Midterm Exam of Fundamentals of Physics (3), Electromagnetism

Open book test from 10:40am to 12:15 pm on April 3, 2023

Instruction to students.

a) This exam paper contains FOUR main questions and TWENTY sub-questions. It comprises THREE printed pages including this page.

b) You can bring two books (at your choice) in the exam.

c) You are allowed to use a basic calculator in the exam.

d) If you carry out a computation please write down how you derived the answer in the answer sheets. If you only write down the answer and it is wrong, no points are given for the question.

Good luck!

Problem 1:

Consider an electric field that the x component is given as $E_x = kxy$, where k is a constant.

(1) Construct a valid form of the electric field for the other two components.

(2) For a given x component of the electric field, $E_x = kxy$, if there is no charge near the origin of the coordinates and the field is translationally invariant along z direction, construct the electric field of the other two components and the potential for the electric field.

(3) Check the divergence theorem of the integral form $\left(\int_V (\nabla \cdot \boldsymbol{V}) d\tau = \oint_S \boldsymbol{V} \cdot d\boldsymbol{a}\right)$ for the electric field found in the problem above (1.(2)) at the cylinder as shown in Fig.1.

(4) Check Stokes theorem of the of the integral form $\left(\int_S (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{a} = \oint_P \boldsymbol{V} \cdot d\boldsymbol{l}\right)$ for the electric field found in the problem above (1.(2)) at a circular ring on the xy plane with the radius a as shown in Fig.2.

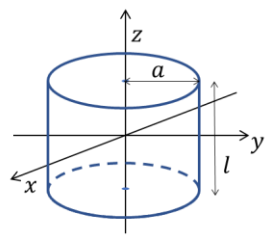


Figure 1

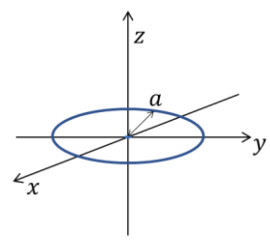


Figure 2

Solution:

(1) Since $\nabla \times \boldsymbol{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\boldsymbol{z}} = \boldsymbol{0}$,

For z-axis, $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \;\rightarrow \frac{\partial E_y}{\partial x} = \frac{\partial (kxy)}{\partial y} = kx, \therefore E_y = \frac{1}{2}kx^2 + f_y(y, z)$

For y-axis, $\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 \;\rightarrow \frac{\partial E_z}{\partial x} = \frac{\partial (kxy)}{\partial z} = 0, \therefore E_z = f_z(y, z)$

For a simple choice, let's put $f_y(y, z) = f_z(y, z) = 0$.

Then $\boldsymbol{E} = kxy\,\hat{\boldsymbol{x}} + \frac{1}{2}kx^2\,\hat{\boldsymbol{y}}$

(2) From above $E_y = \frac{1}{2}kx^2 + f_y(y, z)$ and $E_z = f_z(y, z)$.

Since the field is translationally invariant along z-direction, no electric field along z-direction, that is, $E_z = 0$ and $f_y(y, z) = f_y(y)$.

Since no charge near center, $\nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$, which is, $ky + \frac{\partial f_y(y)}{\partial y} = 0$.

$\therefore f_y(y) = -\frac{1}{2}ky^2 + C$. For simplicity, let's put $C = 0$. Then $E_y = \frac{1}{2}k(x^2 - y^2)$.

Then $\boldsymbol{E} = kxy\,\hat{\boldsymbol{x}} + \frac{1}{2}k(x^2 - y^2)\hat{\boldsymbol{y}}$

For the potential, -$\nabla V = \boldsymbol{E}$.

That is, $\frac{\partial V}{\partial x} = -kxy \rightarrow V = -\frac{k}{2}x^2y + C(y)$ and $\frac{\partial V}{\partial y} = -\frac{k}{2}x^2 + \frac{\partial C}{\partial y} = -\frac{k}{2}(x^2 - y^2)$, then $\frac{\partial C}{\partial y} = +\frac{k}{2}y^2$.

$\therefore C(y) = +\frac{k}{6}y^3$. Finally, $V = -\frac{k}{2}x^2y + \frac{k}{6}y^3$

(3) Since $\nabla \cdot \boldsymbol{E} = 0$, the left side is zero, $\int_V (\nabla \cdot \boldsymbol{E}) d\tau = 0$.

Let's show the right side is also zero, $\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = 0$.

Since $E_z = 0$, we do not need to consider the top and the bottom surfaces of the cylinder in Fig. 1.

Let's use the cylindrical coordinates.

From Eq. (1.74) and Eq. (1.75), $\hat{\boldsymbol{x}} = \cos\phi\, \hat{\boldsymbol{s}} - \sin\phi\, \hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{y}} = \sin\phi\, \hat{\boldsymbol{s}} + \cos\phi\, \hat{\boldsymbol{\phi}}$.

Therefore, $\boldsymbol{E} = kxy\, \hat{\boldsymbol{x}} + \frac{1}{2}k(x^2 - y^2)\hat{\boldsymbol{y}} = k\, s\cos\phi\, s\sin\phi\, \hat{\boldsymbol{s}} + \frac{1}{2}k((s\cos\phi)^2 - (s\sin\phi)^2)\hat{\boldsymbol{\phi}} = \frac{1}{2}ks^2 \sin 2\phi\, \hat{\boldsymbol{s}} + \frac{1}{2}ks^2 \cos 2\phi\, \hat{\boldsymbol{\phi}} = \frac{1}{2}ks^2\left(\sin 2\phi\, \hat{\boldsymbol{s}} + \cos 2\phi\, \hat{\boldsymbol{\phi}}\right)$

We can write $d\boldsymbol{a} = ad\phi dz\hat{\boldsymbol{s}}$, and then

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \frac{1}{2}ks^2\left(\sin 2\phi\, \hat{\boldsymbol{s}} + \cos 2\phi\, \hat{\boldsymbol{\phi}}\right) \cdot ad\phi dz\hat{\boldsymbol{s}} = \frac{1}{2}ka^3 l \int_0^{2\pi} \sin 2\phi\, d\phi$$

$$= \frac{1}{2}ka^3 l \left[-\frac{1}{2}\cos 2\phi\right]_0^{2\pi} = 0$$

(4) Since $\nabla \times \boldsymbol{E} = \boldsymbol{0}$, the left side is zero, $\int_s (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot \boldsymbol{da} = 0$.

Let's show the right side is also zero, $\oint_{\boldsymbol{P}} \boldsymbol{E} \cdot d\boldsymbol{l} = 0$.

We can write $d\boldsymbol{l} = ad\phi\hat{\boldsymbol{\phi}}$, $\oint_{\boldsymbol{P}} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_0^{2\pi} \frac{1}{2}ks^2\left(\sin 2\phi\, \hat{\boldsymbol{s}} + \cos 2\phi\, \hat{\boldsymbol{\phi}}\right) \cdot ad\phi\hat{\boldsymbol{\phi}} = \frac{1}{2}ka^3 \int_0^{2\pi} \cos 2\phi\, d\phi = \frac{1}{2}ka^3 \left[\frac{1}{2}\sin 2\phi\right]_0^{2\pi} = 0$.

Problem 2:

(1–3) Consider a number of point charges, N, each with the charge of Q/N, which are evenly distributed around a circle of radius a as shown in the right figure.

(1) Find out the electric field and the potential at the location of one of the charges, due to all the other charges. (You can leave your answer in the form of a sum.)

(2) Find the electrostatic energy of the whole system.

(3) In the limit of the number of charges increases infinitely, how do the strengths of the electric field, potential and the total energies go? Are they diverge or converge to finite values?

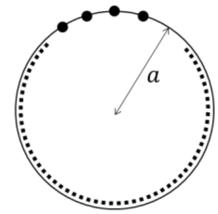


Figure 3

(4–6) Consider a circular ring of a radius a with a uniform line charge density λ and the total charge Q as shown in the right figure.

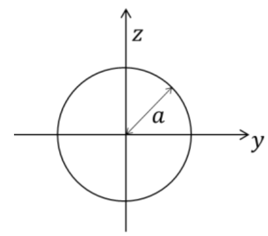


Figure 4

(4) Find out the electric field and the potential along the z-axis when $s \geq a$, where $s = \sqrt{y^2 + z^2}$.

(5) What is the total energy of the system?

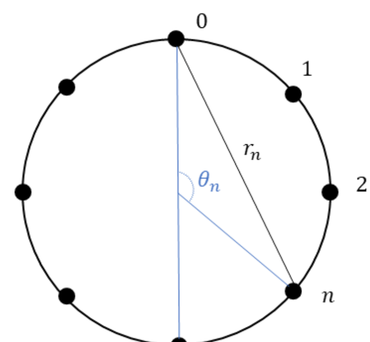
(6) Compare the energies of the systems between point charges and the uniform line charge densities. Are they consistent? If not, discuss the reason.

(Hint) $\int_0^\pi \frac{d\theta}{\sqrt{\beta - \alpha\cos\theta}} = \frac{2\,EllipticK[\frac{2\alpha}{\alpha+\beta}]}{\sqrt{\alpha+\beta}}$, $EllipticK[x] \approx \frac{\pi}{2} + \frac{\pi}{8}x + \frac{9\pi}{128}x^2$

Solution:

(1) Let's first find the potential of the 0th charge in the right figure.

$$V = \frac{1}{4\pi\epsilon_0}\sum_{n=1}^{N}\frac{Q}{N}\frac{1}{r_n} = \frac{1}{4\pi\epsilon_0}\sum_{n=1}^{N}\frac{Q}{N}\frac{1}{|2a\sin\left(\frac{\pi n}{N}\right)|} = \frac{Q}{8\pi\epsilon_0 a}\sum_{n=1}^{N}\frac{1}{N}\frac{1}{|\sin\left(\frac{\pi n}{N}\right)|}$$



$$\theta_n = \frac{2\pi}{N}n$$

$$r_n = 2a\sin\frac{\theta_n}{2} = 2a\sin\left(\frac{\pi n}{N}\right)$$

Let's find the electric field of the 0th charge, which should be vertical direction, or $\hat{\boldsymbol{z}}$ direction.

$$E_z = \frac{1}{4\pi\epsilon_0}\sum_{n=1}^{N}\frac{Q}{N}\frac{1}{r_n^2}\left|\sin\frac{\theta_n}{2}\right| = \frac{1}{4\pi\epsilon_0}\sum_{n=1}^{N}\frac{Q}{N}\frac{1}{\left(2a\sin\left(\frac{\pi n}{N}\right)\right)^2}\left|\sin\left(\frac{\pi n}{N}\right)\right|$$

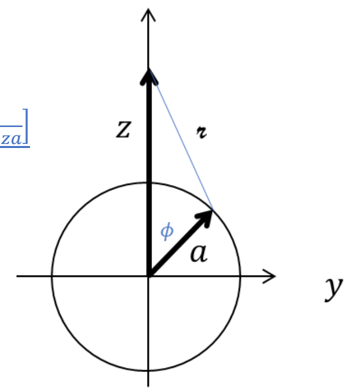
$$= \frac{Q}{16\pi\epsilon_0 a^2}\sum_{n=1}^{N}\frac{1}{N}\frac{1}{|\sin\left(\frac{\pi n}{N}\right)|}.$$

(2) The electrostatic energy for the 0th charge is given by $q_0 V = \frac{Q}{N}V$, which should be same to all the other charges. Therefore, according to Eq. (2.42), the total energy is written as

$$W = \frac{1}{2}NqV = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{8\pi\epsilon_0 a}\sum_{n=1}^{N}\frac{1}{N}\frac{1}{|\sin\left(\frac{\pi n}{N}\right)|}.$$

(3) The potential, electric field and the energy have the same form of the summation as $\sum_{n=1}^{N}\frac{1}{N}\frac{1}{\sin\left(\frac{\pi n}{N}\right)}$.

When N increases (while Q is a constant), we can assume that most of terms $\sin\left(\frac{\pi n}{N}\right)$ can be approximated by $\sin\left(\frac{\pi n}{N}\right) \approx \frac{\pi n}{N}$. This leads to the summation as $\sum_{n=1}^{N}\frac{1}{N}\frac{1}{\sin\left(\frac{\pi n}{N}\right)} \approx \sum_{n=1}^{N}\frac{1}{N}\frac{1}{\frac{\pi n}{N}} = \sum_{n=1}^{N}\frac{1}{\pi n}$, which diverges.



$$\mathscr{r} = \sqrt{z^2 + a^2 - 2za\cos\theta}$$

(4) Let's first find the potential at the point of z.

$$V(z) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda}{\mathscr{r}}dl = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda\, a d\phi}{\sqrt{z^2+a^2-2za\cos\phi}} = \frac{\lambda a}{4\pi\epsilon_0}2\frac{2\ EllipticK\left[\frac{2za}{z^2+a^2+2za}\right]}{\sqrt{z^2+a^2+2za}}$$

$$= \frac{\lambda a}{\pi\epsilon_0}\frac{\frac{\pi}{2}+\frac{\pi}{8}\left(\frac{2za}{(z+a)^2}\right)+\frac{9\pi}{128}\left(\frac{2za}{(z+a)^2}\right)^2+\cdots}{(z+a)} = \frac{\lambda a}{2\epsilon_0}\left(\frac{1}{z+a}+\frac{1}{2}\frac{az}{(z+a)^3}+\frac{9}{16}\frac{(az)^2}{(z+a)^5}+\cdots\right)$$

Along the z axis, the electric field should be also along the z direction.

$$E_z(z) = -\frac{\partial V}{\partial z} = \frac{\lambda a}{2\epsilon_0}\left(\frac{1}{(z+a)^2}+\frac{1}{2}\frac{a(2z-a)}{(z+a)^4}+\frac{9}{16}\frac{za^2(3z-2a)}{(z+a)^6}+\cdots\right)$$

(5) The total energy of the system is given by the Eq. (2.43),

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{1}{2}\int\lambda V\,dl = \frac{2\pi l\lambda}{2}V = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\left(\frac{1}{z+a}+\frac{1}{2}\frac{az}{(z+a)^3}+\frac{9}{16}\frac{(az)^2}{(z+a)^5}+\cdots\right)\Big|_{z=a}$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\left(\frac{1}{2a}+\frac{1}{2}\frac{a^2}{(2a)^3}+\frac{9}{16}\frac{(a^2)^2}{(2a)^5}+\cdots\right) = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\frac{1}{2a}\left(1+\frac{1}{4}+\frac{9}{256}+\cdots\right)$$

(6) Although the total charge Q are same, the total energies of the rings with point charges and with line charge density λ are different. The energy of point charged ring diverges, but that of line charged ring has a certain value. The difference comes mainly from the fact that the point charge has infinitely small distance. When the distances between point charges are getting smaller, the energy diverges. But the line charge density does not have such an unphysical problem.

(*) This statement is not correct, since $V(z)$ in problem 2.(4) is also diverging. In this situation, it is difficult to compare even in qualitatively. I am sorry for my mistake and I should give all of you full score (5) for this problem.

Problem 3:

Consider two semi-infinite grounded conducting planes that meet at right angles. In the region between them, there is a dipole $\boldsymbol{p}$, situated as shown in Fig. 5.

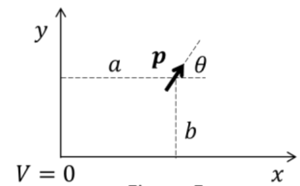


Figure 5

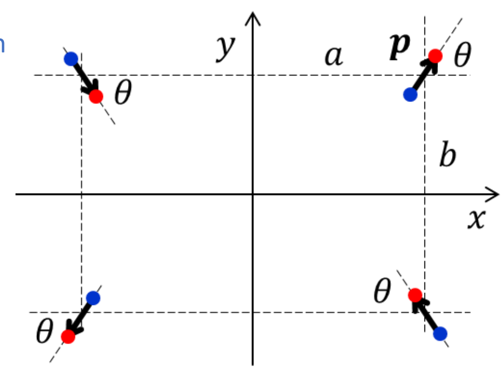
(1) Find the positions and angles of image dipoles and write down (2) the potential and the electric field from all the dipoles. And show that the potential satisfies all the boundary conditions.

(3) Find the force on the dipole $\boldsymbol{p}$.

(4) How much work did it take to bring the dipole $\boldsymbol{p}$ in from infinity?

Solution:

(1) We can consider the dipole as infinitely close two-charge system as shown in the right figure. Then similar to the problem 3.11, we can set the image charges as shown in the right figure.
The original dipole at $\boldsymbol{s_1} = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$ is written as
$\boldsymbol{p_1} = p(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}}) = p_x\hat{\boldsymbol{x}} + p_y\hat{\boldsymbol{y}}$
Therefore, the image dipoles can be constructed as follows.
At $\boldsymbol{s_2} = -a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}, \boldsymbol{p_2} = p_x\hat{\boldsymbol{x}} - p_y\hat{\boldsymbol{y}}$,
at $\boldsymbol{s_3} = -a\hat{\boldsymbol{x}} - b\hat{\boldsymbol{y}}, \boldsymbol{p_3} = -p_x\hat{\boldsymbol{x}} - p_y\hat{\boldsymbol{y}}$,
at $\boldsymbol{s_4} = a\hat{\boldsymbol{x}} - b\hat{\boldsymbol{y}}, \boldsymbol{p_4} = -p_x\hat{\boldsymbol{x}} + p_y\hat{\boldsymbol{y}}$.



(2) Let's construct the potential and make sure the boundary conditions. And let $k = \frac{1}{4\pi\epsilon_0}$ for simplicity.

The potential from $\boldsymbol{p_1}$ is given by $V_1 = k\frac{\boldsymbol{p_1}\cdot\widehat{(\boldsymbol{r}-\boldsymbol{s_1})}}{|\boldsymbol{r}-\boldsymbol{s_1}|^2} = k\frac{\boldsymbol{p_1}\cdot(\boldsymbol{r}-\boldsymbol{s_1})}{|\boldsymbol{r}-\boldsymbol{s_1}|^3} = k\frac{p_x(x-a)+p_y(y-b)}{((x-a)^2+(y-b)^2+z^2)^{\frac{3}{2}}}$

Similarly, $V_2 = k\frac{p_x(x+a)-p_y(y-b)}{((x+a)^2+(y-b)^2+z^2)^{\frac{3}{2}}}$, $V_3 = k\frac{-p_x(x+a)-p_y(y+b)}{((x+a)^2+(y+b)^2+z^2)^{\frac{3}{2}}}$, and $V_4 = k\frac{-p_x(x-a)+p_y(y+b)}{((x-a)^2+(y+b)^2+z^2)^{\frac{3}{2}}}$

Boundary conditions (a) at $x = 0$, total $V = V_1 + V_2 + V_3 + V_4 = 0$ and (b) at $y = 0$, total $V = 0$.
At $x = 0$, $V_1 = k\frac{-p_xa+p_y(y-b)}{(a^2+(y-b)^2+z^2)^{\frac{3}{2}}}$, $V_2 = k\frac{p_xa-p_y(y-b)}{(a^2+(y-b)^2+z^2)^{\frac{3}{2}}}$, $V_3 = k\frac{-p_xa-p_y(y+b)}{(a^2+(y+b)^2+z^2)^{\frac{3}{2}}}$, and $V_4 = k\frac{p_xa+p_y(y+b)}{(a^2+(y+b)^2+z^2)^{\frac{3}{2}}}$.

Therefore $V = 0$ since $V_1 + V_2 = 0$ and $V_3 + V_4 = 0$.

At $y = 0$, $V_1 = k\frac{p_x(x-a)-p_yb}{((x-a)^2+b^2+z^2)^{\frac{3}{2}}}$, $V_2 = k\frac{p_x(x+a)+p_yb}{((x+a)^2+b^2+z^2)^{\frac{3}{2}}}$, $V_3 = k\frac{-p_x(x+a)-p_yb}{((x+a)^2+b^2+z^2)^{\frac{3}{2}}}$, and $V_4 = k\frac{-p_x(x-a)+p_yb}{((x+a)^2+b^2+z^2)^{\frac{3}{2}}}$.

Therefore $V = 0$ since $V_1 + V_4 = 0$ and $V_2 + V_3 = 0$.

The electric filed is obtained by using $\boldsymbol{E} = -\nabla V = -(\nabla V_1 + \nabla V_2 + \nabla V_3 + \nabla V_4)$.

Here, $-\nabla\left(\frac{\boldsymbol{p}_i\cdot(\boldsymbol{r}-\boldsymbol{s}_i)}{|\boldsymbol{r}-\boldsymbol{s}_i|^3}\right) = \boldsymbol{p}_i\cdot(\boldsymbol{r}-\boldsymbol{s}_i)\frac{(\boldsymbol{r}-\boldsymbol{s}_i)}{|\boldsymbol{r}-\boldsymbol{s}_i|^4} - \frac{\boldsymbol{p}_i}{|\boldsymbol{r}-\boldsymbol{s}_i|^3}$,

since $-\nabla\left(\frac{p_x x + p_y y}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{2(p_x x + p_y y)}{(x^2+y^2+z^2)^3}(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) - \frac{p_x\hat{\boldsymbol{x}} + p_y\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{3/2}}$.

Therefore, $\boldsymbol{E} = k\sum_{i=1}^{4}\left[\boldsymbol{p}_i\cdot(\boldsymbol{r}-\boldsymbol{s}_i)\frac{(\boldsymbol{r}-\boldsymbol{s}_i)}{|\boldsymbol{r}-\boldsymbol{s}_i|^5} - \frac{\boldsymbol{p}_i}{|\boldsymbol{r}-\boldsymbol{s}_i|^3}\right]$.

(*) Because the problems of (3) and (4) are outside the contents of the exam, the full score will be given to all of the students. And if anyone solves the problems will have extra points.

(3) The force on the dipole can be calculated from the Eq. (4.5), $\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E}$ in Chapter 4. It is quite complicated. Let's first consider a simple case that $\theta = 0$, which leads $\boldsymbol{p}_1 = p_x\hat{\boldsymbol{x}}$ and $\boldsymbol{F} = p_x\frac{\partial}{\partial x}\boldsymbol{E}$. There should not be any force along z-axis at $z = 0$, and let's ignore z coordinate.

$$\boldsymbol{E}_2 = -\nabla V_2 = \frac{2p_x(x+a)((x+a)\hat{\boldsymbol{x}} + (y-b)\hat{\boldsymbol{y}})}{((x+a)^2+(y-b)^2)^{5/2}} - \frac{p_x\hat{\boldsymbol{x}}}{((x+a)^2+(y-b)^2)^{3/2}}$$

$$\boldsymbol{E}_3 = -\nabla V_3 = \frac{-2p_x(x+a)((x+a)\hat{\boldsymbol{x}} + (y+b)\hat{\boldsymbol{y}})}{((x+a)^2+(y+b)^2)^{5/2}} - \frac{-p_x\hat{\boldsymbol{x}}}{((x+a)^2+(y+b)^2)^{3/2}}$$

$$\boldsymbol{E}_4 = -\nabla V_4 = \frac{-2p_x(x-a)((x-a)\hat{\boldsymbol{x}} + (y+b)\hat{\boldsymbol{y}})}{((x-a)^2+(y+b)^2)^{5/2}} - \frac{-p_x\hat{\boldsymbol{x}}}{((x-a)^2+(y+b)^2)^{3/2}}$$

$$\frac{\partial}{\partial x}\left[\frac{2x}{(x^2+y^2)^{5/2}}(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}) - \frac{1}{(x^2+y^2)^{3/2}}\hat{\boldsymbol{x}}\right] = \left(-\frac{10x^3}{(x^2+y^2)^{7/2}} + \frac{7x}{(x^2+y^2)^{5/2}}\right)\hat{\boldsymbol{x}} + \left(-\frac{10x^2y}{(x^2+y^2)^{7/2}} + \frac{2y}{(x^2+y^2)^{5/2}}\right)\hat{\boldsymbol{y}}$$

, where we can replace $x \to (x \pm a), y \to (y \pm b)$ depending on the locations of the dipoles.

At $\boldsymbol{s}_1 = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$, from $\boldsymbol{s}_2$, $x \to 2a, y \to 0$.

$$\frac{\partial}{\partial x}\boldsymbol{E}_2 = kp_x\left(-\frac{10x^3}{(x^2+y^2)^{7/2}} + \frac{7x}{(x^2+y^2)^{5/2}}\right)\hat{\boldsymbol{x}} = -\frac{3k}{16}\frac{p_x}{a^4}\hat{\boldsymbol{x}}.$$

From $\boldsymbol{s}_4$, $x \to 0, y \to 2b$

$$\frac{\partial}{\partial x}\boldsymbol{E}_4 = k\frac{2y}{(x^2+y^2)^{5/2}}\hat{\boldsymbol{y}} = \frac{k}{8}\frac{p_x}{b^4}\hat{\boldsymbol{y}}.$$

From $\boldsymbol{s}_3$, $x \to 2a, y \to 2b$

$$\frac{\partial}{\partial x}\boldsymbol{E}_3 = kp_x\left(\frac{-3a^3+7ab^2}{16(a^2+b^2)^{7/2}}\hat{\boldsymbol{x}} + \frac{6a^2b+b^3}{8(a^2+b^2)^{7/2}}\hat{\boldsymbol{y}}\right)$$

Therefore the force $\boldsymbol{F} = kp_x^2\left[\left(\frac{-3a^3+7ab^2}{16(a^2+b^2)^{7/2}} - \frac{3}{16a^4}\right)\hat{\boldsymbol{x}} + \left(\frac{6a^2b+b^3}{8(a^2+b^2)^{7/2}} + \frac{1}{8b^4}\right)\hat{\boldsymbol{y}}\right]$.

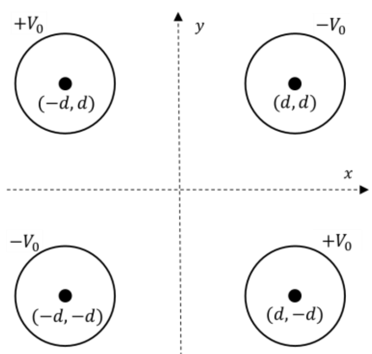
(4) The energy taken to bring the dipole $\boldsymbol{p}$ in from infinity should be same to the energy of the system, which is given as $U = -\boldsymbol{p}\cdot\boldsymbol{E}$ in Eq. (4.6). Therefore, the total energy is

$$U = -k\boldsymbol{p}_1\cdot\sum_{i=1}^{4}\left[\boldsymbol{p}_i\cdot(\boldsymbol{r}-\boldsymbol{s}_i)\frac{(\boldsymbol{r}-\boldsymbol{s}_i)}{|\boldsymbol{r}-\boldsymbol{s}_i|^4} - \frac{\boldsymbol{p}_i}{|\boldsymbol{r}-\boldsymbol{s}_i|^2}\right]$$

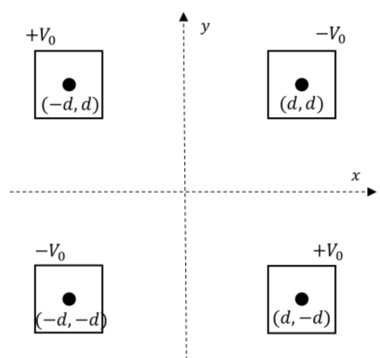
Problem 4:

Consider conductors of the following shapes that are infinitely long in the z direction as shown below. ***Qualitatively*** draw equipotential lines and electric field lines. When drawing, ***focus on two areas: near the center and one of the electrodes. Draw at least 10 lines for each region and describe the features of the picture as much as possible.*** Consider that the potential is a property of the solution of Laplace's equation, i.e. the value of the potential at a point is the average of the values around it. In the case of an electric field, the density clearly shows that the field lines are proportional to the strength of the field.

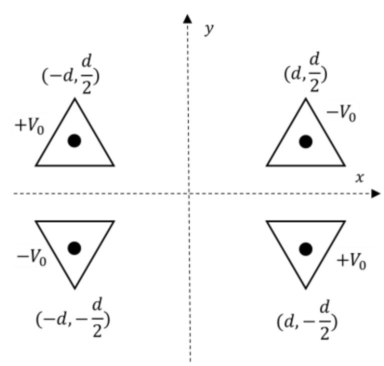
(1) Equipotential lines near the center and one of the electrodes
(2) Electric field lines near the center and one of the electrodes



(3) Equipotential lines near the center and one of the electrodes
(4) Electric field lines near the center and one of the electrodes



(5) Equipotential lines near the center and one of the electrodes
(6) Electric field lines near the center and one of the electrodes



Solutions:

1. General idea about equipotential line.
1) The equipotential line cannot be crossed.
2) A conductor is an equipotential (P98).
3) Density of equipotential lines is proportional to the strength of the potential.
4) The potential is a property of the solution of Laplace's equation, i.e. the value of the potential at a point is the average of the values around it.

2. General idea about electric field line.
1) Electric field line starts from positive voltage to negative voltage.
2) Electric filed line cannot be crossed.
3) Electric field line should be perpendicular to the surface of the conductor.
4) Electric field line should be perpendicular to the equipotential lines.

(*) If you can submit the equipotential lines and electric field lines by using computer program or any way, we will include them on the score of this mid-term exam.