Midterm Exam of Fundamentals of Physics (3), Electromagnetism

Open book test on April 6, 2022

Instruction to students.

a) This exam paper contains FIVE main questions and EIGHTEEN sub-questions. It comprises THREE printed pages including this page.

b) This is one-book (at your choice) open exam.

c) You are allowed to use a basic calculator in the exam.

d) If you carry out a computation please write down how you derived the answer. If you only write down the answer and it is wrong, no points are given for the question.

Good luck!

1. The electric field due to a static charge distribution has x component $E_x = kx^2yz$.

(a) Construct a valid form of the electric field for the other two components

(b) What are the potential and charge distribution for the electric field?

Answers

(a) Since $\nabla \times \boldsymbol{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\boldsymbol{z}} = \boldsymbol{0}$,

For z-axis, $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \rightarrow \frac{\partial E_y}{\partial x} = \frac{\partial(kx^2yz)}{\partial y} = kx^2z, \therefore E_y = \frac{1}{3}kx^3z + f_y(y,z)$

For y-axis, $\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 \rightarrow \frac{\partial E_z}{\partial x} = \frac{\partial(kx^2yz)}{\partial z} = kx^2y, \therefore E_z = \frac{1}{3}kx^3y + f_z(y,z)$

For x-axis, $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0 \rightarrow \frac{1}{3}kx^3 + \frac{\partial f_z(y,z)}{\partial y} = \frac{1}{3}kx^3 + \frac{\partial f_y(y,z)}{\partial z}$

Let's choose $f_y(y,z) = f_z(y,z) = 0$

Then, $\boldsymbol{E} = kx^2yz\,\hat{\boldsymbol{x}} + \frac{1}{3}kx^3z\,\hat{\boldsymbol{y}} + \frac{1}{3}kx^3y\,\hat{\boldsymbol{z}}$

(b) Charge distribution $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0 \rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial(kx^2yz)}{\partial x} + \frac{\partial\left(\frac{1}{3}kx^3z\right)}{\partial y} + \frac{\partial\left(\frac{1}{3}kx^3y\right)}{\partial z} = 2kxyz$

$$\rho = \varepsilon_0 \nabla \cdot \boldsymbol{E} = 2\varepsilon_0 kxyz$$

Potential $\nabla V = -\boldsymbol{E}$

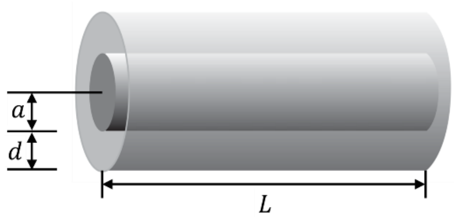
For x-axis, $\frac{\partial V}{\partial x} = -kx^2yz \rightarrow V = -\frac{1}{3}kx^3yz + f(y,z)$

For y-axis, $\frac{\partial V}{\partial y} = -\frac{1}{3}kx^3z + \frac{\partial f(y,z)}{\partial y} = -\frac{1}{3}kx^3z \rightarrow \frac{\partial f(y,z)}{\partial y} = 0 \;\therefore f = f(z)$

For z-axis, $\frac{\partial V}{\partial z} = -\frac{1}{3}kx^3y + \frac{\partial f(z)}{\partial z} = -\frac{1}{3}kx^3y \rightarrow \frac{\partial f(z)}{\partial z} = 0 \;\therefore f = const$

Finally, $V = -\frac{1}{3}kx^3yz + const$

2. Consider two coaxial metal cylindrical tubes of radii a and $a+d$ with length L as shown below.



(a) Find the capacitance of two coaxial metal cylindrical tubes. Here, ignore the edge effect.

(b) Which length L provides the largest capacitance if the total area of two metal tubes and d are fixed?

(c) Given total area of metal plate A and fixed distance d, which shape will provide the largest capacitance among plate, cylinder and sphere? Please list them in descending order and explain how you obtain the conclusion.

Answers

(a) Let's assume the total charge density Q for inside the tube and take a tube surface to use Gauss Law with the radius s and length L. Then, from the Gauss Law

If $a < s < a+d$, $\oint \boldsymbol{E} \cdot d\boldsymbol{a} = E\ 2\pi s L = Q/\varepsilon_0$, $\boldsymbol{E} = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{s}\hat{\boldsymbol{s}}$

The potential difference between tubes is given as

$$V = -\int_a^{a+d} \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{s}ds = -\frac{Q}{2\pi\varepsilon_0 L}\ln\frac{(a+d)}{a}$$

The capacitance $C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln\frac{(a+d)}{a}}$

(b) Let the total area of metal plate A, then $A = 2\pi a L + 2\pi(a+d)L = 2\pi(2a+d)L \ \rightarrow L = \frac{A}{2\pi(2a+d)}$

$$C = \frac{2\pi\varepsilon_0 L}{\ln\frac{(a+d)}{a}} = \frac{\varepsilon_0 A}{(2a+d)\ln\frac{(a+d)}{a}} = \frac{\varepsilon_0 A}{a(2+\frac{d}{a})\ln\left(1+\frac{d}{a}\right)} = \frac{\varepsilon_0 A/d}{\frac{a}{d}(2+\frac{d}{a})\ln\left(1+\frac{d}{a}\right)}$$

If we define $x = \frac{d}{a}$, $C = \frac{\varepsilon_0 A}{d}\frac{x}{(2+x)\ln(1+x)}$.

Since $\frac{dC}{dx} = \frac{-x(2+x)+2(1+x)\ln(1+x)}{(1+x)(2+x)^2[\ln(1+x)]^2} < 0 \ \because x > \ln(1+x)$, it has the maximum at $x = 0$, that is $a \rightarrow \infty$ or $L \rightarrow 0$. In the limit of $L \rightarrow 0$, the capacitance of the tubes becomes same to that of plate capacitor.

(c) The capacitance of sphere is shown in Example 2.12, which is given as

$$C_{sphere} = 4\pi\varepsilon_0\frac{ab}{(b-a)} = \varepsilon_0\frac{4\pi a(a+d)}{d} = \varepsilon_0\frac{4\pi a^2\left(1+\frac{d}{a}\right)}{d} = \frac{\varepsilon_0 A}{d}\frac{\left(1+\frac{d}{a}\right)}{\left(1+\left(1+\frac{d}{a}\right)^2\right)}.$$

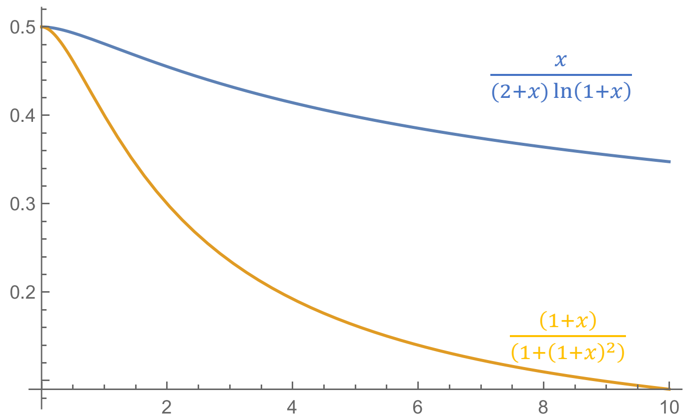
Since $A = 4\pi a^2 + 4\pi(a+d)^2 = 4\pi a^2\left(1+\left(1+\frac{d}{a}\right)^2\right)$ for sphere.

Let $x = \frac{d}{a}$, then $C_{sphere} = \frac{\varepsilon_0 A}{d}\frac{(1+x)}{(1+(1+x)^2)}$, which as maximum at $x = 0$, that is $a \rightarrow \infty$.

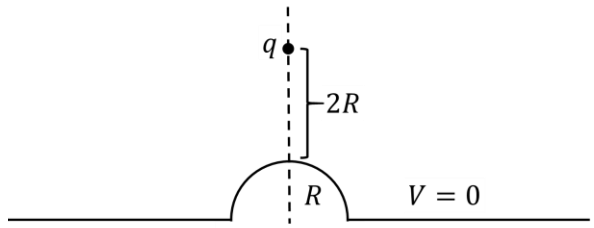
Therefore, plate capacitor has the largest capacitance.

Now let's compare the tube and sphere capacitor near $x = 0$. We can compare them by using graphs or Taylor expansion near zero as shown below

$$\frac{x}{(2+x)\ln(1+x)} - \frac{(1+x)}{(1+(1+x)^2)} = \frac{x\left(1+(1+x)^2\right)-(1+x)(2+x)\ln(1+x)}{(2+x)\ln(1+x)(1+(1+x)^2)} \sim \frac{\frac{5}{6}x^3}{(2+x)\ln(1+x)(1+(1+x)^2)} > 0$$



3. An infinite conducting plane has a hemispherical bump on it with radius R. A point charge q is located a distance $2R$ above the top of the hemisphere, as shown below. The conductor is grounded. Let's try to use image charge method to solve the following problems.



(a) Find the position and amount of charges of image charges needed to make the electric field perpendicular to the plane and the hemisphere at all points or to satisfy the boundary condition that $V = 0$ on the conducting plane.

(b) Find the potential and (c) electric field above the conducting plane.

(d) Find the force on the charge q.

(e) Compared to the case without bump and the same height ($3R$ from the plane), which force is larger and how much larger?

Answers

(a) We need the following three image charges to satisfy the boundary conditions of hemisphere and plane

- Image charge to satisfy the boundary condition of the hemisphere: $q_h = -\frac{R}{3R}q = -\frac{1}{3}q$, the distance from the center of the hemisphere: $b_h = \frac{R^2}{3R} = \frac{1}{3}R$
- Image charge to satisfy the boundary condition of the plane for charge q: $q_p = -q$, the distance from the plane: $b_p = -3R$
- Image charge to satisfy the boundary condition of the plane for charge q_h: $q_{hp} = -q_h = \frac{1}{3}q$, the distance from the plane: $b_{hp} = -b_h = \frac{1}{3}R$

(b) Potential

$V(s,z) = V_q + V_{q_h} + V_{q_p} + +V_{q_{hp}}$,

For cylindrical coordinate,

$V_q(s,z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{s^2+(z-3R)^2}}$, $V_{q_h}(s,z) = \frac{-q/3}{4\pi\varepsilon_0}\frac{1}{\sqrt{s^2+(z-R/3)^2}}$,

$V_{q_p}(s,z) = \frac{-q}{4\pi\varepsilon_0}\frac{1}{\sqrt{s^2+(z+3R)^2}}$, $V_{q_{hp}}(s,z) = \frac{q/3}{4\pi\varepsilon_0}\frac{1}{\sqrt{s^2+(z+R/3)^2}}$

For spherical coordinate,

$$V_q(r,\theta) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+(3R)^2-2r(3R)\cos\theta}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+9R^2-6rR\cos\theta}}$$

$$V_{q_h}(r,\theta) = \frac{-q/3}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+\left(\frac{1}{3}R\right)^2-2r\left(\frac{1}{3}R\right)\cos\theta}} = -\frac{1}{3}\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+\frac{1}{9}R^2-\frac{2}{3}rR\cos\theta}} = -\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{9r^2+R^2-6rR\cos\theta}}$$

$$V_{q_p}(r,\theta) = \frac{-q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+9R^2+6rR\cos\theta}}$$

$$V_{q_{hp}}(r,\theta) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{9r^2+R^2+6rR\cos\theta}}$$

(c) Electric field

$s > R$

Let's use the cylindrical coordinate

$$\boldsymbol{E} = -\nabla V(s,z) = -\frac{\partial V}{\partial s}\hat{\boldsymbol{s}} - \frac{\partial V}{\partial z}\hat{\boldsymbol{z}}$$

$$-\frac{\partial V}{\partial s} = \frac{qs}{4\pi\varepsilon_0}\left(\frac{1}{(s^2+(z-3R)^2)^{\frac{3}{2}}} - \frac{1}{3}\frac{1}{\left(s^2+\left(z-\frac{R}{3}\right)^2\right)^{\frac{3}{2}}} - \frac{1}{(s^2+(z+3R)^2)^{\frac{3}{2}}} + \frac{1}{3}\frac{1}{\left(s^2+\left(z+\frac{R}{3}\right)^2\right)^{\frac{3}{2}}}\right)$$

$$-\frac{\partial V}{\partial z} = \frac{q}{4\pi\varepsilon_0}\left(\frac{(z-3R)}{(s^2+(z-3R)^2)^{\frac{3}{2}}} - \frac{1}{3}\frac{\left(z-\frac{R}{3}\right)}{\left(s^2+\left(z-\frac{R}{3}\right)^2\right)^{\frac{3}{2}}} - \frac{(z+3R)}{(s^2+(z+3R)^2)^{\frac{3}{2}}} + \frac{1}{3}\frac{\left(z+\frac{R}{3}\right)}{\left(s^2+\left(z+\frac{R}{3}\right)^2\right)^{\frac{3}{2}}}\right)$$

The conductor plane should be $z = 0$.

$$-\frac{\partial V}{\partial s}\Big|_{z=0} = 0$$

$$-\frac{\partial V}{\partial z}\Big|_{z=0} = \frac{q}{4\pi\varepsilon_0}\left(\frac{(-6R)}{(s^2+9R^2)^{\frac{3}{2}}} + \frac{\frac{1}{9}R}{\left(s^2+\frac{1}{9}R^2\right)^{\frac{3}{2}}}\right)$$

$s < R$

Let's use the spherical coordinate.

$$\boldsymbol{E} = -\nabla V(r,\theta) = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}}$$

$$-\frac{\partial V}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left(\frac{r-3R\cos\theta}{(r^2+9R^2-6rR\cos\theta)^{\frac{3}{2}}} - \frac{9r-3R\cos\theta}{(9r^2+R^2-6rR\cos\theta)^{\frac{3}{2}}} - \frac{r+3R\cos\theta}{(r^2+9R^2+6rR\cos\theta)^{\frac{3}{2}}} + \frac{9r+3R\cos\theta}{(9r^2+R^2-6rR\cos\theta)^{\frac{3}{2}}}\right)$$

$$-\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(\frac{3Rr\sin\theta}{(r^2+9R^2-6rR\cos\theta)^{\frac{3}{2}}} - \frac{3Rr\sin\theta}{(9r^2+R^2-6rR\cos\theta)^{\frac{3}{2}}} - \frac{3Rr\sin\theta}{(r^2+9R^2+6rR\cos\theta)^{\frac{3}{2}}} + \frac{3Rr\sin\theta}{(9r^2+R^2+6rR\cos\theta)^{\frac{3}{2}}}\right)$$

The conductor plane should be $r = R$

$$-\frac{\partial V}{\partial r}|_{r=R} = \frac{q}{4\pi\varepsilon_0}\left(\frac{10R}{(10R^2-6R^2\cos\theta)^{\frac{3}{2}}} - \frac{10R}{(10R^2+6R^2\cos\theta)^{\frac{3}{2}}}\right)$$

$$-\frac{1}{r}\frac{\partial V}{\partial \theta}|_{r=R} = 0$$

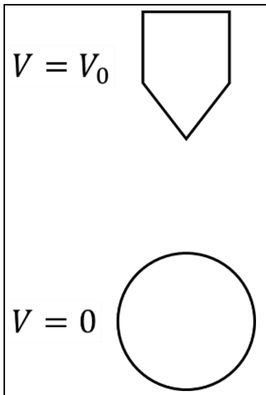
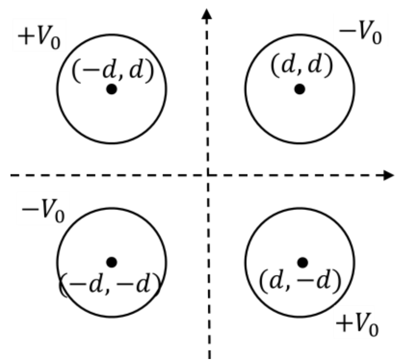
(d) Force on charge q is from all the image charges

$$F = \frac{q}{4\pi\varepsilon_0}\left(\frac{q_h}{\left(3R-\frac{1}{3}R\right)^2} + \frac{q_p}{(3R+3R)^2} + \frac{q_{hp}}{\left(3R+\frac{1}{3}R\right)^2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{-\frac{1}{3}q}{\left(\frac{8}{3}R\right)^2} + \frac{-q}{(6R)^2} + \frac{+\frac{1}{3}q}{\left(\frac{10}{3}R\right)^2}\right) = -\frac{643}{14400}\frac{q^2}{4\pi\varepsilon_0 R^2}$$

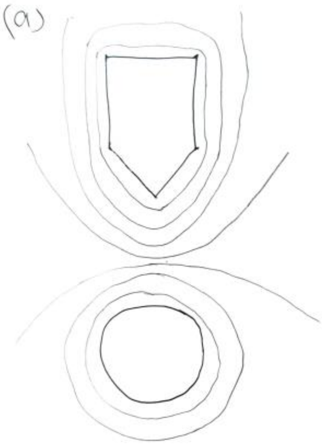
(e) Force on charge q without hemisphere is

$$F = \frac{q}{4\pi\varepsilon_0}\left(\frac{-q}{(3R+3R)^2}\right) < \frac{q}{4\pi\varepsilon_0}\left(\frac{-\frac{1}{3}q}{\left(\frac{8}{3}R\right)^2} + \frac{-q}{(6R)^2} + \frac{+\frac{1}{3}q}{\left(\frac{10}{3}R\right)^2}\right)$$

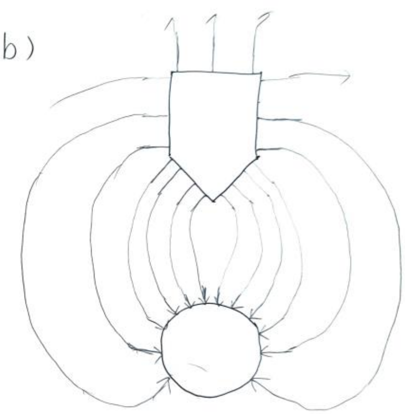
4. Consider the conductors of infinitely long stick and tubes, where the shapes are as shown below. **Qualitatively** draw the equipotential lines and electric field lines. When you draw them, please pay attention on three regions, near to the center and near to each electrode. For the potential, consider that the property of solution of Laplace equation, which is, the value of potential at a point is the average of those around the point. And for the electric field, clearly show that the density is the filed lines is proportional to the strength of the field.

	Draw (a) a few equipotential lines and (b) electric field lines for the left geometries of the conductors in 2D.		Draw (c) a few equipotential lines and (d) electric field lines for the left geometries of the conductors in 2D. (e) Find the exact mathematical form of electric field near the center. Represent it with unit vectors. Consider $\nabla \cdot \boldsymbol{E} = 0$.

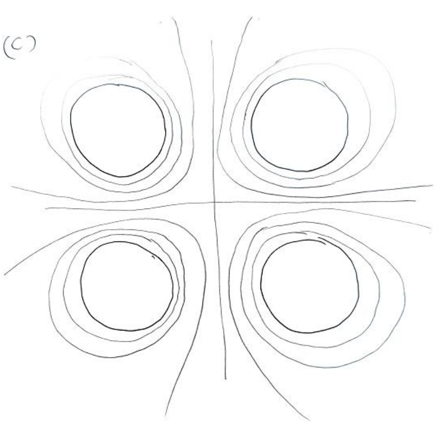
(a)



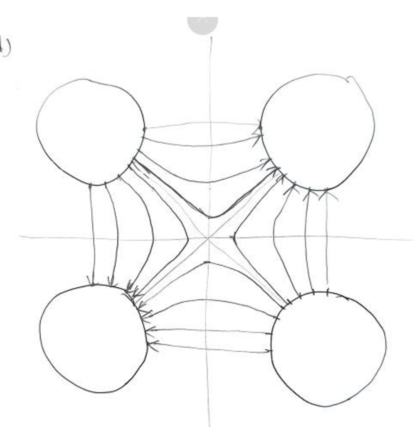
(b)



(c)



(d)



(e) The electric field should not have z-component and no z-dependence. Therefore, it can be written as $\boldsymbol{E} = E_x(x, y)\hat{\boldsymbol{x}} + E_y(x, y)\hat{\boldsymbol{y}}$.

From $\nabla \cdot \boldsymbol{E} = 0$, $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$, which leads to $\frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y} = A$

Therefore, $E_x = Ax + f_x(y)$ and $E_y = -Ay + f_y(x)$.

Let's consider the curl $\nabla \times \boldsymbol{E} = \boldsymbol{0}$, $\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0 \rightarrow \frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x} = B$

Therefore, $f_x(y) = By + C_x$ and $f_y(x) = Bx + C_y$

That is, $E_x = Ax + By + C_x$ and $E_y = -Ay + Bx + C_y$.

At $E_x(x = 0, y = 0) = E_y(x = 0, y = 0) = 0$, it leads to $C_x = C_y = 0$.

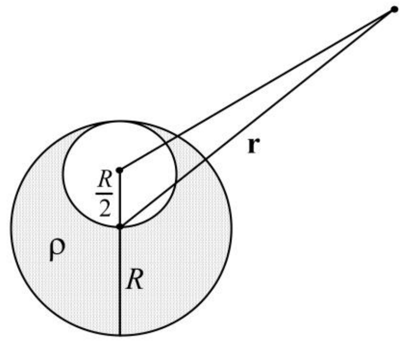
From the Fig. (d), the solution should be $\boldsymbol{E} = B(y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}})$.

5. Consider a hollowed charged sphere with radius R and uniform charge density ρ as shown in the following figure. The inner radius of the spherical cavity is $R/2$.

(a) When $r > R$, find the exact potential V.

(b) Find the dipole moment $\boldsymbol{p}$.

(c) Find the electric field $\boldsymbol{E}$ up to the dipole term.



Answers

We can understand the problem as the whole sphere is filled with uniform charge density ρ, and the hollow region is filled with $-\rho$.

(a) $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\frac{4}{3}\pi R^3\rho}{r} - \frac{\frac{4}{3}\pi\left(\frac{R}{2}\right)^3\rho}{\left(r^2+\left(\frac{R}{2}\right)^2 - rR\cos\theta\right)^{1/2}}\right) = \frac{R^3\rho}{3\varepsilon_0}\left(\frac{1}{r} - \frac{1}{8}\frac{1}{\left(r^2+\left(\frac{R}{2}\right)^2 - rR\cos\theta\right)^{1/2}}\right)$

(b) $\boldsymbol{p} = \sum_i q_i \boldsymbol{r}'_i = \frac{4}{3}\pi R^3\rho\, \mathbf{0} + \left(-\frac{4\pi}{3}\frac{R^3}{8}\rho\right)\frac{R}{2}\hat{\boldsymbol{z}} = -\frac{\pi R^4\rho}{12}\hat{\boldsymbol{z}}$

(c) Total charge q $= \frac{4}{3}\pi R^3\rho - \frac{4}{3}\pi\left(\frac{R}{2}\right)^3\rho = \frac{7}{6}\pi R^3\rho$

$V_{mono}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\frac{7}{6}\pi R^3\rho}{r} = \frac{7}{24\varepsilon_0}\frac{R^3\rho}{r}$

$V_{dip}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{p}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\frac{\pi R^4\rho}{12}\cos\theta}{r^2} = \frac{1}{48\varepsilon_0}\frac{R^4\rho\cos\theta}{r^2}$

$\boldsymbol{E}_{mono} = -\frac{\partial V_{mono}}{\partial r} = \frac{7}{24\varepsilon_0}\frac{R^3\rho}{r^2}\hat{\boldsymbol{r}}$

$\boldsymbol{E}_{dip} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) = \frac{R^4\rho}{48\varepsilon_0 r^2}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$

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$\boldsymbol{E}$