Ex. $p>0$, 讨论 $\int_0^{+\infty}\frac{\ln(1+x)}{x^p}dx$ 的收敛性.

解: $\int_0^{+\infty}\frac{\ln(1+x)}{x^p}dx$ 收敛

$\Leftrightarrow \int_0^{1}\frac{\ln(1+x)}{x^p}dx$ 与 $\int_1^{+\infty}\frac{\ln(1+x)}{x^p}dx$ 同时收敛.

$\lim\limits_{x\to 0^+} x^{p-1}\cdot\frac{\ln(1+x)}{x^p}=1$, $\int_0^{1}\frac{1}{x^{p-1}}dx$ 收敛 $\Leftrightarrow p-1<1$,

故 $\int_0^{1}\frac{\ln(1+x)}{x^p}dx$ 收敛 $\Leftrightarrow p<2$.

当$0 < p \le 1$时，$\int_1^{+\infty} \frac{1}{x^p} dx$发散，从而$\int_1^{+\infty} \frac{\ln(1+x)}{x^p} dx$发散.

当$p > 1$时，$\forall q \in (1, p)$，$\lim\limits_{x \to +\infty} x^q \cdot \frac{\ln(1+x)}{x^p} = 0$，$\int_1^{+\infty} \frac{1}{x^q} dx$收敛，

从而$\int_1^{+\infty} \frac{\ln(1+x)}{x^p} dx$收敛.

综上，$\int_0^{+\infty} \frac{\ln(1+x)}{x^p} dx$收敛$\Leftrightarrow 1 < p < 2$. □

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