Mechanics and
Introduction to Special Relativity

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Forewords on the Course

This is your first physics course in college. It focuses on the classical mechanics and an introduction to theory of special relativity. For the mechanics part, we shall have a thorough discussion on the Newtonian Formalism of classical mechanics.

I realized the majority of you had already learned quite a lot (if not all) materials covered in mechanics part of this course. However, you probably will still find my treatment may be quite different from what you learned in high school. Such treatment is not a show-off or put the old stuff in new dress; it is the orthodox one which is most efficient or best to handle the subjects and you will find its shadow (I mean similar methods or models) in all branches of physics you are going to study. During the study of this course, I hope you may pay attention to the following aspects:

1) Use the 'orthodox' methods taught in the class, such as vectors, differentiation and integration, solving ordinary differential equations etc., even though the problems may be handled by what you learned in high school. These methods will accompany you a long way, so learn it, master it and apply it.

2) Understanding the fundamental physical principles and building up "physical pictures", instead of memorizing formula. There will be too many formulas to remember and they are all derived from physical

principles. So understand the principle and apply it is the most efficient way of learning. The “physical picture” is a little abstract. It is your understanding of the physical principle, your modeling based on the principles etc. It may easily get lost in the calculation and reduce the physics (mathematician will argue reversely☺) to a bunch of math formulas and forget what the formula is trying to tell you. A clear physical picture will guide you through or even explain phenomena without computation. Paul. A. Dirac once said (my recollection, may not be exact word by word): I think I understand a physical problem is when I know the answer before I solve the differential equations.

3) Adopting reasonable approximation whenever applicable. Physics is to reduce complexity into simple models with reasonable approximations. So in the calculation, you will need reasonable approximation too. Whenever you need it, do not hesitate using it. There aren’t many problems in real world you can solve it exactly. Here by approximation what I mean is not replacing g=9.8 with 10 in the computation (this kind of order of magnitude estimation is definitely useful and needed, but not my focus here). The approximation is to discard the lesser effect at the beginning in order to get simple solutions, such as the application of Taylor expansion, using harmonic approximation for potential wells close to the equilibrium point (see chapter 6 of the notes).

4) Practicing with what you learned in the lecture. I will try to explain

principles and applications as clear as possible, so that I hope most of you will feel understanding the lecture afterwards. But this is no guarantee that you master the materials. You need practice for the completion of the learning cycle. The practice will be in forms of homework problems and application by you trying to explain physics of some daily experience or phenomena. In the preparation of the lecture note, I read books and thought that I understood the stuff. I still had troubles in solving some problems or explaining some phenomena. The reason is two fold: one is due to skill or mathematical methods, you need practice to be skillful, and you need certain math technique in solving the problem. This is obvious but the less important part. The more important part is this: when I have trouble in problem solving, 99% is due to wrong physical modeling or negligence of some subtleties in the physical principles which escaped attention in the first reading. You could only find and understand these subtleties in problem solving or applications. I hope I will uncover most of these subtleties in lecture, but to fully understand them require practice yourself. So do the homework yourself or discuss physical problems with your peer or TA or me.

The lecture is composed of two distinct and related parts: Newtonian Mechanics which will be taught in about 10-week span of time (I hope), covering first 10 chapters of my notes, so about one week per chapter (The chapter 2,4,5 will take less than a week, while 6,7 will be more, if we are short in time, chapter 9 on the motion in central field may be skipped); the rest will be devoted to introduction of special relativity (1 week on chap.11 and 2-week for chap. 12, 13 each)which may be a complete new to most of you (I presume).

Grade system for the course (theoretical part):

5% homework: Note the TA will not grade the homework. You still need to hand in the homework so that TA knows that you finished it. I will not ask TA to grade them. Instead the TAs need to workout the solutions for the homework and post them on internet so that you can ‘grade’ your homework yourself. To test that you indeed workout the homework without just copying from others, there will be 20 points worth in exams where the problem in the exam is just the homework or a simple variation.

45%: Midterm (taken at 9^{th} week)

50%: Finals

Both exams will be open-book. You can bring in the following:

a) Copy of my lecture notes
b) Pieces of paper where you make summary yourself
c) Calculator, dictionary.

These are all you can bring, and NO more other materials such as homework answers.

Contents

Chapter 1 What is Mechanics? Concepts of Particle; Motion and Force in Newtonian Mechanics

1.1 Description of motion

Mechanics is the study of the **motion** of **particle** (or particles). Here I may need to talk a bit more on what is particle and what is motion, though I believe all of you had already developed a feeling for these terms.

The particle in this course is an infinitesimal point stripped off its geometric property, a pure idealization to simplify the problem. It is the building block of the real material body and an idealization (a *model*) of real fundamental particles. The point representing the particle will have intrinsic properties which will govern its interaction with other particles or the fields (modern treatment for the interaction with the field is interactions between the particles). These properties are mass (will be defined later), charge (related to the interaction with electric field) and spin (an angular momentum related to interaction with magnetic field). The above is applied to all mechanics, below the discussion will be limited to Newtonian mechanics (also called classical mechanics for historical reason to distinguish it from quantum and relativity theory).[1]

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[1] Besides the original Newtonian formalism of classical mechanics, there are other equivalent formalisms developed in 19th century, based on the least action principle (also called Hamilton principle). The classical

The motion of the particle is described by the space-time events. We choose a reference frame, a coordinate system to locate the positions of the particle at different time. In classical mechanics, the space is 3 dimensional Euclid-space which can be represented by rectilinear orthogonal (means perpendicular) coordinate system, such as Cartesian or other equivalents (Spherical, Cylindrical etc.). Time is another independent variable (a parameter that is not affected by the space) that changes evenly in all frames (linearly). [2]

With this rectilinear Euclid-space and steady time flow assumption, the Newtonian mechanics describes the motion of a particle by obtaining the position of the particle at different time: for example $(x_1,t_1);(x_2,t_2);(x_3,t_3)\ldots$ in 1-dimesion case, or more succinctly by a function of time $x(t)$, which is called **trajectory** of the particle. The motion is fully described by this trajectory (the variation of space with time), and other properties of motion can be derived from it, such as **velocity** (the rate of change of position vs. time):

$$v=\frac{dx(t)}{dt}\equiv\dot{x} \qquad (1\text{-}1)$$

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mechanics by this formalism shall be covered in the course of theoretical mechanics (also called analytical mechanics, a bad terminology in my opinion) which will pave the way to quantum mechanics, i.e. through that formalism, the links between classical and quantum mechanics is much clearer.

[2] Of course, this Euclid space and independent steady time flow in all reference frame assumption adopted in Newtonian Mechanics breaks down in relativity. The time flow will depend on the speed of the reference frame in special relativity, which we shall discuss later in the special relativity section. There the time will mix with space and will be treated as another dimension in space-time. The rectlinear Euclid space will break down in general relativity, the massive object will 'bent' the space-time.

and **acceleration** (the rate of change of velocity vs. time; or second order variation of position vs. time)

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \equiv \ddot{x} \qquad (1\text{-}2)$$

This description, though complete, is not very satisfactory, it requires infinite pairs of (x_i, t_i). How did we obtain the trajectory $x(t)$? By experiments of course! That is how every theory (at least scientific theory) develops. Through the observation (celestial motion of the stars and planets) and experiments (balls rolling down the slope or dropped from Pisa tower), regularities will appear and summary will be made (Kepler's laws of planets motion and Galileo's idea of inertia), and finally all the pieces will be put together like jigsaws to form a nice theory (Newtonian mechanics), with minimum but fundamental assumptions or postulates (also termed laws in old days); then applying logic derivation (in the language of math), phenomena explained and predictions made, and the theory will be subjected to the rigorous and wide test through further experiments. If and only if no contradictions observed, people will accept the theory. The theory will be modified or completely overhauled and reformulated as new and better experimental data become available. All theories developed follow these steps (only exception to my knowledge is probably general relativity, brain child of a weirdo and genius with no compelling experimental facts at the time of its birth [3]). The power of a successful theory lies in the facts not only explaining the existing experimental evidences (this is a test for the correctness of any theory), but also in making correct predictions which can be tested by experiments.

Now back to the motion of a particle, from the observation of its trajectory, Newtonian mechanics developed. Now the question is: can we predict the motion of a particle with minimum knowledge (knowing the complete trajectory is nice but may be impossible in some situations). Of course you know the answer is yes, but let's proceed at first with this prediction of motion to see how far we can go just with common logic and without resorting to Newtonian theory.

To predict the motion, you will need a starting point obviously, this is called initial conditions. Suppose (still in the 1-d space for simplicity) we know the initial starting position (x_0, t_0). This is not enough information to proceed to make any prediction. Suppose we add initial velocity (v_0, t_0) to our knowledge, we then can predict accurately the position of the particle a short time (very short, approaches to 0) later (x_1, t_1), with $x_1 = x_0 + v_0 \Delta t, \Delta t = t_1 - t_0 \to 0$. If we want to proceed further, (x_0, v_0) at initial time would not be sufficient again, because we need to find the

[3] There were experimental facts indeed, such as the precession motion of Mercury unaccounted by the classical mechanics, but that is not very compelling. I should also mention that using 'weirdo' in the note means no negative as in common use. All genius (which means exceptional uncommon, or extraordinary) are weirdo in some aspects. But be warned don't imitate genius by behaving weird, the reverse is definitely untrue, not all weirdo are genius☺

velocity v_1 at x_1 in order to proceed, and the v_1 can be obtained if we add the initial acceleration a_0. So without any help from Sir Newton, we can proceed to the third position at t_2 by knowing the initial position, velocity and acceleration. This argument will go on, so if you want to find the 4th position (x_3, t_3), you will need position and velocity at t_2; which in turn requires position, velocity and acceleration at t_1; which requires $x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0$ at the starting point. The situation will quickly go out of hand if you want to predict motion after reasonable steps. You will need many time derivatives initially. These initial time derivatives would be same amount of information as $x(t)$ [4], so not much progress from knowing the complete trajectory.

Now Sir Newton comes to rescue by telling you that you can calculate the acceleration of the particle *provided* you know two things, the **force** and the **mass**. The force[5] is due to the interaction with the surrounding which may have a simpler distribution formula over space-time.[6] The

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[4] You may find this description of motion from a starting point is equivalent to the Taylor expansion of functions in calculus.

[5] The force in Newtonian mechanics is a very vague concept based largely on daily experience and it plays a central role. Roughly speaking, it is a measure of interaction between different particles. Modern physics tends to avoid force by replacing it with energy (though the energy itself is also a pretty elusive idea, see the discussion in Feynman's vol.1 chap.4.1.) But force though vague in definition, still a very useful concept. The forces considered in this course are within classical limit, such as gravitation and electromagnetic.

[6] Newton reveals one form of force, the gravitation force; its modern view is interaction with gravitation field and such interaction is conducted by the so called gravitons. (i.e. the two massive objects interact by exchanging gravitons) Other fundamental forces in physics are electromagnetic force (Coulomb's law), and it is interaction of charged particles with the electro-magnetic field (the interaction is by exchanging photons); and nuclear force within the short distance (10^{-15}m, the dimension of the nuclei), the interaction (so called strong interaction) is by exchanging gluons. Browsing the WIKI under 'fundamental interaction' for some background.

mass will be an intrinsic property of the particle which is independent of space and time (a constant in Newtonian). Thus allow you to determine the acceleration of particles over space and time. So in Newtonian mechanics, you only need to know the initial velocity and position, and the rest of the motion of the particle shall be determined provided the force distribution and the mass is known. The Newton equation of motion (his second law) is essentially a second order differential equation of position over time, and knowing the initial position and velocity will give you a specific solution of the trajectory. Sometimes we restate the above by saying that the **state** of a particle in space-time is described by specifying its position and velocity[7], and its motion (i.e. its state at later time) can be predicted with the knowledge of the force distribution.

1.2 Dependence of Force on Variables

1.2-1 The General Form of Force; Explicit and Implicit Dependence on Variables

From the above discussion, we only stated that in Newtonian mechanics, the motion will be affected by interaction between particles. This interaction can be described as force and it will determine the

[7] This is the reason that in other formalisms of classical mechanics, the fundamental functions to replace the force are functions of position, velocity and time but no more, such as Lagrangian and Hamiltonian function you will encounter in theoretical mechanics.

acceleration of the motion. Once the force and mass and initial conditions are fixed, the motion shall be fixed too. We did not know however what the force exactly is, especially how its distribution over space and time, expressed in function with variables of space and time. I do not intend to give you all the function forms of forces, some of which you already know probably, such as Hook's law, Universal gravitation, Coulomb's law, etc. Here I would rather talk a little about the general dependence of the force on the space and time variables within classical limit (low velocity and macroscopic world).

Suppose our system only consists of two particles existing in an ideal environment that no other interactions exist except that between the particles (not one particle this time, there will be no force if our complete system only has one particle). Such system is called a **closed system**, it is obvious an idealization but a useful one, it is equivalent to put our particles in outer space far away from any mass object in universe and also in vacuum with electro-magnetic shield.

The most general form of force would be a function of all the possible variables describing the mechanical state of the particle, i.e. position, velocity and time. So $F = f(x^A(t), x^B(t), v^A(t), v^B(t), t)$, A, B specify the particles, the position and velocity will also change with time. If there is a time explicitly in the force function, such as $F = (x^A - x^B) + (v^A - v^B) + \cos t$, then we say the force has **explicit**

dependence on time[8]. If there is no time variables in the function, the force will still change with time, but this time it is implicitly through the change of position and velocity, such dependence of time would be called **implicit dependence**.

The general form of force above would let us determine the motion of the two particles, provided we know the initial states (given by initial position and velocity) of the particles, we shall start there, from the force knowing the acceleration, and thus knowing the position and velocity of both particles at later time, so we can know the states of the particles at all time. Just from the mechanical point of view, the force in a completely closed system can be a function of position, velocity and time.

However, you probably know that the fundamental forces between two particles are only functions of position (*In this course, we only consider the classical domain and the fundamental forces are gravitation and electrostatic forces*), actually only upon relative position, i.e. $F = f(x_A - x_B)$ in 1-dimensional case here, not explicitly depend on time and velocity. This is indeed how the nature works, and have you ever wondered what is the reason?

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[8] In this case, the change of force with respect to time would be: $\frac{df}{dt} = \frac{\partial f}{\partial x^A}\frac{\partial x^A}{\partial t} + \frac{\partial f}{\partial x^B}\frac{\partial x^B}{\partial t} + \frac{\partial f}{\partial v^A}\frac{\partial v^A}{\partial t} + \frac{\partial f}{\partial v^B}\frac{\partial v^B}{\partial t} + \frac{\partial f}{\partial t}$

1.2-2 Homogeneous and Isotropic Space-Time, Inertial Frames and Relativity Principle

The reason of the forces above only depend on relative positions of the particles is what we called homogeneous and isotropic of space-time in the reference frame we choose, such reference frame is called **inertial frame**; another reason is **relativity principle** in physics. So the following talks may be a little advanced for this opening course in mechanics[9], but I shall put it as plain as possible.

Homogeneous in physics and chemistry usually means the property would be same in all positions in space. i.e. it is related to the translational symmetry. For example, a tank of water, its density (the number of water molecules inside a small volume) and other physical, chemical properties on the average speaking would be same everywhere inside the tank (except at the edge of tank, but we may think the tank is infinitely large), then we say the water is homogeneous in the tank. **Homogeneous in space-time** means that all the positions (in x and in t) would be equivalent for the closed system. This means **translational invariance**. i.e. the closed system behave equivalently no matter what is the origin you choose for the space or time, you translate the whole system along the x-axis or time axis, and making the observation, the results would be same before and after the translation.

[9] That means it won't appear in the test.

To put this more clearly, suppose you have a closed system of two particles, you observe their motion in Beijing on Monday; then you repeat the experiment with same initial conditions (i.e. same initial state of particles) on Tuesday, you would get same result (Here, the closed system means isolated from the outside world, so weather change or the motion of earth…will all be shielded and won't interfere with the experiment). This is what you believe and tested true in all experiments. This is the translational invariance along time or homogeneity of time. Now you translate the whole system to Tianjin, and repeat the experiment there with same initial conditions, you will get same results as in Beijing. This is called translational invariance of space, or homogeneity of space[10]. The above is not to prove that space and time is homogeneous, but from past experience and experimental tests, we believe this should be the case. So this is a fundamental postulate which has far more profound impact in all branches of physics. You may like to know that from the homogenous of space (translational invariance, sometimes also called translational symmetry), we can see the momentum of a closed system will be conserved---conservation of momentum. From the homogeneous of time (translational invariance in time), the energy of the closed system will be

[10] The strict definition would be the Lagrangian (a function of space-time and will be defined in theoretical mechanics) will be invariant with translation. For more general discussions like in this notes, please refer to Feynman's Vol.1, chap.52.

conserved---conservation of energy.[11]

Isotropic means the physical properties and interactions would be same in all directions, i.e. independent of direction. It is related to the rotational symmetry. For example, that tank of water is homogeneous but also isotropic in an average sense. If you plant a bomb inside, the explosion sets off the sound wave will travel equivalently in all directions. This won't be true in ice crystal where the interaction will be directional dependent due to the ordered lattice structure of the crystal. **Isotropic in space** means the choice of the coordinate axis of our frame won't affect the physical properties for the closed system. This is **rotational invariance**. Noticed the rotation of the coordinate axis would be equivalent to rotation of the whole system in the reversed direction of same amount with fixed coordinate axis. So the above can also be stated that if you rotate the whole closed system with arbitrary angle along any direction, the physical property would be same.

For example, in our 1-dimensional case, we rotate the x-axis 180-degree, the motion of the particle would be same (or equivalently you rotate the two particle 180 degrees, make A particle ahead, B particle behind, same results). Of course rotation is better illustrated at least in 2-dimension.

[11] To prove the conservation laws from the invariance of space and time would be best dealt with the Lagrangian in theoretical mechanics. If interested, please read Landau and Lifshitz "Mechanics" 3rd edition, Chap. 1 and 2. The Newtonian formalism won't fit for this task. Of course we can work out the conservation relations for mechanical system from the Newton's laws, just not easy to see the link with the invariance in space and time. A general theorem, relating the symmetry and conservation of physical quantity is called Noether's theorem, see Goldstein, "classical mechanics" 2nd edition, 12-7.

Suppose you carried the two particle experiment along the North-South direction initially and then along the East-West direction, both with same initial conditions, the results would be expected and be tested same. This is the meaning of rotational invariance. (A student may argue that if you rotate the two particles and make it along up-down direction, the gravity would change the results. Yes, but don't forget we are talking about the closed system which will be isolated from outside interactions. By introducing the gravity, the system would not be closed anymore, but subjected to an outside field. Of course we shall deal with this, but not now. So to negate the gravity, we may do the experiment in a weightless environment, such as a free fall elevator, and the student who argued would be in the elevator to make observation☺). Isotropic in space (rotational invariance) will give us conservation of angular momentum for the closed system. **Isotropic in time** would mean time reversal invariance, since time is in its own dimension, 'rotation' in time means making a reversal of time flow, to make time go backward. i.e. t will be replaced by –t in all functions and relations. Just imagine you record the motion of particles in a closed system, make it a movie and play the movie backward, and all motions playing backward would also obey the same physical law. i.e. two particles initially attracted each other and accelerated towards each other and collide; if you play it in reversed order, the two particles may appear initially with larger speed and flying apart under attraction force will slow down. Both obey same physical laws. In this sense, the motions of a closed system in classical mechanics would all be reversible. This is the meaning of isotropic in time.[12]

Relativity principle means that the physical laws govern the motion would be same for all inertial frames (the frame which satisfies homogeneous and isotropic of space-time). The different inertial frames can be different by a constant velocity. For example, you carry the two particle closed system, aboard a train moving with constant velocity and carry out the experiment on the moving train. Your observation would be same as that carried on the ground. This is to say you cannot know you are on a moving train by doing physical experiments on the train (throw the ball, play the pool etc), because all the observation would be same as in another inertial frame such as on the ground, and all the inertial frames are equivalent in studying physical laws. This implies you cannot determine the absolute velocity of the frame but only the relative velocity between different inertial frames. The common example is that two moving trains run across parallel with each other, the guy on the trains won't be able to tell whether it is his train is moving or his train is stationary but the other train is moving in the opposite direction, only the

[12] Someone may raise objection here, that something (actually a lot of things) in real world are not reversible. A glass fall on the floor and broke into pieces never put back again (if you argue your girlfriend left you and never come back, I cannot answer and you probably are sitting in the wrong class). Actually from the mechanical point of view, the reversed process is certainly possible, but since in this case, it involves so large number of particles, and it requires thermodynamics and statistical mechanics to calculate the probability for the reversed process. That turns out to be extremely small, close to zero in any practical sense.

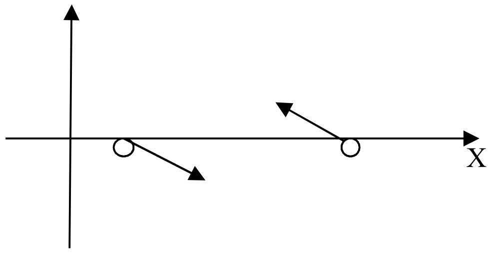
relative motion can be determined. Such relativity principle was first pointed out by Galileo and is called Galileo Relativity principle which only applies to mechanical system. Einstein extended this to all physical laws, and we shall come back to this principle when we talk about special relativity.

1.2-3 The Force between Particles in a Closed System Depends Explicitly on Position

Now let's come back to the question at the end of section 1.2-1, the force between the particles in a *closed system* in an *inertial frame* is in the form of $F = f(x_A - x_B)$, only depend explicitly on positions of the particle. Base on the discussion of last section, we now can give the reason for such dependence.

First, the force will only depend on the relative positions of the two particles. If it depends on the absolute positions of individual particle, then this will violate the homogeneous of space (translational invariance). Suppose we translate the system by an amount of d along the x-axis (this can be done by choosing a new origin of the coordinate system with –d, or shuffle the whole system down the axis by d). Then the form of force would become $f(x^A + d, x^B + d)$, if it depends on velocity and time, this translation will not affect the velocity since d is a constant, and time can be chosen the same as before. If the dependence is not only on the relative position in which the translation of d could be cancelled, then there will be a change of force due to the translation. Then the motion of our two and only two particles system would be different, will not be translational invariant anymore. So the dependence of force could only be on the relative positions. We can further show that the force is also along the line connecting the two particles. Of course we have to go to higher dimension for this. Now suppose that the force between the two particles is not along the connecting line, but with an angle, see the figure below. Then the force will make the two particles rotate counterclockwise. Now if we rotate the two particles around X axis with 180-degree, then the rotation would become clockwise. But from symmetry, the system should be rotational invariant around the X-axis. That is possible only if the force is collinear with the two particles.

(Comment: This is true if we only consider the two particles and the two particles do not have any internal degrees of freedom, such as spin of electron etc, that will vary upon the rotation. Of course this is an approximation only valid in cases when we do not need to consider such internal degrees of freedom in classical mechanics. There are cases that the forces between the particles do not along the line of connections if the above condition is not valid)



Similar argument would also apply to the time dependence of force. The force in a closed system cannot depend explicitly on time. Otherwise, it will violate the homogeneous of time. i.e. if force explicitly depend on time, $F = f(...t)$. Then if we translate in time (reschedule the experiment from Monday to Tuesday), the force will be different and the following motions of the particles will be different too.

How about the force dependence on velocity of the particles? The argument will be a little different and complicated from above, and I shall only give a brief discussion here and leave the detail later (by the end of this course when we have learnt special relativity). In Newtonian mechanics, the interaction between the particles (the propagation of the force) is considered instantaneous, takes no time. i.e. if the particle B is introduced into a system originally only has A, the A will immediately feel the force from B. Such instantaneous is a built-in in the Newtonian mechanics, an assumption which turns out faulty (the modern view is that the interaction between particles is not instantaneous, but through the field created by the particle as source, such as Gravitation filed by massive particle, and electro-magnetic field by charged particle. The propagation is actually the propagation of the field, and the force experienced by the other party are due to the interaction with the field. An analogy will be throwing a stone into a pond of water, the disturbance will propagate as water wave and when this wave reaches the a piece of wood a distance away, the wood will subject to the force and start shaking). Such instantaneous interaction is also termed as **non-locality**. It implies the force will not depend on the velocity of the particle, whether they travel fast or slow or to any direction, the force always propagates with infinite velocity.

We shall derive the transformation of forces by the end of this course using special relativity, i.e. for one inertial observer the force is F, and for another observer in another inertial frame (say a train moving with constant velocity v relative to the first frame), the force is F', there will be relations between F and F' which I call transformation relation between the forces. We shall find there that *if we can neglect relativistic effect* (v/c~0), the forces will take **same form in all inertial frames**(this is equivalent to say c is infinity mathematically and in accordance with non-locality of classical mechanics, i.e. interaction is instantaneous); however strictly speaking, the force will depend on the velocity due to the relativistic effect. In this sense here the independence of force on velocity is result of neglecting such relativistic effect in Newtonian mechanics. Then the forces in particle A, B system, say the force felt by A created by the field from particle B. We can choose a frame in which the B is stationary (motionless) and thus the field of B only depends on the relative position to B, so will the force felt by A.

So In conclusion, all fundamental forces are in forms of $f(r_i - r_j)$, depends explicitly only on relative positions in classical physics. Of course due to the motion of particles, the position will change with time and the force will change too, so it depends implicitly with time.

1.2-4 Superposition Principle of Force and Force in an Open System

A smart you may raise objection once again that you know there are forces depending explicitly on velocity (such as frictional force, air resistance, etc), and on time (such as a charge in a time varying electric field). The difference in this situation is we are dealing with an **open system**. We have to include interactions of the part of the world that we are interested with the rest of the world. The closed system we are dealing with in previous discussion (only two particles) is a very simplified idealization. The real problem will involve too many parties to treat it as a closed system. You still can include everything into a closed system, but the treatment with that many particles is just not simple anymore.

Consider we extend our two-particle closed system to a 3 particle system, with particles A, B, C. Now the force on one of the particles will have two contributions from the other two:

$$F_A = F_{BA} + F_{CA} = f_1(x_B - x_A) + f_2(x_C - x_A) \qquad (1\text{-}3)$$

If there are more than 3 in the system, the general form of force on A:

$$F_A = \sum_j f_j(x_j - x_A) \qquad (1\text{-}4)$$

The relations expressed in 1-3 and 1-4 are called ***superposition principle***, that the forces exerted by many parties are just summation of forces by each individual contribution as if it is acting alone. It is simple and straightforward and greatly simplified interactions. Science would be quite different (more complicated) without it. This superposition is also a postulate, a principle not from derivation but based on experimental facts, from peoples pulling heavy objects to atomic interactions. This principle implies if we understand the interaction between two particles, we can in principle understand many particles interaction. But as particles number increases, the analysis becomes more complicated even with the help of superposition principle.

Back to our idealized closed system, but with 3 particles. To find out the motion of particles, we need to have initial conditions, including positions and velocities of the three particles (6 variables in 1-dimensional, and 18 in 3-D). For example, to determine the motion of A, from its initial state, we can calculate its position and velocity at later time. Then from the positions of all particles at this later time, we can calculate the force on A and calculate further motions at even later time, it goes on like this. This procedure may workout for small number of particles, but quickly goes out of control in real world, where number of particles would be on the order of Avagadro number 10^{23}. Of course there is simplification even for macroscopic object: if the object does not have any internal change (geometric shape, mass etc) and we do not need to consider the interactions between the atoms/molecules that the object is made of, it may be treated as a mass point. But generally if we include everything into our system, making it closed, the number of interacting parties would be too big to handle, so approximation has to be made.

Let's come back again to the simplified model of 3 particles. If I am only interested in the motion of A, I shall include A in my system, and treat B,C as outside world. My system with only A will be an open system, and interaction will not be restricted within the system like the closed one. Then I shall try to approximate the force from B,C (these forces are very often called interaction with external field). If the B, C are doing some periodic motion and I do not care about them. Then the force on A will appear as a function of position of A and time. So the force experienced by a body in an open system may have explicit dependence of time and even velocity. For example, in the situation of air resistance, a bullet is flying through air. The bullet-only is an open system, it is interacting with billions air molecules, whose interaction with the bullet will be averaged and approximated to give a simplified force form. The number of molecules colliding with the bullet per unit time would certainly depend on the velocity of the bullet, so the empirical force form would depend on velocity of the bullet. Most empirical force expressions ($F = -kx$ for elastic extension; $F = \mu N$ for friction force; $F = -bv^2$ for air resistance etc.) are the results of such averaged and approximated effect of the outside world (the molecules in the spring, on the surface or in the air) to the object of interest.

The beauty of Newtonian (or classical) mechanics is that if we know the force, we can determine the motion of the object of interest provided with initial state. So given a force (or equivalently the potential field) and predict the motion is the task for mechanics; knowing and understanding the form of force will be tasks in other branches of physics and chemistry.

1.3 The Limitation of Newtonian Mechanics

You probably learned that classical mechanics breaks down at high speed and in microscopic world. That is quite true, we now know the limitations and possibly the reason of the breakdown of the Newtonian mechanics. There are faulty assumptions in the formalism and these lead to the breakdown.

1) The space-time is absolute and independent, time is flowing evenly in

all reference frames, the interaction is non-local.

Such assumption turns out faulty at high speed, i.e. v/c (c speed of light in vacuum) is not <<1, the interaction would be local, propagate with a limiting speed c. This leads to the special relativity.

2) The space-time is rectilinear Euclid type.

This turns out untrue in gravitation field (or accelerated frames). The massive object will make a curved space-time. This is treated in the theory of general relativity. Of course you have to be close to a massive object to see this effect (gravitational lensing of celestial object) or conducting very fine experiment on earth (Pound's experiment for frequency shift of light in a gravitational field)

3) The position and velocity can be determined and thus the trajectory description is valid.

We have talked extensively on this classical description of motion in previous sections. Such description is so natural from our common sense (well, the absolute and rectilinear space-time is also very natural from common sense), it breaks down in the microscopic world, where the little devil Planck constant $h = 6.6 \times 10^{-34} Joule \bullet \sec ond$ will creep in, and we will have the 'bizarre' uncertainty relation which rules out the possibility of determining the velocity and position simultaneously. This would make the classical description of the state invalid. New description and new formalism of mechanics is needed, and this is the realm of quantum mechanics.

A demanding you would argue now if above are true, then why one still needs to learn classical mechanics. I probably can think of 3 reasons:

1) The classical mechanics is not wrong. It is incomplete, just cannot handle all observations and experimental results. Provided suitable situations (loosely speaking, macroscopic world, low speed and far away from massive stars, which basically dealt with 99% of average person's daily life), the classical mechanics works with charm.
2) New theory does not appear from nowhere like magic. The study of mechanics will pave the way and equip you with necessary background (both physical and mathematical) to advance into modern theories.
3) All new theories have to be tested by the classical theory. They have to be able to reproduce the results of classical (show equivalence with classical theory) under the condition where classical theory prove to be correct. This is called **correspondence principle**. The special relativity should reproduce Newtonian mechanics when the $v/c \ll 1$; and quantum mechanics should reproduce Newtonian mechanics (actually it turns out more easy to show the relation between the quantum and Hamilton's formalism of classical mechanics) when one can neglect h, treat it as zero.

Finally I will give a very brief comment on the current accomplishment of physical theories. The general relativity deals with gravitational field; and quantum theory deals with electro-magnetic field (quantum mechanics combined with special relativity, it handles all phenomena in chemistry, and large part in physics, except gravitation and inside nuclei). The quantum theory is also successful to explain the nuclei dynamics---so called chromodynamics. However, the attempt to include general relativity with quantum theory (i.e. use quantum mechanics to treat gravitation field) is unsuccessful. This part of theory is properly termed GUTs (Grand Unification Theories; Technically the unification of quantum description of nuclear physics is called GUTs, but I reckon it is more proper to name GUTs as unification to all physical interactions known to man), since it would be a theory including all known interactions, it also implies you really need guts to dive into it.

Chapter 2 Kinematic in One-Dimension

Kinematic is a study of motion of some party by describing its state, i.e. position and velocity, and how the state changes with time. This does not concern the 'cause' (the force or potential field) of the motion. The study of the cause of the motion and its effect is called **dynamics**. However, such division in my opinion is not important. In these early chapters, before formally introducing the Newton Laws, I will focus on the kinematics. I shall use it as an illustration of some basic concepts and mathematical techniques. In this chapter, only consider a single particle moving in one-dimension, this is the simplest system you can get. The physics and math are relatively easy, but I shall use this to introduce **differentiation** and **integration** of calculus. Please also refer to the supplementary 1 for some details on the math. Next chapter, I shall treat the motion in higher dimensions and an important concept **vector** will be introduced.

As stated in Chap.1, for a particle the motion is described by its position and velocity. The position changes with time and if we know how this change is, we can use a function x(t) to describe it. This is the trajectory of the particle in 1-D. The rate of change of the position function over time is called velocity (its magnitude is called speed), it can be positive (travelling along the defined positive direction +x) or negative (along the –x direction). The velocity is also a function over time. The relation between the velocity and position is given in (1-1) but I shall repeat it here.

$$v(t) = \frac{d}{dt}x(t) = \dot{x} \qquad (2\text{-}1)$$

This states that velocity is the time derivative of position function. Here the v(t) is the **instantaneous** velocity at time t. So knowing the trajectory x(t), finding the velocity is straightforward, just differentiation. For the details of differentiation and derivatives of some basic functions, please see the supplementary materials or books on calculus.

Reversely, if we know the velocity v(t), we can also calculate the position change over time. x(t) is an antiderivative of the velocity function.

$$x(t) = \int v(t)dt + C \qquad (2\text{-}2)$$[13]

C is a constant that can be determined by the initial condition, i.e. at $t=t_0$, $x=x_0$, then the C will be fixed. The simple example is that if we know v(t)=gt, then from (2-2), we will get $x(t) = \int gtdt + C = \frac{1}{2}gt^2 + C$. And if our initial condition is at t=0, $x=x_0$, then $C=x_0$. The above would be:

$x(t) = x_0 + \frac{1}{2}gt^2$. The distance of free fall of an object.

The (2-2) can also be written in a definite integral form:

$$x(t) - x(t_0) = \int_{t_0}^{t} v(t)dt \qquad (2\text{-}3)$$[14]

So knowing either function of position or velocity, we can calculate the other.

Just as the velocity is the rate of change of position over time, we can define the rate of change of velocity too, that is of course called acceleration. i.e.

$$a(t) = \frac{d}{dt}v(t) \qquad (2\text{-}4)$$

It is first order time derivative of velocity, and put the expression of velocity in terms position, then the acceleration is the second order time

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[13] See the supplement under antiderivative for detail.

[14] This is the fundamental theorem of calculus, also see that in the supplementary.

derivative of position:

$$a(t) = \frac{d}{dt}(\frac{d}{dt}x(t)) = \frac{d^2x(t)}{dt^2} = \ddot{x} \qquad (2\text{-}5)$$

Listed in (2-5) are some commonly used notations. There is no need to introduce higher order time derivatives, because physical laws only related to the acceleration.

With the definition of above, the kinematics of 1-dimension motion will be calculations involving differentiation and integration.

(1) If we know x(t), then solving v(t) by (2-1), take time derivative of x(t). Then taking time derivative of v(t) (2-4) to get a(t).

(2) If we know v(t), then its time derivative would give us acceleration a(t), and its definite integration (2-3) will give us x(t), of course we will need initial position to determine the x(t) completely in this case.

(3) If we know the a(t), then we can use integration to get v(t) first, this will require a initial velocity (i.e. v_0 at t_0). Then using integration of v(t) we will get position, here we will need initial position. So in general, we will need two initial conditions in this case.[15]

The third case may need an example, consider the familiar motion with constant acceleration. If all we know acceleration is constant, $a(t) = A$.

Then

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[15] This is because we are dealing with second order differential equation. And the specific solution to remove the constants involved in integration will require two (the same number as the order of differential equation) conditions

$$v(t) = \int A dt + C_1 = At + C_1$$

$$x(t) = \int v dt + C_2 = \int (At + C_1) dt + C_2 = \frac{1}{2}At^2 + C_1 t + C_2$$

There are two undetermined constants, because we only know the acceleration. Suppose now if we know the initial velocity, v_0 at t=0; then $C_1=v_0$; if we know the initial position x_0 at t=0, then $C_2=x_0$.

The above relation would be reduced to the familiar form of constant acceleration:

$$x(t) = x_0 + v_0 t + \frac{1}{2}At^2 \qquad (2\text{-}6)$$

$$v(t) = v_0 + At \qquad (2\text{-}7)$$

You can eliminate t from the above two and get:

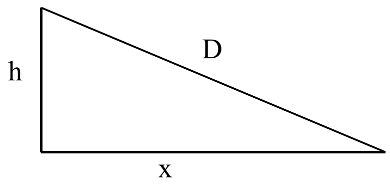
$$v^2 - v_0^2 = 2A(x - x_0) \qquad (2\text{-}8)$$

We shall see that (2-8) express the work-energy theorem in 1-dimension.

For those who are familiar with ordinary differential equation, the x(t) can be solved directly from: $\ddot{x} = A$. The general solution of x(t) would be $x(t) = \frac{1}{2}At^2 + Bt + C$ (B, C constant need to be determined from initial conditions), the results are same.

If the acceleration is not constant, but some function of time, the calculation would be a little complicated, but still straightforward integration. However, if the acceleration not only depends on time but on position or velocity too, then it is the problem of solving differential equations and won't be covered here.

Another example (this kind of problem is generally called related rate):



As the figure above, if we know the car moving along the x direction, at certain point, the velocity viewed along the D is v, a constant. Then what is the velocity and acceleration of the car along x direction? (In the problem, h is a constant, D, x will change over time, and the changing rate of D is given as constant)

Ans:
$$x = \sqrt{D^2 - h^2}$$
$$\frac{dx}{dt} = \frac{d}{dt}\sqrt{D^2 - h^2} = \frac{1}{2}\frac{2D}{\sqrt{D^2 - h^2}}\frac{dD}{dt} = \frac{D}{\sqrt{D^2 - h^2}}v$$

The method is the implicit derivative discussed in the supplementary.

You can further take time derivative to get acceleration along x:

$$a_x = \frac{d}{dt}\left(\frac{D}{\sqrt{D^2 - h^2}}v\right) = \frac{v}{\sqrt{D^2 - h^2}}\frac{dD}{dt} + Dv\left(\frac{-1}{2}\right)\frac{2D}{(D^2 - h^2)^{3/2}}\frac{dD}{dt}$$
$$= \frac{v^2}{\sqrt{D^2 - h^2}} - \frac{D^2 v^2}{(D^2 - h^2)^{3/2}} = \frac{-h^2 v^2}{(D^2 - h^2)^{3/2}}$$

Comment: Whenever we are dealing with integration or solving differential equations, the specific solution will depend on the initial conditions, as the motion with constant acceleration demonstrated. Another example is KK's Example 1.11 (pg 22), the motion of a free charge under an oscillating electric field.

Example 1.11 **Nonuniform Acceleration—The Effect of a Radio Wave on an Ionospheric Electron**

The ionosphere is a region of electrically neutral gas, composed of positively charged ions and negatively charged electrons, which surrounds the earth at a height of approximately 200 km (120 mi). If a radio wave passes through the ionosphere, its electric field accelerates the charged particle. Because the electric field oscillates in time, the charged particles tend to jiggle back and forth. The problem is to find the motion of an electron of charge $-e$ and mass m which is initially at rest, and which is suddenly subjected to an electric field $\mathbf{E} = \mathbf{E}_0 \sin \omega t$ (ω is the frequency of oscillation in radians per second).

The law of force for the charge in the electric field is $\mathbf{F} = -e\mathbf{E}$, and by Newton's second law we have $\mathbf{a} = \mathbf{F}/m = -e\mathbf{E}/m$. (If the reasoning behind this is a mystery to you, ignore it for now. It will be clear later. This example is meant to be a mathematical exercise—the physics is an added dividend.) We have

$$\mathbf{a} = \frac{-e\mathbf{E}}{m}$$
$$= \frac{-e\mathbf{E}_0}{m} \sin \omega t.$$

$\mathbf{E}_0$ is a constant vector and we shall choose our coordinate system so that the x axis lies along it. Since there is no acceleration in the y or z directions, we need consider only the x motion. With this understanding, we can drop subscripts and write a for a_x.

$$a(t) = \frac{-eE_0}{m} \sin \omega t = a_0 \sin \omega t$$

where

$$a_0 = \frac{-eE_0}{m}.$$

Then

$$v(t) = v_0 + \int_0^t a(t')\, dt'$$
$$= v_0 + \int_0^t a_0 \sin \omega t'\, dt'$$
$$= v_0 - \frac{a_0}{\omega} \cos \omega t' \Big|_0^t = v_0 - \frac{a_0}{\omega} (\cos \omega t - 1)$$

and

$$\begin{aligned} x &= x_0 + \int_0^t v(t')\,dt' \\ &= x_0 + \int_0^t \left[v_0 - \frac{a_0}{\omega} (\cos \omega t' - 1) \right] dt' \\ &= x_0 + \left(v_0 + \frac{a_0}{\omega} \right) t - \frac{a_0}{\omega^2} \sin \omega t. \end{aligned}$$

We are given that $x_0 = v_0 = 0$, so we have

$$x = \frac{a_0}{\omega} t - \frac{a_0}{\omega^2} \sin \omega t.$$

The result is interesting: the second term oscillates and corresponds to the jiggling motion of the electron, which we predicted. The first term, however, corresponds to motion with uniform velocity, so in addition to the jiggling motion the electron starts to drift away. Can you see why?

The reason lies in the initial condition given in the problem, the charge's trajectory will be an oscillation plus a drift. Could you figure out under what condition (initial condition) that the charge will be only an oscillation? This example also demonstrated that given the acceleration, the trajectory (solved by integration or solving differential equation) will depend on the initial conditions. (The solution to differential equations depends on the equation as well as initial conditions or boundary values)

Chapter 3 Motions in 2-Dimension

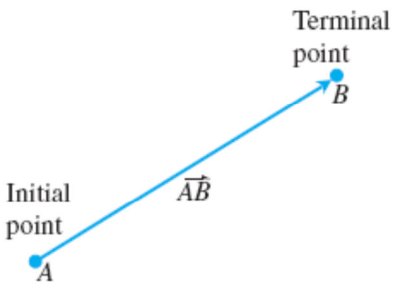
In this chapter we are going to discuss the motion in higher dimensions than the 1-D case in Chapter 2. Though the focus is in 2-Dimension, the results can be generated to the 3-D or even higher dimensions.

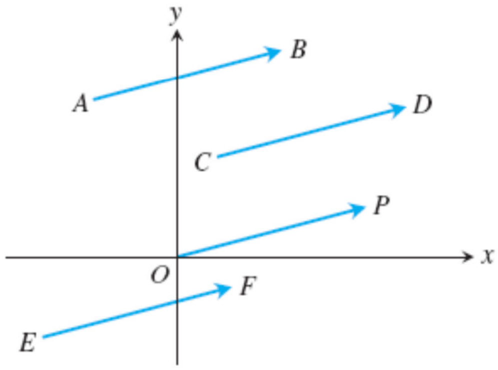
In the higher dimension, with the increase of degree of freedom, there are physical quantities that cannot be represented by a single number. Consider the position and displacement for instance: In 1-D case, suppose we start at some initial point A, and travel a distance d along the x-axis, the new position B would be $x_A + d$, if d is positive >0, the B will be to the right of A; if d is negative <0, then it is to the left of A. So in one dimension, the positions can be just labeled by a single number, either positive or negative; or a single value d would define the relation between the two positions A and B, d is called displacement between A and B.

In higher dimensions, the displacement between positions cannot be specified by a single number. If we travel a distance d from A, that won't specify the final position, it could be on a circle or sphere centered at A. To specify the final position, thus the displacement between A and B, we need not only distance but also direction. So the displacement between AB will be a quantity with information on both value and direction. There are many physical quantities similar to the displacement, such physical quantities are **vectors**. There are other physical quantities which only have a value, such as mass, temperature, etc. Such quantities are called **scalar**.

3.1 Definition of Vector and Geometric Representation

A vector is a quantity with both value and direction. It is represented geometrically with a pointing 'arrow', the length of the arrow tells the magnitude (that is how big the arrow is), and the arrow is pointing towards certain direction. The two vectors are equal require both same magnitude and same direction.





The figure above on the left is a geometric representation of a vector labeled as $\vec{AB}$, |AB| represents the magnitude (also called Norm of the vector or Module or length), and direction is the arrow from A to B. Conventionally a single bold faced letter or a letter with arrow head are used to represent the vector. i.e. $\mathbf{R}=\vec{a}=\vec{AB}$ are both legitimate symbols for vector. The vectors shown in the figure are all equal, because they have same length (magnitude) and same direction. You can treat them as equal displacement between the two points.

The vectors are quantities with magnitude and direction, is the reverse true also? i.e. is any value with magnitude and direction a vector? The answer is not necessary. The vectors have to satisfy certain algebra, i.e. the rules of their linear combination which we shall discuss next.[16]

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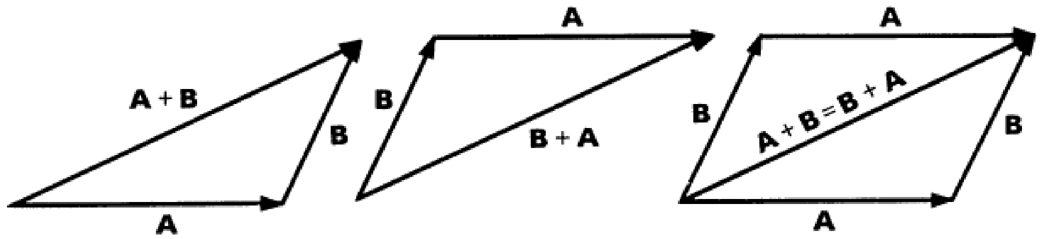
[16] A good example of physical quantity with magnitude and direction but is not a vector will be finite angle of

3.2 Linear Combination of Vectors

Linear combination means a**A**+b**B,** where a, b are some scalar number. The linear combination of vectors will give you another vector. First the addition of vectors obeys parallelogram rule



As shown in the figure (the one on the right), vectors **A,B** forms a parallelogram, the addition of two results in the vector **A+B** which is the diagonal. Or as shown in the figure on the left, put **A,B** head to tail, and **A+B** is shown in the figure. This is also called triangle rule. It is easy to

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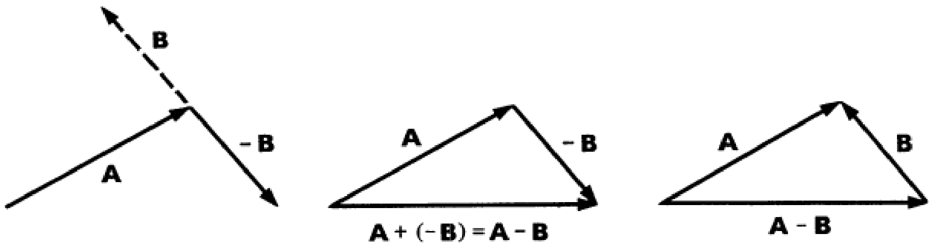
rotation. A not very good example of physical quantity that appears has both value and direction but not a vector is current. Consider a Y shaped 3 branches tube (or wire in electricity, the top 2 are labeled as 1,2, the lower branch is 3), the current flow from the upper two branches into the lower one. For incompressible fluid case, we have $I_1+I_2=I_3$. I's are currents in each branch. Sometimes this is used as an example to show that current is not a vector because it does not obey the parallelogram rules of addition. I say this is not good example. Because from the definition of current, I is not a vector at all, it does not contain information of direction. The current is defined as a quantity (the number of water molecules or mass; or number of electron) passing a unit area per unit time, the math form is: $I=\iint_{area} \vec{j}\bullet d\vec{s}$. It is defined as an area integral of scalar product between two vectors. $\vec{j}$ is a vector representing the density of flow.

You may further argue that $\vec{j}$ does not appear like a vector, because it seems does not follow the vector addition. If the 3 branches of Y have equal areas of cross section, then $|\vec{j}_1|+|\vec{j}_2|=|\vec{j}_3|$, this seems cannot be true from addition of vectors: $\vec{j}_1+\vec{j}_2=\vec{j}_3$. Indeed the $|\vec{j}_1|+|\vec{j}_2|=|\vec{j}_3|$ relation is not from the addition of vectors (the physics here is not a simple addition of vectors) but from the continuity requirement, i.e. what is flowing into a small volume enclosed by a surface would be same as that of flowing out, i.e. $\oiint_{\substack{closed\\surface}} \vec{j}\bullet d\vec{s} = 0$

see from this definition, **A+B**=**B+A**, the addition is commutative. And if we add more than two vectors **(A+B)+C**=**A+(B+C)**, the addition is also associative.

Second the product of a scalar number with vector. c**A** is also a vector who has the same direction (for c>0) as ***A***, but the magnitude is |c||**A**|; for c<0, the direction is reversed from **A**. With this rule the **–B** will be well defined, and **A-B**=**A+(-B)**, as the figure below shows.



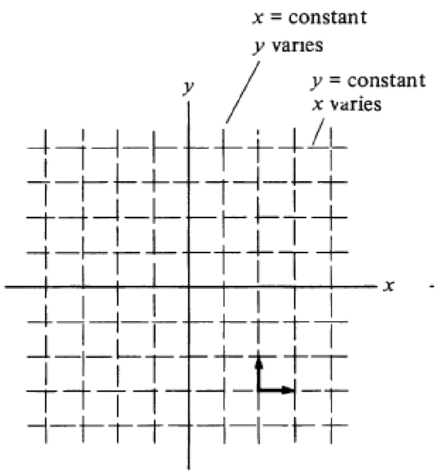
So a vector should be defined as a quantity with direction and they obey the linear combination rules discussed above.

3.3 Coordinate System and Algebra Expression of Vector

The above geometric representation of vector and their linear combination is straightforward. In practical applications, the calculation many-times are better carried out with algebra form of the vector. In order to put vectors in algebraic form, we first need to specify a coordinate system.

3.3-1 Coordinate System and Position Vector

Coordinate system is man-made choice to specify quantities. In the 2-dimension case, the simplest one is orthogonal x-y Cartesian, it is coordinate system with rectangular grids[17]:

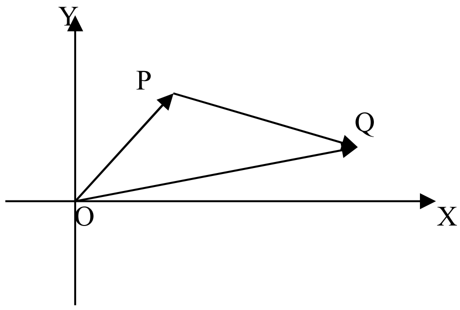


Cartesian

Every point in the plane thus will be specified by its 'grid numbers', it is the coordinates (x,y).

We will also define a special vector called position vector, it is nothing but a vector starting from origin and points toward a point P.

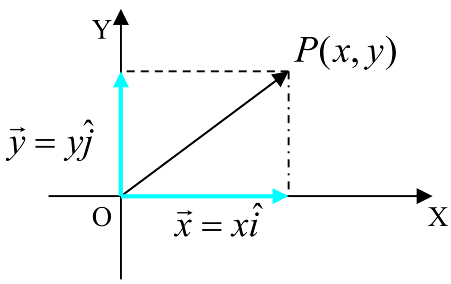
[17] For the 3-Dimensionm the conventional rule is to make x-y-z satisfies the right hand rule. Of course there are other choices of coordinate system, such as another orthogonal one: the polar coordinate we shall discuss later. You may in principle choose an oblique coordinate system (two axis which are not parallel will do), but that will make calculation much more complicated. So people prefer orthogonal system whenever such choices available. The advantage of orthogonal will be shown later.



What are shown in the figure above are a position vector **OP**, and another position vector **OQ**. The displace vector **PQ** is **OQ-OP** (O cancels and P changes order with this choice of symbols) from addition rule of vector. Thus any displacement vector can be expressed as linear combination of position vectors. Actually the position vector is just a special case of displacement vector between the origin of the coordinate system O and another point, the conventional symbol for a position vector is $\vec{r}$, i.e. $\vec{r}_P \equiv OP, etc.$

3.3-2 Analytical Expression of Vector in a Coordinate System

With the choice of a coordinate system (the origin and axis), any vector has an algebra (analytical) expression in such system. Let's consider the position vector first, because other vectors can be expressed as linear combination of these position vectors.



The position vector **OP** is the addition of two component vectors $\vec{x} = x\hat{i}, \vec{y} = y\hat{j}$, where x, y are coordinate numbers of point P, $\hat{i}$ is a unit vector (with length 1) along the X-direction; $\hat{j}$ is a unit vector along the Y-direction. The above relation (addition of vectors) can be written as:

$$OP \equiv \vec{r} = x\hat{i} + y\hat{j} \qquad (3\text{-}1)$$

More succinctly, we use <x,y> to represent the position vector $\vec{r}$. With this analytical form of the position vector, any vectors can be expressed this way, for example the displacement vector PQ would be:

$$\begin{aligned} PQ &= OQ - OP = \vec{r}_Q - \vec{r}_P = x_Q\hat{i} + y_Q\hat{j} - (x_P\hat{i} + y_P\hat{j}) \\ &= (x_Q - x_P)\hat{i} + (y_Q - y_P)\hat{j} \end{aligned} \qquad (3\text{-}2)$$

Or it is represented as $< x_Q - x_P, y_Q - y_P >$. You can check the figure on pg 36 to see that indeed the X component of the vector **PQ** is $(x_Q - x_P)\hat{i}$ (this is also called **projection** of vector along X) and its Y component is $(y_Q - y_P)\hat{j}$ (projection along Y).

In the 3-dimensional case, the above can be easily extended with an extra Z-axis and unit vector $\hat{k}$ along this direction as shown in the figure below (taken form Thomas Calculus).

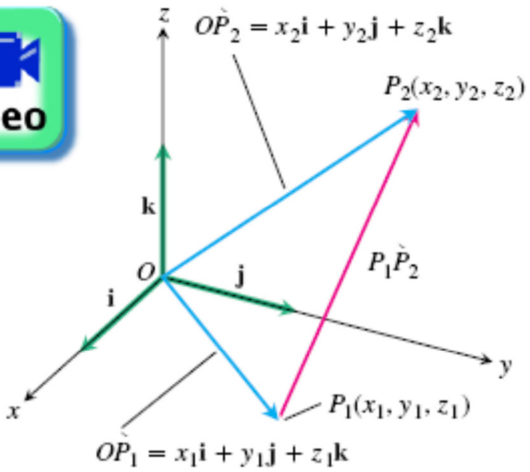


FIGURE 12.16 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

The linear combination of vectors can all be expressed as in algebraic forms. For any vector **A**:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \qquad (3\text{-}3)$$

More succinctly, (3-3) can also be expressed as $\langle A_x, A_y, A_z \rangle$. The coordinates $A_{i,j,k}$'s are also called the coefficients of the expansion, it is a number indicating how big the component is along the specific base direction. These numbers are called projection coefficients for the reasons discussed in the section of scalar product.

One example is that if we write two vectors are equal, i.e. **A**=**B,** this is true only if their components are equal, so 2 equations in 2-D and 3 in 3-D.

$$\vec{A} = \vec{B} \rightarrow A_x = B_x; A_y = B_y; A_z = B_z \qquad (3\text{-}4)$$

The magnitude (norm, module) of a vector is:

$$|A| = (\sum_i A_i^2)^{1/2} \qquad (3\text{-}5)$$

In 2-D, $|A| = \sqrt{A_x^2 + A_y^2}$ (Pythagoras theorem)

A unit vector along the direction of **A** can be written as**:**

$$\hat{a} = \frac{\vec{A}}{|A|} \qquad (3\text{-}6)$$

3.3-3 Base Vectors

The unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ are called base vectors in the 3-D space represented by Cartesian. They are the most fundamental vectors because they span the whole space in a sense that any vectors in this space can be written as linear combination of these base vectors, We say that they form a basis for the 3-D space. In the Cartesian, $\hat{i} = <1,0,0>$, $\hat{j} = <0,1,0>$ and $\hat{k} = <0,0,1>$.

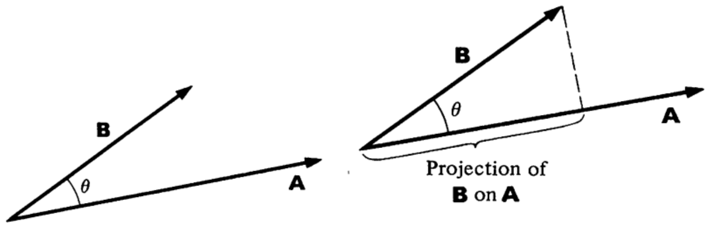
3.4 Product of Vectors

There are two kinds of product of vectors when we 'multiply' vectors. Both have important applications in physics. We shall discuss both the geometric and analytical formula for these vector products.

3.4-1 Scalar (Dot, inner) Product

The scalar product between two vectors is defined as (Geometrically):

$$\vec{A} \bullet \vec{B} \equiv |A||B|\cos\theta \qquad (3\text{-}7)$$



The meaning is clear from the figure. $|B|\cos\theta$ is the projection of **B** on **A**, and similarly you can treat $|A|\cos\theta$ is projection of **A** on **B**. ***So the dot product between two vectors will give a number, a scalar***.

An immediate result from this definition is that for the two perpendicular vectors (orthogonal, $\theta=\frac{\pi}{2}$), their dot product would be 0; and the dot product of vector with itself ($\theta=0$)would be its module squared. So in the orthogonal coordinate system, such as Cartesian, we have important relation among the base vectors, dot product between them is 0, between themselves is 1:

$$\begin{aligned}&\hat{i}\cdot\hat{i}=1,\hat{j}\cdot\hat{j}=1,\hat{k}\cdot\hat{k}=1\\&\hat{i}\cdot\hat{j}=0,\hat{i}\cdot\hat{k}=0,\hat{j}\cdot\hat{k}=0\end{aligned}\qquad(3\text{-}8)$$

In compact notation, the above can be written as:

$$\hat{m}\cdot\hat{n}=\delta_{mn}\ (m,n=i,j,k)\qquad(3\text{-}9)$$

δ_{mn} is Kroneck delta, $\delta_{mn}=1$ when m=n; $\delta_{mn}=0$ when m≠n.

The seemingly simple relations in (3-8) would have profound applications in math and physics. The expansion coefficients of any vector in a basis are indeed the dot product between a base vector and the vector, thus making a dot product of $\hat{i}$ with any vector A in (3-3):

$$\hat{i}\cdot A=\hat{i}\cdot(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})=A_x \qquad (3\text{-}10)$$

Similar relations exist for the y and z components.

So A_x the coefficient of x component is the projection of the vector onto the x-direction, this is also very intuitive from the figure in last page.

With the dot product of base vectors defined, we can work out the algebraic form for dot product between any two vectors:

$$\begin{aligned}\vec{A}\cdot\vec{B}&=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\cdot(B_x\hat{i}+B_y\hat{j}+B_z\vec{k})\\&=A_xB_x+A_yB_y+A_zB_z=\sum_m A_mB_m\end{aligned} \qquad (3\text{-}11)$$

(3.11) could also be proved from laws of cosine and definition of (3-7), this would be left as a practice. (hint: construct a triangle out of A, B and A-B)

Properties of the Dot Product

If **u**, **v**, and **w** are any vectors and c is a scalar, then

1. $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$
2. $(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})=c(\mathbf{u}\cdot\mathbf{v})$
3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
4. $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$
5. $\mathbf{0}\cdot\mathbf{u}=0.$

The above properties can be proved from (3-9) straightforward, and you do not need to memorize it, they are intuitive.

The dot product would have many applications in physics, such as work will be defined as the dot product between force and displacement vectors, current of flow is defined as dot product of current density and area element. Here I only discuss some general application of dot product

(1) Finding the length of vector and angle between vectors

The dot product of vector with itself will give the length squared. A rewriting of (3-7) definition and using (3-11) would enable us to calculate the angle between the vectors if we know their algebraic forms.

(2) Detection of orthogonality

If we want to know whether the two vectors are perpendicular or not, compute their dot product. If it is zero, then the two vectors are orthogonal.

Example: The analytical equation x+2y+3z=0, the x,y,z satisfies this relation is what shape in the 3-D space?

The answer is a plane. The relation is simply <1,2,3>• <x.y.z>=0. <1,2,3> is a vector, and <x,y,z> are vectors in a plane perpendicular with the <1,2,3> and the plane passes the origin. Question: how about x+2y+3z=5, what the x,y,z in this equation are? (It is also a plane, but not passing the origin anymore, can you give reasoning?)

(3) Component of vector (projection) along a direction $\hat{u}$

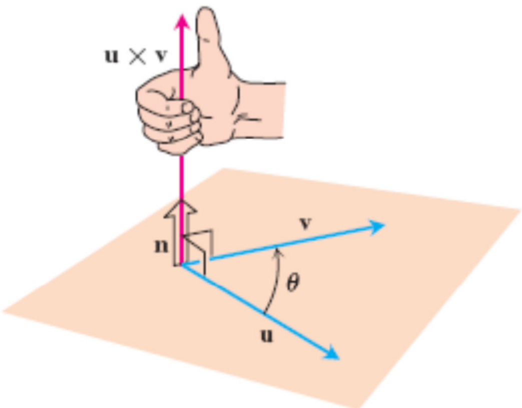
$\hat{u}$ is a unit vector specifies a direction, then the projection of any vector along this direction is simply:

$$A_u = \hat{u} \cdot \vec{A} \qquad (3\text{-}12)$$

A specific example is $A_x = \hat{i} \cdot \vec{A}$, etc. These simple relations will play important roles when we talk about transformations of coordinate system.

3.4-2 Vector (cross) Product

The cross product of two vectors **u, v** will result in **another vector n**, as shown in the figure below. The new vector formed by cross product has the direction defined by the right hand rule: first point the right hand fingers along the **u,** then rotate the fingers towards the positive direction of **v**, the thumb will give you the direction of **n**.



The magnitude of **n** is given by:

$$|n| \equiv |u||v|\sin\theta \qquad (3\text{-}13)$$

This definition means the new vector will be perpendicular with respect to the original two. i.e. **n** will be perpendicular to the plane formed by **u** and **v**. The magnitude also has a clear geometric meaning, it is the area of the parallelogram formed by **u** and **v** (example 1.4 in K,K).

From the definition of cross product, it has following basic properties:

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

The cross product is distributive but not commutative (It is anti-commute from property 4 above). The distributive property (2,3 in the table) may not be easily proved by using geometry only[18].

From this definition, you should verify the following relations between the base vectors:

$$\hat{i} \times \hat{j} = \hat{k}; \hat{j} \times \hat{k} = \hat{i}; \hat{k} \times \hat{i} = \hat{j} \qquad (3\text{-}14)$$

The above relation is of course because we choose a right hand orthogonal Cartesian.[19] It follows a simple pattern, i-j-k-i (a cycle), the cross product between the adjacent two will give you the next. Noticed the order of cross product is important, if you reversed order, the cross product will have a reversed sign (representing the reversed direction). And the cross product of vector with itself is zero. With (3-14) we can give the algebraic expression for cross product.

---

[18] See the Appendix 6 in Thomas 'Calculus' for proof.

[19] The cross product is also only applicable in our 3-D space. In higher dimensional space, the cross product between two vectors defined above would meet difficulty. It even can not specify a unique direction. Because in dimension>3, there are many directions perpendicular with respect to a plane.

$$
\begin{aligned}
C = A \times B &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\
&= A_x B_y \hat{i} \times \hat{j} + A_y B_x \hat{j} \times \hat{i} + A_y B_z \hat{j} \times \hat{k} + A_z B_y \hat{k} \times \hat{j} + A_z B_x \hat{k} \times \hat{i} + A_x B_z \hat{i} \times \hat{k} \\
&= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}
\end{aligned}
$$

Put it in component form, the above is:

$$
\begin{aligned}
C_x &= A_y B_z - A_z B_y \\
C_y &= A_z B_x - A_x B_z \qquad (3\text{-}15) \\
C_z &= A_x B_y - A_y B_x
\end{aligned}
$$

Another useful expression to remember the relation is using determinant:

$$
C = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (3\text{-}16)
$$

We shall see the application of cross product when we deal with rotation, the angular momentum, the torque etc.

3.5 Transformation of Basis

Let me first make clear of the meaning of some terms used in this section:

Basis: A set of base vectors (usually orthogonal and unit length) that spans a space, in which any other vector can be written as linear combination of these base vectors. These base vectors also form the coordinate system, so coordinate system and basis means same thing here; people (at least I) also use word ‘frame’ meaning the same thing.

Transformation of basis: Another choice of different set of base vectors, another coordinate system (frame)

Analytical form (expression) of vectors: the labeling of vectors in a particular basis, the coefficients of linear expansion of the vector in terms of base vectors. It is similar to the coordinate of points in a coordinate system. (see relation 3-3)

We have discussed geometric and analytical representation of vector. This part is the axiomatic point of view on vectors. To put the vector in a specific analytical form, we have to choose a coordinate system. The choice of coordinate system is man-made, and sort of arbitrary, generally we choose a coordinate system in which the expression and calculation would be the simplest. If we change the coordinate system, such as shifting the origin, or with fixed origin but rotating the axis, or even choose a completely different coordinate (eg. Polar instead of Cartesian), the analytical expression of the vector would be changing as the coordinate changes. However, the vector itself (geometric representation) does not change[20], it is **invariant** upon change of the coordinate system.

This is better be illustrated by an example, taking the displacement vector between Beijing and Tianjin. It is with fixed length and starting from Beijing ending at Tianjin. If you choose the earth coordinate with east-north as x-y axis, the vector would be something like <100km,

[20] To avoid confusion, I should point out that the vector here are displacement-vector-like. The position vector which is defined as displacement between points and origin will change if you shift the origin. However, the old position vector which is displacement between the point and old origin does not change. The statement here that the geometric representation of a vector does not change as coordinate system changes, does not imply that vectors are always constant. It certainly can changes over time. That is a different issue.

-100km>. If you choose a different axis for the coordinate, say south-east and north-east as x-y axis. In this coordinate, the Beijing-Tianjin would be <$100\sqrt{2}$,0>. The vector itself is fixed[21], but its expression within the coordinate system depends on the choice of coordinate. Because of this invariance of vector, i.e. the vector itself is independent of choice of coordinate, the *physical laws that can be expressed in term of vectors and scalars*[22] *will be independent of coordinate system too*, which should be since the choice of coordinate is man-made and somewhat arbitrary. Of course the detailed formula expressed with analytical form would depend on the coordinate (as we shall see that expression of acceleration would be quite different in Cartesian and Polar coordinate).

The change of coordinate system is called transformation of basis. Because of the invariance of vector itself, the expression of the vector will change accordingly as the coordinate changes. Since any vector can be expressed as linear combination of base vectors, we shall see that the relation between the base vectors in different coordinate system is the most important. Once we find out the transformation (how the new base vectors related to the old ones) between the base vectors, the expression

---

[21] Someone may argue that the length of vector is fixed alright, but isn't the direction depending on the choice of coordinate axis. Well, here with only one vector, to specify its direction indeed we need coordinate axis. However, if more vectors involved, the direction can be defined as directions relative to other vectors, and will independent of coordinate axis. Taking Beijing-Tianjin vector, we add another Beijing-Shanghai vector, their length and relative angles do not depend on the choice of coordinate system.

[22] Scalar is a special case for the general vector and tensor analysis. It is just the same number independent of coordinate system.

of any vector in the changed coordinate system can be determined from the previous one. The change of the expression of vectors with the coordinate system is called covariance of the vector.[23]

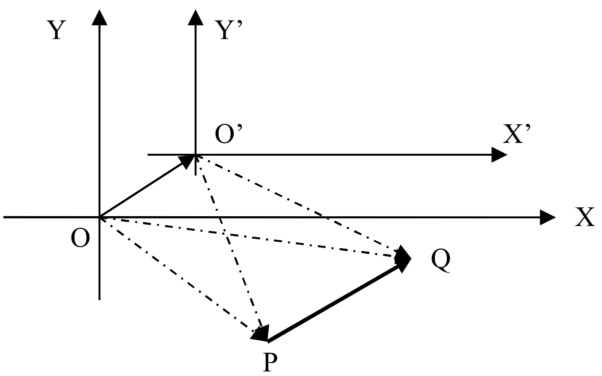
Here lies the reason why the vector is so important in physics, so it is worth to almost repeat myself once more. Because we want to have physical laws (a math relation between physical quantities or variables) to be independent of choice of coordinate (Now you know the jargon is: transformation between basis), and since the vectors themselves are independent of coordinate system, thus we want to express the physical laws in term of vectors (actually in terms of tensors to be strict, while scalar is 0^{th} order and vector is 1^{st} order tensors, we shall see higher order tensors in this course, but I shall only focus on vector for now). The physical laws thus expressed in terms of vectors are same for all coordinate of choice (Jargon: Invariance upon transformation); However because the expression of vectors (its projection coefficients along base vectors) depends on the choice of basis, and does change from one base to another. Such change of its expression is not arbitrary but follow certain rules (jargon: covariance of vector expression upon transformation) and we shall focus on two important transformation below: translation and rotation (another important transformation: constant velocity motion

[23] Actually, if we use oblique coordinates (i.e. the base vectors are not orthogonal), there are two kinds of expressions for a vector. One is called covariant and the other is called contravariant. Such difference does not exist in the orthogonal coordinate system.

called boost transformation will be delayed till special relativity, and we shall see a 'strange' 4-vector arise there), and see how a vector's expression will change (the relation between expressions in one coordinate system to the expressions in another) upon translation and rotation of the coordinate system. The vector expression has to change this way in order to keep the relations among them invariant upon transformation.

3.5-1 Translation of the Coordinates

Here we consider the simplest transformation, a shift of the origin. The coordinate is still Cartesian, but with the origin shifted to a new location, (a, b) in the old Cartesian, i.e O' is at (a,b) in X-Y system:



The new shifted coordinate system is X'-Y' in the figure. Just from the geometric point of view:

PQ=OQ-OP=(OO'+O'Q)-(OO'+O'P)=O'Q-O'P

PQ is invariant with coordinate change. Let's now take a look of its analytical form in different coordinate system. The relations between the coordinates in the initial and shifted systems are:

$$x' = x - a$$
$$y' = y - b$$

The expression of PQ in the X-Y is:

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (x_Q - x_P)\hat{i} + (y_Q - y_P)\hat{j} = \Delta x\hat{i} + \Delta y\hat{j}$$

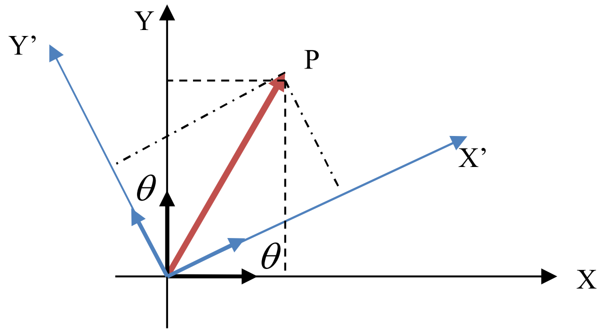
The expression of PQ in the X'-Y' is:

$$\vec{PQ} = \vec{O'Q} - \vec{O'P} = (x_Q' - x_P')\hat{i} + (y_Q' - y_P')\hat{j}$$
$$= [x_Q - a - (x_P - a)]\hat{i} + [y_Q - b - (y_P - b)]\hat{j} = \Delta x\hat{i} + \Delta y\hat{j}$$

So the analytical forms are same in both coordinate. This is not surprising at all, because in the translation, the *base vectors are unchanged*, and the projections (dot product) of the vector along the base vectors are also unchanged. This is the reason that we can shift (translate) vectors in the calculation without worrying about the change, such as the addition of vectors using triangle rule. We see that the translation is a very simple transformation, the vector's expression in the original and shifted frames are same. This is rather special than general as we shall see next in rotation, where the expressions in different frames are related but not the same.

3.5-2 Rotation of the Coordinate

As the figure shown, the X'-Y' system is rotated counterclockwise with angle θ relative to the X-Y system (the angle is defined positive for counterclockwise rotation, a right hand rule), the origins are overlapped.



The analytical form of vectors would be different in these two basis as illustrated for OP (same as O'P). The problem we are facing now is this: If we know the expression of a vector in one basis, say X-Y, what is its expression in the other?

This is not a difficult question, and just by simple geometry, you probably worked out already the relation between (x,y) and (x',y'), the coordinate of P in two systems:

$$x' = x\cos\theta + y\sin\theta$$
$$y' = x(-\sin\theta) + y\cos\theta$$

However, I shall workout the relation from point of view of vectors, starting from the most important relation between the base vectors (the **unit** vector along the axis). This may appear slow and complicated at the beginning for this simple problem, especially I will work out the example with different 'flavors', first by just calculating the components of the vector (the x, y and x',y's) directly from transformation of basis and specific to the rotation; then redo the same thing by expressing the vectors in terms of linear combinations of base vectors, and express its form in some general transformation, so it is more general and probably more abstract; and finally put all these in forms of matrices. It is like shooting a mosquito with cannon, but would prove fruitful in general treatment of transformations, which you will use in theoretical mechanics and quantum mechanics.

(1) The treatment from base vectors

The rotation will change the base vectors, this is in contrast to the translation. The relation between the base vectors are (from geometry and trigonometry):

$$\begin{aligned}\hat{i}' &= \cos\theta\hat{i} + \sin\theta\hat{j} \\ \hat{j}' &= -\sin\theta\hat{i} + \cos\theta\hat{j}\end{aligned} \qquad (3\text{-}17)$$

Or expressed the base vectors for X-Y as:

$$\begin{aligned}\hat{i} &= \cos\theta\hat{i}' - \sin\theta\hat{j}' \\ \hat{j} &= \sin\theta\hat{i}' + \cos\theta\hat{j}'\end{aligned} \qquad (3\text{-}18)$$

For any vector (the vector will be represented by $\vec{r}$, it is invariant under transformation, but its analytical expression will change) if its expression in the X-Y system is given by:

$$\vec{r} = x\hat{i} + y\hat{j}$$

Then using 3-18, we can rewrite the expression of the vector in component of base vectors in X'-Y':

$$\begin{aligned}\vec{r} &= x\hat{i} + y\hat{j} = x(\cos\theta\hat{i}' - \sin\theta\hat{j}') + y(\sin\theta\hat{i}' + \cos\theta\hat{j}') \\ &= (x\cos\theta + y\sin\theta)\hat{i}' + (x(-\sin\theta) + y\cos\theta)\hat{j}' = x'\hat{i}' + y'\hat{j}'\end{aligned} \qquad (3\text{-}19)$$

Now we have the expression of components of vector in the X'-Y':

$$\begin{aligned}x' &= x\cos\theta + y\sin\theta \\ y' &= x(-\sin\theta) + y\cos\theta\end{aligned} \qquad (3\text{-}20)$$

Reversely, if you know the x’,y’, you can work out the x,y (either by using 3-17, or directly from 3-20, I prefer from 3-17)

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \qquad (3\text{-}21)$$

Now we know the transformation relations due to the rotation of coordinates. If we know the vector expression in one, we can calculate its form in another.

(2) Express the Expansion Coefficients as Dot Product

I still want to press the issue a little further[24] by applying the orthogonality of the base vectors (relation 3-8 and 3-10) and dot product: Let’s relook the problem in this way: we have a vector, and it can be expressed in either coordinate system. In the X-Y or X’-Y’, it is a linear combination of the base vectors $\hat{i}$ and $\hat{j}$ or $\hat{i}'$ and $\hat{j}'$:

$$\vec{r} = x\hat{i} + y\hat{j} = x'\hat{i}' + y'\hat{j}'$$

The component of the projection can be expressed as dot product (a rewriting of 3-10):

$$x = \hat{i}\cdot\vec{r}, y = \hat{j}\cdot\vec{r} \qquad (3\text{-}22)$$

The component of vector in X’-Y’ would be similarly:

$$x' = \hat{i}'\cdot\vec{r}, y' = \hat{j}'\cdot\vec{r} \qquad (3\text{-}23)$$

The vector can be rewritten as:

$$\vec{r} = (\hat{i}\cdot\vec{r})\hat{i} + (\hat{j}\cdot\vec{r})\hat{j} = (\hat{i}'\cdot\vec{r})\hat{i}' + (\hat{j}'\cdot\vec{r})\hat{j}' \qquad (3\text{-}24)$$

---

[24] The following would not be required for this course, but the treatment would be useful in your later study in quantum mechanics.

The 3-24 is just a rewriting the vector as linear combination of base vectors. The significance of 3-24(as well as 3-22 and 3-23) is that it is explicitly showing that the coefficient is the projection (dot product) along the direction of the base vector. If you replace the general vector $\vec{r}$ with a specific one, such as the base vectors $\hat{i}'$or$\hat{j}'$, you should get back relation 3-17:

$$\begin{aligned} \hat{i}' &= (\hat{i}\cdot\hat{i}')\hat{i} + (\hat{j}\cdot\hat{i}')\hat{j} \\ \hat{j}' &= (\hat{i}\cdot\hat{j}')\hat{i} + (\hat{j}\cdot\hat{j}')\hat{j} \end{aligned} \qquad (3\text{-}25)$$

You should check that the above relation is exactly same as 3-17, by using the definition of dot product (here the dot product between unit vectors are extremely simple, they give you the cosine of the angle, referring the figure on pg 51). Similarly if you replace $\vec{r}$ with $\hat{i}$ or $\hat{j}$, you will get back 3-18(check it yourself).

$$\begin{aligned} \hat{i} &= (\hat{i}'\cdot\hat{i})\hat{i}' + (\hat{j}'\cdot\hat{i})\hat{j}' \\ \hat{j} &= (\hat{i}'\cdot\hat{j})\hat{i}' + (\hat{j}'\cdot\hat{j})\hat{j}' \end{aligned} \qquad (3\text{-}26)$$

We can also work out the equivalent of 3-20 and 3-21, the relation between the expansion coefficients (projection coefficients) in different basis. Using 3-22 and 3-23:

$$\begin{aligned} x' &= \hat{i}'\cdot\vec{r} = x(\hat{i}'\cdot\hat{i}) + y(\hat{i}'\cdot\hat{j}) \\ y' &= \hat{j}'\cdot\vec{r} = x(\hat{j}'\cdot\hat{i}) + y(\hat{j}'\cdot\hat{j}) \end{aligned} \qquad (3\text{-}25)$$

$$\begin{aligned} x &= \hat{i}\cdot\vec{r} = x'(\hat{i}\cdot\hat{i}') + y'(\hat{i}\cdot\hat{j}') \\ y &= \hat{j}\cdot\vec{r} = x'(\hat{j}\cdot\hat{i}') + y'(\hat{j}\cdot\hat{j}') \end{aligned} \qquad (3\text{-}26)$$

The conclusion is that if we know the relations between the base vectors (the dot product between them) of different basis, and the expression of vector in one basis, the expression in the other basis can be determined. This works for all transformations (not limited to the rotation case) between basis with orthogonal unit vectors (there is another jargon for these vectors, **orthonormal**=orthogonal and unit length), because in the above derivation orthonormal base vectors is the only property I used.

(3) Matrix Representation and Unitary Transformation[25]

The previous relations between vectors under transformation are best represented in a matrix form, and the powerful linear algebra is best suited to analyze these transformations. Since many of you may not have a background in linear algebra yet, so I shall present the above in matrix form, and try to use just a little linear algebra.

The vectors in a certain coordinate is represented by its components, and I use $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \equiv < A_x, A_y, A_z >$ to represent the vector in the space spanned by base vectors i,j,k, and $\hat{i} =< 1,0,0 >$, $\hat{j} =< 0,1,0 >$ $\hat{k} =< 0,0,1 >$. I could also use a matrix to do the same thing. The convention is to use a column matrix (n x1, n row, 1 column) to represent a vector. In such representation, the i,j,k will take the forms of:

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (3\text{-}27)$$

Then for any vector in this coordinate:

[25] For those are not familiar with linear algebra, please skip this part first. This is not required for early stage of this course, do come back reading this after you take the linear algebra.

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = A_x\begin{bmatrix}1\\0\\0\end{bmatrix} + A_y\begin{bmatrix}0\\1\\0\end{bmatrix} + A_z\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}A_x\\A_y\\A_z\end{bmatrix} \qquad (3\text{-}28)$$

Up to now this is just a reformulate the expression in matrix form (another bookkeeping). For the simple case of 2-D, the relation of 3-20 can be expressed as:

$$\vec{r} = \begin{bmatrix}x'\\y'\end{bmatrix}_{X'-Y'} = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}_{X-Y} \qquad (3\text{-}29)$$

(3-29) means the expression of vector r in the X'-Y' basis is related to the expression in X-Y by another matrix, $\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$. This matrix is called transform matrix (here just representing a rotation counterclockwise with certain angle), and I shall use symbol ***R(XY→X'Y')*** for this matrix and specify that it is a transformation from XY to X'Y' system. Similarly the relation 3-21 can be written in a matrix form:

$$\vec{r} = \begin{bmatrix}x\\y\end{bmatrix}_{X-Y} = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix}_{X'-Y'} \qquad (3\text{-}30)$$

Where ***R'(X'Y'→XY)***$=\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$, and it is interesting to put the (3-30) back into (3-29) and this will show the relation between **R** and **R':**

$$\begin{bmatrix}x'\\y'\end{bmatrix} = R(XY\to X'Y')\begin{bmatrix}x\\y\end{bmatrix} = R(XY\to X'Y')R'(X'Y'\to XY)\begin{bmatrix}x'\\y'\end{bmatrix}$$

The above equation means, first transform from X'Y' to XY and then from XY back to X'Y' (the order of matrix is important), so the two transformations would cancel each other, the vector (which is invariant) would get back its expression in X'-Y' basis. In matrix language this requires:

$$R(XY \to X'Y')R'(X'Y' \to XY) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (3\text{-}31)$$

I is called identity matrix. The matrix forms of rotational transformation do satisfy this requirement, since $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Similarly the ***R'R***=***I*** too. So we see that ***R'***=$\boldsymbol{R}^{-1}$ ($\boldsymbol{R}^{-1}$is called inverse matrix of ***R***). This is not surprising because the transformation from XY-X'Y' corresponds to the rotation of angleθ, and the transformation from X'Y'-XY is just another rotation but with angle $-\theta$. What is interesting is the matrix form of **R** and **R'**, they are transpose to each other, i.e. the 1^{st} row of matrix **R** is the 1^{st} column of matrix **R'**, etc. We use the symbol **R'**=$\mathbf{R}^{\mathbf{T}}$ for this relation. Combined with the inverse property, we see that:

$$\mathbf{R}^{\mathbf{T}}=\mathbf{R}^{-\mathbf{1}} \qquad (3\text{-}32)$$

The transformation satisfying 3-32 is called **Unitary Transformation**[26]**,** it is the most important transformation in physics. One important property of this unitary transformation is that the length (magnitude) of the vector

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[26] A math rigorous person will raise objection that I abuse the term Unitary here, which I admit. Strictly speaking, (3-32) represents orthogonal transformation where all matrix elements are real numbers. There are cases the matrix elements are complex numbers(such as in quantum), and the relation (3-32) should be modified by equating the adjoint of matrix with inverse of matrix, and that is unitary transformation in linear algebra (adjoint is a matrix transposed and complex conjugate to the original one). The essence of these are same, orthogonal transformation in real number world, and unitary in complex number world, so I abuse the term a little bit to call both unitary(In the real number world in this course, there is no difference between them).

will not change in the transformation.[27]

It is also illustrating to express the transformation matrix in terms of dot product of base vectors, using 3-25, we see that:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \hat{i}'\cdot\hat{i} & \hat{i}'\cdot\hat{j} \\ \hat{j}'\cdot\hat{i} & \hat{j}'\cdot\hat{j} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

To make it easier to remember, I shall use 1,2 and 1',2' to replace the symbol i, j and i',j', this labeling is also in accordance with labeling of matrix element. With this notation change, the transform matrix **R**:

$$R(XY \to X'Y') = \begin{bmatrix} \hat{1}'\cdot\hat{1} & \hat{1}'\cdot\hat{2} \\ \hat{2}'\cdot\hat{1} & \hat{2}'\cdot\hat{2} \end{bmatrix} \qquad (3\text{-}33)$$

The transformation matrix is determined by the dot products of the base vectors! Take a close look of each column of this matrix, you may recognize the first *column* is the column matrix representing the unit vector $\hat{1}$ in the $\hat{1}'-\hat{2}'$basis, and the second column is the column matrix representing unit vector $\hat{2}$ in the $\hat{1}'-\hat{2}'$basis,[28] i.e.

$$\hat{1}_{in\ X'Y'} = \begin{bmatrix} \hat{1}'\cdot\hat{1} \\ \hat{2}'\cdot\hat{1} \end{bmatrix};\ \hat{2}_{in\ X'Y'} = \begin{bmatrix} \hat{1}'\cdot\hat{2} \\ \hat{2}'\cdot\hat{2} \end{bmatrix}$$

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[27] To prove this would require the matrix definition of dot product. <r|r>, here the symbol |r> is a vector represented by a column matrix (as defined in the notes), where <r| is the row matrix representation of the vector, which is the transpose matrix of |r>. i.e. $|r\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$, $\langle r| = [x \ \ y]$. $\langle r|r\rangle = |r|^2 = x^2+y^2$. In a transformation where $|r'\rangle = \boldsymbol{R}|r\rangle$, then $\langle r'| = \langle r|\,\mathbf{R^T}$ (the proof of this is also in linear algebra). Then the magnitude of vector in the transformed basis is: $\langle r'|r'\rangle = \langle r|\mathbf{R^T R}|r\rangle$, if we require the $\langle r'|r'\rangle = \langle r|r\rangle$, for any vectors. Then $\mathbf{R^T R = I}$ or $\mathbf{R^T = R^{-1}}$.

[28] In linear algebra, there is a special term for this kind of matrix, i.e. the matrix with columns that are orthogonal to each other and with unit module, as the ***R*** here (the $\hat{1}$ and $\hat{2}$ are certainly orthonormal), such matrix is called orthogonal matrix.

So the relation:

$\vec{r} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \hat{1}'\cdot\hat{1} & \hat{1}'\cdot\hat{2} \\ \hat{2}'\cdot\hat{1} & \hat{2}'\cdot\hat{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is nothing but a matrix representation of the relation $\vec{r} = x\hat{i} + y\hat{j} = x'\hat{i}' + y'\hat{j}'$ in the basis of 1'-2' (here i,j… are replaced by 1,2…), because the above matrix form can be rewritten as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \hat{1}'\cdot\hat{1} & \hat{1}'\cdot\hat{2} \\ \hat{2}'\cdot\hat{1} & \hat{2}'\cdot\hat{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x \begin{bmatrix} \hat{1}'\cdot\hat{1} \\ \hat{2}'\cdot\hat{1} \end{bmatrix} + y \begin{bmatrix} \hat{1}'\cdot\hat{2} \\ \hat{2}'\cdot\hat{2} \end{bmatrix}$$

$$x'\hat{1}' + y'\hat{2}' = x\hat{1} + y\hat{2}$$

The last equation used fact that in the basis of 1'-2', the matrix forms of 1' and 2' are just $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the second equation is derived from the first by using column picture of matrix multiplication.

Similarly, using 3-26, we can work out the matrix expression for transformation ***R'***:

$$R'(X'Y' \to XY) = \begin{bmatrix} \hat{1}\cdot\hat{1}' & \hat{1}\cdot\hat{2}' \\ \hat{2}\cdot\hat{1}' & \hat{2}\cdot\hat{2}' \end{bmatrix} \qquad (3\text{-}34)$$

Comparing with 3-33 (or by seeing that the first *row* of ***R'*** is the expansion coefficient of $\hat{1}$ in the basis of $\hat{1}' - \hat{2}'$), ***R'*** is indeed the transpose matrix form of ***R***, i.e. $\boldsymbol{R'}=\boldsymbol{R}^T$. From the physical reasoning, the ***R'*** should also be the inverse transform of ***R*** (see the argument on pg56 leads to 3-31), i.e. $\boldsymbol{R'}=\boldsymbol{R}^{-1}$.[29] So we see that again $\boldsymbol{R}^{T}=\boldsymbol{R}^{-1}$, the transformation is ***unitary***. However this time it is not limited to the rotation (I only used orthonormal conditions for base vectors in the above argument, nothing specific to the rotation), generally the transformation between basis with orthonormal base vectors are unitary.[30]

Indeed, whenever possible, physicists choose orthonormal base vectors, so you are sure to see quite a few unitary transformations in the future.

3.6 A Summary and Other Vectors besides Displacement Vector

We have a relative long discussion of vectors, it may be better to give a brief summary before continue.

A vector is physical quantity with both value and direction. Its geometric representation is a pointed 'arrow', the linear combination of vectors follows parallelogram rule (or triangular rule).

They also have two kinds of product. The dot product is related to the projection; and the cross product defines a new vector which is perpendicular to the original ones.

The vectors are independent of the coordinate systems we choose, and the physical laws expressed as relations between vectors would also be independent of coordinate system. However, a choice of coordinate

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29 You should really check this by taking $\boldsymbol{R'R}$ given by 3-33 and 3-34, to see whether it is identity matrix. (You will need the fact that the base vectors are orthonormal. i.e. relation 3-8)

30 In linear algebra, this result is stated that for an orthogonal matrix Q, $Q^T=Q^{-1}$

system may be necessary and convenient in actual computation and application.

The vectors can be expressed as linear combination of base vectors in a particular coordinate system (this is also called decompose the vector into its components, or projection of vectors onto base vectors), the projection coefficients forms the analytical form of the vector in the basis. Knowing the base vectors and the projection coefficients, we know everything about the vector, its length and direction can be determined (3-5and 3-6); the addition and product of vectors can also be computed conveniently using the analytical expression (3-2, 3-11,3-15,3-16). This analytical expression does vary as basis is changed; to find out how the vector's expression changes as we change the basis is solving the problem of transformation of basis, and it is determined by the relations of base vectors in different basis.

All the above discussion I use only the displacement vector as example, it is a quantity with value (the distance) and direction, it is also following the linear combination rules, in short, it is a vector. It is actually a 'prefect' of vector, a 'Lei Feng' of vector. Other vectors in mechanics can be generated from this displacement vector, so it is also the 'mother' of other vectors.

Take the velocity for instance, it is generally taken for granted that velocity is a vector, and we seldom ask why. The reason that velocity is a vector is from its definition.

$$v = \frac{d\vec{r}}{dt}$$

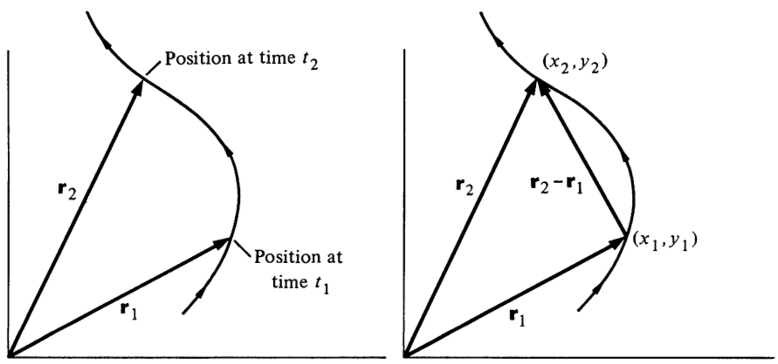
Where $d\vec{r}$ is an infinitesimal displacement vector, small it is, still a vector, dt is a infinitesimal time interval which is a scalar, so this is just a vector times a scalar, then from linear combination rules the velocity defined above would also be a vector. Once the velocity is a vector, then similar argument would lead the acceleration is a vector too, so will be the momentum and angular momentum (a cross product of displacement and velocity vector).

How about force? Well the argument is not straightforward as the one above, you may accept the force is a vector as a fact. But since we have a discussion in Chapter 1, that fundamental forces are functions of relative positions of interacting parties, it is a function of displacement vectors and its direction is along the displacement vector. The force also obeys the principle of superposition which is the addition of vectors (this is not a derivation but a postulate from experimental facts), so the fundamental force is a vector. Other force forms will be in fact the linear combination of the fundamental forces, so in general force is a vector.

3.7 Kinematics of Motion in 2-Dimension

The KK's book gives a very good and clear discussion on this (1.6-1.8), so I will only give a brief discussion here. The general strategy of the treatment here is: with higher dimension, we do need the vector forms to describe the motion. However, in real computation, we try to reduce the vector form into its component form, so reduce the problem into a couple 1-dimension-alike problems and all we stated in Chap.2 can be applied then. After solving the components (such as its change over time), we can construct the vector from its components if necessary.

The position of a particle in 2-D is described by a position vector (displacement relative to the origin of choice) $\vec{r}$, at certain time, this vector can be expressed in a Cartesian as: $\vec{r} =< x(t), y(t) >$. As time changes, the position vector changes too. The vector will generally make a trace of curve in the 2-D X-Y plane (the trajectory), the curve is given by the <x(t),y(t>, a **parametric** description of the trajectory. It may be reduced to some function y=f(x), but not necessarily (see the example in next section). Knowing the x(t) and y(t), it is sufficient to determine the trajectory. This is like reducing the problem in 2-D into two 1-D problems, by decomposing the vector into x and y components.



As the figure shows:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{r}(t_1 + \Delta t) - \vec{r}(t_1) \qquad (3\text{-}35)$$

And into its components:

$$\Delta\vec{r} = [x(t+\Delta t) - x(t)]\hat{i} + [y(t+\Delta t) - y(t)]\hat{j} = \Delta x\hat{i} + \Delta y\hat{j} \qquad (3\text{-}36)$$

3.7-1 Velocity and Acceleration Vectors

As the time interval approaches zero, the direction of $\Delta\vec{r}$ will approach the direction of the tangent line at point of (x(t),y(t)), and its value would approach the arc length of the curve ds (s is the arc length of the curve between the two points). The velocity is defined as the time derivative of the position vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \left|\frac{ds}{dt}\right|\hat{T} \qquad (3\text{-}37)$$

This is the geometric definition of velocity, its magnitude is given by the arc length change rate, and its direction is along the tangent line, indicated by the direction vector (a unit vector) $\hat{T}$.

More useful is the analytical expression of velocity in its x and y components:

$$\vec{v} = v_x\hat{i} + v_y\hat{j} = \frac{d}{dt}[x(t)\hat{i} + y(t)\hat{j}] = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} \qquad (3\text{-}38)$$

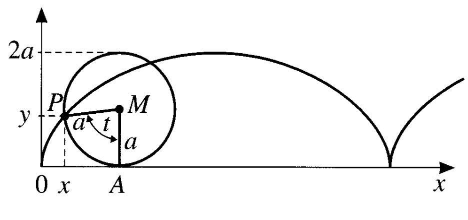
$$v_x = \frac{dx(t)}{dt}; \quad v_y = \frac{dy(t)}{dt} \qquad (3\text{-}39)$$

In the above derivation, it is vital that the base vectors do not change over time which greatly simplifies the time derivative. This is the most important property of a Cartesian system and thus the favorite one in many cases.

Reversely from velocity vector, we can find the position vector <x(t), y(t)>, it is a problem of integration. This is a sheer extension of the cases discussed in 1-D in Chapter 2, but you need two equations (thus two differentiation or integration) in 2-D. Similarly we can proceed from velocity vector to define the acceleration.

Examples are given in K&K, example 1.7-1.11. They are straightforward, pay attention to example 1.8 to see how we get familiar result of constant velocity motion along a circle from vector treatment.

Here is another example: the cycloid motion. It is the trajectory of a fixed point on a circle. As the circle rolls with constant velocity and rolls without slipping, the point will trace out a trajectory, and that is a cycloid. (a light attached to a bicycle wheel, as bicycle moves at constant velocity, the light will trace out a cycloid)



As the figure shows, the point on the wheel with radius *a* is labeled as P, we may take P at origin at time 0. i.e. P starts at <0,0>, as time goes on and the wheels rolls without slipping, the position vector of P will change too. To find out its trajectory is just to find out <x(t),y(t)> of OP:

$OP \equiv \vec{r} = OA + AM + MP$ simple vector addition

ω is the angular velocity of the wheel (the angle (in radian) wheel rotates per unit time), it is a constant here so that the rotated angle $\theta = \omega t$. It rolls without slipping, meaning |OA|(the distance traveled by the center)= $\widehat{PA}$ (the arc length of rotation) or $vt = a\omega t \to v = a\omega$ (v is the value of velocity of the wheel as whole, or the velocity of M). For simplicity, let's say $\omega = 1$ rad/sec, so the angle is just t. This gives:

$|OA| = PA(arc - length) = a\omega t = at$, $OA = < at, 0 >$

$AM = < 0, a >$

$MP = < -a\sin t, -a\cos t >$

So $OP = < at - a\sin t, a - a\cos t >$, this is the parametric function of a cycloid curve. You may try to find the expression of the velocity and acceleration for point P.

Below is a list of differentiation rules for vectors and functions of vectors:

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$

2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

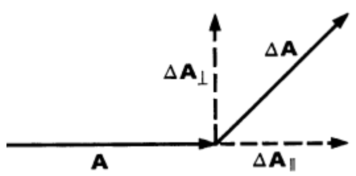
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

3.7-2 More about Derivative of a Vector

(K&K 1.8)

Because vector has magnitude and direction, so the change of vector can be seen from two parts: its magnitude change and direction change. We have discussed derivative of a vector by taking derivative of its components, reducing to the ordinary derivative of functions (3-38). Here we shall investigate the change of vector from its magnitude and direction.



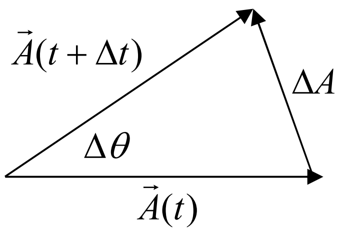
As the figure shows, the change of the vector (which is also a vector) can be decomposed into parallel component and vertical component. In the limit of small ΔA (which will be the case when we take derivative of **A**), we shall see that the parallel component is due to the magnitude change, and the vertical component is due to the directional change of vector. Let me rewrite the vector as:

$$\vec{A} = |A|\,\hat{a} \qquad (3\text{-}40)$$

This is just a rewriting of (3-6), |A| is the amplitude and $\hat{a}$ is the unit vector along direction of **A**, so |A| carries information of magnitude and $\hat{a}$ carries information on direction of the vector. When we take derivative of A over some variable, say time:

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}(|A|\,\hat{a}) = \frac{d\,|A|}{dt}\hat{a} + |A|\frac{d\vec{a}}{dt} \qquad (3\text{-}41)$$

The first part is the rate of change of magnitude, and its direction along the $\hat{a}$,it is (at least a part of) the parallel component of ΔA. Let's now focus on the second part which is clearly the directional change over time. It is the change of a vector with fixed length (unit length here, but it could also be looked upon as |A| fixed but direction changes). We shall show that the infinitesimal change of a fixed length vector is perpendicular to the original vector. Please refer to the figure below:



|A| is a constant here, and the change of vector over time interval is:

$$|\Delta A| = 2|A|\sin\frac{\Delta\theta}{2}$$

And the direction of ΔA is $\pi - (\frac{\pi}{2} - \frac{\Delta\theta}{2}) = \frac{\pi}{2} + \frac{\Delta\theta}{2}$. In the infinitesimal change, where the Δt is small, $\Delta\theta$ will be small too. We see right way that in such case:

$|\Delta A| \doteq |A|\Delta\theta$, and its direction is $\frac{\pi}{2}$ to the $\vec{\mathrm{A}}$ (3-42)

This proves that indeed the second part in (3-41) corresponds to the vertical component of vector change. For the unit length directional vector (|*a*|=1), put the relation into (3-41):

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}(|A|\hat{a}) = \frac{d|A|}{dt}\hat{a} + |A|\frac{d\vec{a}}{dt} = \frac{d|A|}{dt}\hat{a} + |A|\frac{d\theta}{dt}\hat{a}_{\perp} \qquad (3\text{-}43)$$

$\hat{a}_{\perp}$ is a unit vector perpendicular to the $\hat{a}$, and pointing toward the direction that θ increases[31]. Later we shall define a vector of rotational speed $\vec{\omega} = \frac{d\theta}{dt}\hat{k}$ (the vector is along the z direction perpendicular to the paper in this case, generally of course it can along any direction). Then

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[31] For $\hat{a}_{\perp}$, though it is perpendicular to $\hat{a}$, it still can take two directions. This definition will remove the ambiguity. Now the positive direction of $\hat{a}_{\perp}$ will depend on how we define the increase of angle, the convention is if the angle changes counterclockwise, it is increasing.

the directional change of vector can be written in a more compact form:

$$(\frac{d\vec{A}}{dt})_{\perp} = |A|\frac{d\theta}{dt}\hat{a}_{\perp} = \omega \times \vec{A} \qquad (3\text{-}44)$$

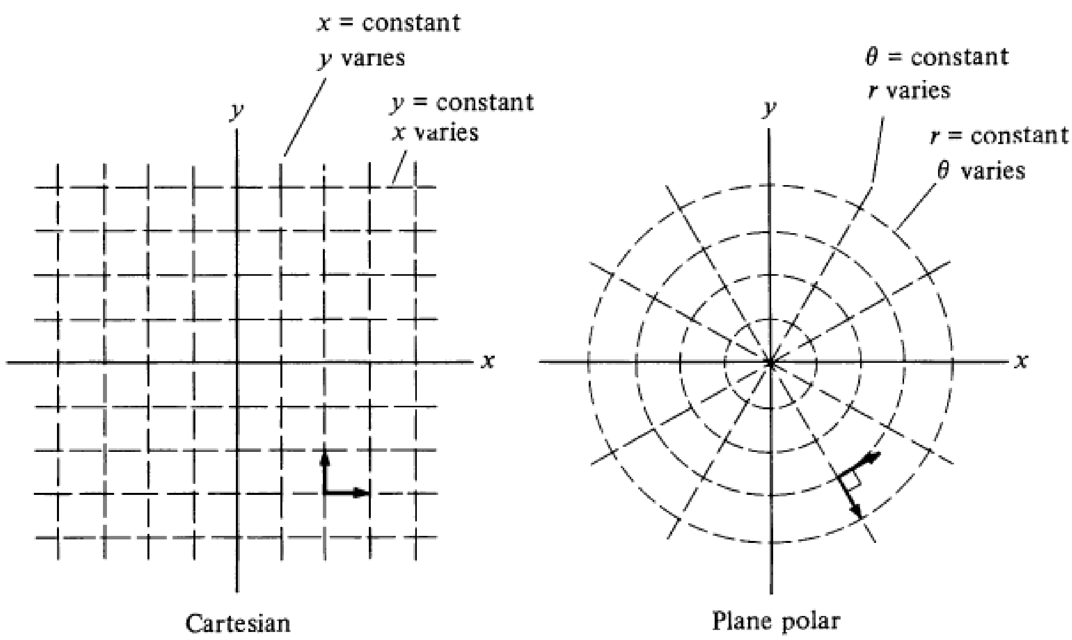
This relation also express the change rate of a vector of constant length, i.e. |A| is constant and then (3-44) is the change rate of the vector **A**. For instance, if we put the position vector $\vec{r}$ in place of **A**, and make the |r| not change with time, it is a circular motion in 2-D. From (3-44), we shall get familiar result, $\vec{v} = \frac{d\vec{r}}{dt} = \omega \times \vec{r}$, $|v| = |r||\omega|$, direction perpendicular with r. If the $|\omega|$ is a constant, the velocity will be a vector with constant magnitude too, and you figure out the expression of acceleration, of course you will get the same result as in example 1.8. (3-44) and the fact that it expresses the rate of change for a vector with constant magnitude will play important role when we deal with rotation later, so it may be worth remembering it.

Generally if the instantaneous change of a vector only has vertical component, it will change direction only, the vector will rotate, but the magnitude will not change. The change of vector over time is given by (3-44). Another example will be the Lorentz force in magnetic field. It will not alter the speed, but change the direction of velocity of a charge.

3.8 Polar Coordinate

(KK 1.9)

Up to now, the only coordinate system we discussed is the rectangular Cartesian, it is simple and the most important property is that: the base vectors do not change over time. In practice, sometimes it is useful to introduce other coordinate system, such as the polar coordinate here (in 3-D, you may use spherical and cylindrical system, that will be introduced when we need them[32]). When the motion involves rotation around center, it is very likely that polar coordinate becomes convenient to describe the motion, such as the motion of planets around star, etc, it is the coordinate we shall use when we discuss the motion in a central force field.



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[32] You may refer to Greiner's 'classical mechanics-particles and special relativity', chap 10 for discussion on spherical and cylindrical coordinates in 3-D.

As the figure shows that the grid system in polar coordinate is spider-web like. Every point in the 2-D has coordinates (r,θ) instead of (x,y) in Cartesian.

(1) Polar Coordinate vs. Cartesian Coordinate

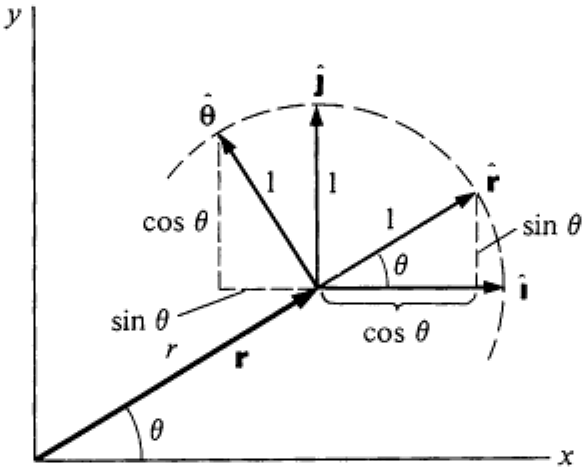
$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad (3\text{-}45)$$

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned} \qquad (3\text{-}46)$$

Relation 3-45 is used to get Cartesian from polar coordinates, and 3-46 is used vice versa (θ's value is determined within the range of $0 \to 2\pi; or\ -\pi \to \pi$, the value will be determined uniquely by the signs of x and y)

(2) Base Vectors in Polar Coordinate System

As we stressed before, when we change the coordinate system, the old expression are related to the new expression, and the most important is the relations between the base vectors of the two coordinate systems.



The base vectors in the polar coordinate is unit vectors $\hat{r},\hat{\theta}$, it is defined as the figure shows[33]. We see immediately this is just a rotated transformation as in 3.5. From (3-17) and (3-18), we have:

$$\begin{aligned} \hat{r} &= \cos\theta\hat{i} + \sin\theta\hat{j} \\ \hat{\theta} &= -\sin\theta\hat{i} + \cos\theta\hat{j} \end{aligned} \qquad \text{(3-47)}$$

And:

$$\begin{aligned} \hat{i} &= \cos\theta\hat{r} - \sin\theta\hat{\theta} \\ \hat{j} &= \sin\theta\hat{r} + \cos\theta\hat{\theta} \end{aligned} \qquad \text{(3-48)}$$

(3) Expression of Position Vector in Polar Coordinates

Direct from the figure above, we can see that the expression for position vector $\vec{r}$ in polar is just:

$$\vec{r} = r\hat{r} \qquad \text{(3-49)}$$

Where *r* is a shorthand for |r|, the module. It is **NOT** $r\hat{r} + \theta\hat{\theta}$ if you

[33] The definition is actually more subtle. It is the unit vector pointing to the direction of tangent line of the contour curve. In polar coordinate, the radial base vector is the tangent line of the curve with constant angle (the contour of angle, itself is a line, so the tangent line is just in the direction of the line). The angular base vector is the tangent line of the contour of the radius(the contour is a circle, so the angular base vector is along the tangent line of the circle). All these can be put in a math formula involving partial derivative: $\hat{q}=\frac{\partial\vec{r}/\partial q}{|\partial\vec{r}/\partial q|}$. For detail, refer to Afken's 'Mathematical Methods for Physicists" chap.2.

blindly use the results in Cartesian. Actually the result of (3-49) can be derived from the Cartesian expression since we know the relation between the base vectors:

$$\vec{r} = x\hat{i} + y\hat{j} = x(\cos\theta\hat{r} - \sin\theta\hat{\theta}) + y(\sin\theta\hat{r} + \cos\theta\hat{\theta})$$
$$= r\cos\theta(\cos\theta\hat{r} - \sin\theta\hat{\theta}) + r\sin\theta(\sin\theta\hat{r} + \cos\theta\hat{\theta}) = r\hat{r}$$

(You surely can also prove the same with a fancy method using matrix of transformation 3-33)

It may appear on the surface (3-49) is a simpler expression than Cartesian. Watch out, there is a snake lurking in the little $\hat{r}$! $\hat{r}$ (and $\hat{\theta}$) is not a constant unit vector like the one in Cartesian, (3-47) shows that they change as angle changes. In fact (3-49) is just a rewriting of general expression of vectors in (3-40). This fact is important when we derive the expression of velocity and acceleration expression in polar coordinates, because both r, θ and $\hat{r}, \hat{\theta}$ may depend on time.

(4) Velocity and Acceleration in Polar Coordinate

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = (\frac{dr}{dt})\hat{r} + r(\frac{d\hat{r}}{dt})$$

The expression explicitly displays the change of both r and $\hat{r}$ over time. The derivative of unit vector $\frac{d\hat{r}}{dt}$ is just what we discussed in section 3.7, relation 3-44 in which $\vec{A}$ (a constant length vector) is replaced by $\hat{r}$, and $\hat{a}_{\perp}$ is $\hat{\theta}$ here. We have:

$$r(\frac{d\hat{r}}{dt}) = r\frac{d\theta}{dt}\hat{\theta}$$

Then:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad (3\text{-}50)$$

The proof here is basically the method 1 in KK's book and it is the simplest one. The book also offers extra two nice methods for this proof. There can be more method, such as the following: If you are really a Cartesian guy, you may start from velocity expression in Cartesian $\vec{v} = v_x\hat{i} + v_y\hat{j}$, and throw in (3-48) to replace the base vectors, and v_x and v_y part, using the relation: $v_x = \frac{dx}{dt} = \frac{dr\cos\theta}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}$ …(try it yourself, a good practice for your derivative ability anyway)

Similar argument as in section 3.7 could also apply to the unit vector $\hat{\theta}$, so we have (using cross product in 3-44 to get sign correct):

$$\begin{aligned} \frac{d\hat{r}}{dt} &= \frac{d\theta}{dt}\hat{\theta} = \dot{\theta}\hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= -\frac{d\theta}{dt}\hat{r} = -\dot{\theta}\hat{r} \end{aligned} \qquad (3\text{-}51)$$

This relation will be used in the derivation of acceleration from velocity:

$$\begin{aligned} \vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = (\frac{d\dot{r}}{dt})\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{\theta}\hat{\theta}\frac{dr}{dt} + r\frac{d(\dot{\theta}\hat{\theta})}{dt} \\ &= (\frac{d\dot{r}}{dt})\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{\theta}\hat{\theta}\frac{dr}{dt} + r\hat{\theta}\frac{d(\dot{\theta})}{dt} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\ &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \end{aligned} \qquad (3\text{-}52)$$

The expression of acceleration is quite a beast in polar coordinate. This is because the chosen base vectors are not constant over time. You see why Cartesian is usually the preferred one. However, the polar coordinate is useful in some applications and example 1.14 in KK illustrates it. The $-r\dot{\theta}^2$ term along the $\hat{r}$ (it is actually along the direction of -$\hat{r}$, with magnitude $r\dot{\theta}^2$) is called centrifugal acceleration; and $2\dot{r}\dot{\theta}$ along the $\hat{\theta}$ is called Coriolis acceleration. The KK's book also give description of how you visualize these different terms, please read it yourself.

Don't get panicked by relations like (3-50) and (3-52), you can always derive them from the Cartesian results and relation between the base vectors. The most fundamental relations between the Cartesian and polar are relation (3-45) to (3-48).

An example of use of acceleration expression is a problem in central field, i.e. the force only along the direction of $\hat{r}$. Then we have: $r\ddot{\theta}+2\dot{r}\dot{\theta}$=0. This form is still not much illustrating, so play a math trick: $r\ddot{\theta}+2\dot{r}\dot{\theta}=\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=0$, this means in the central force field, there is a quantity that does not change over time, i.e. $r^2\dot{\theta}=\text{costant}$, we shall see that this is conservation of angular momentum.

Chapter 4.   Newton's Laws

We have spent quite some time on kinematics (the description of motion), now we come to dynamics, that is what changes the motion and how the motion evolves with time. The cause of the change of motion is the Force exerted on the party, and the motion will obey the Newton's laws. Only a brief discussion is presented here, since all of you already know the laws and there are details in the KK's book. However I'd stress on some points which are buried under these appealing simple forms.

4.1 Newton's Laws and Premises

(1) First Law (the law of inertia): In an inertial frame, the party will remain at rest or rectilinear motion with constant velocity if subject to no force.

This raised a question, what is an inertial frame? We will see this later.

(2) Second Law(equation of dynamics): In an inertial frame, we have:

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \qquad (4\text{-}1)$$

$\vec{P} = m\vec{v}$ is the mechanical momentum, m is the mass. So (4-1) also in form of:

$$\vec{F} = \frac{d\vec{P}}{dt} \qquad (4\text{-}2)$$

If the mass does not change over time, we have the familiar form:

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \qquad (4\text{-}3)$$

Naturally this raised the question of what are forces and mass? This will also be discussed later.

(3) Third Law (action-reaction): The forces exerted by the interacting parties to each other are equal in magnitude and reversed in direction:

$$\vec{F}_{1\to 2} = -\vec{F}_{2\to 1} \qquad (4\text{-}4)$$

(4) The principle of superposition of force. This is a postulate that is almost important as the other 3 laws:

$$\vec{F}_i = \sum_{j\neq i}\vec{F}_{j\to i} \qquad (4\text{-}5)$$

The relation states that the force experienced by the *i*th party equals to the sum of forces exerted by the other parties on it. $\vec{F}_i$ is the total force experienced by the *i*th party; $\vec{F}_{j\to i}$ is the force exerted by the *j*th party *alone*, that means as if only *i* and *j*th parties exist. The sum also obeys the parallelogram of vectors, so the force is a vector.

There are premises for these laws, some of which we had already discussed in Chapter 1. The premises are about the space, time and mass, all these parameters appears in the equations.

Premise 1: The time is absolute. The time is a parameter that varies continuously at the same rate in all reference frames (coordinate system).

Premise 2: Absolute and rectilinear space. This means there is a rectilinear Euclid space is absolutely at rest. This defines an inertial reference frame. It is independent of the objects in this space, like a stage where stars (the celestial body not movie stars, the universe won't change much when a movie star passed away) play; the stage is independent and unaffected by the players. This absolute space is also termed 'ether' in old physics.

Premise 3: Mass is independent of velocity.

As we have discussed at the end of chapter 1, these premises turn out faulty under certain circumstances, and thus put a limitation on the Newtonian mechanics.

4.2 Inertial Frame, Force and Mass

Newton's laws are fundamental postulates that cannot be proved from math and logic derivation, it is subjected to the test of experiments. However, there are important concepts and subtle points in these postulates and we shall take a closer look on these in this section.

(1) Inertial Frame

The first and second law only applies to the inertial reference frame. Newton took the absolute rest space as the 'mother' of inertial frame, and any reference frame which travels at a constant velocity with respect to this absolute inertial frame is also an inertial frame. This is derived from Galileo's relative principle: For the two frames only translate with constant velocity to each other (which one is at rest and which one in motion is not important, the relative velocity is a constant), if one reference frame (say X) is an inertial frame, the other one (say X')with constant velocity relative to X is also a inertial frame. This is because there is simple transform of coordinates in these two frames (**Galileo transformation**):

$$\begin{aligned} x' &= x - v_0 t \\ t' &= t \end{aligned} \qquad (4\text{-}6)$$

x' and x, t' and t are space and time coordinate in the X' and X frames, v_0 is the constant relative velocity between them. In the expression, assumed that X' is traveling with v_0 to the positive x-direction relative to X. (X appears for the observer in X' traveling with -v_0. i.e. toward the negative direction with speed of v_0)

If Newton's law holds for X frame, it also holds in the X':

$$\frac{dm'v'}{dt} = m'\frac{d^2x'}{dt^2} = m'\frac{d^2}{dt^2}(x - v_0 t) = m\frac{d^2x}{dt^2}$$

In the above derivation, besides Galileo transformation relation, we also applied the mass is independent of velocity. So the Newton's second law would apply in both frames and so is the first law by let F=0.

This implies that the absolute rest is redundant; if you have one inertial frame, you can have infinite numbers of other inertial frames. It may also appear the first law is just a special case for the second as F=0.

The value of the first law is that *it offers a method to test whether a* *reference frame is inertial or not.* Suppose we exclude all the forces (isolate the body), and observe the velocity of a particle, and if it stays the same, we conclude that we have an inertial frame.

In a airplane during take off or landing, you feel the push force but not change speed relative to the airplane, or the luggage on the floor of the plane suddenly rolls back or forward without subjecting to any forces horizontally, you know that the airplane during take off or landing is not an inertial frame. It is accelerating with respect to the inertial frame, and such frames are called unimaginatively non-inertial frame. We shall treat the non-inertial frames later and see how the Newton's laws modified in such frames (we have to introduce an inertial force, a fictitious force to make Newton's second law work properly). But as stated in the KK's book (pg 63), it may cause more confusion than their worth at this early stage, so we shall delay the discussion on the non-inertial frame. Our treatment of inertial frame is from experimental point of view (an empirical rather than paradigmatic), i.e. whether the 1st law is tested correct or not will tell us whether the frame is inertial or not.

In the theory of general relativity, an inertial frame is in which the space and time is homogeneous and isotropic. We had a discussion on this in chapter 1 and I told you there without proof that this homogeneous and isotropic (also called translational and rotational invariance) lead to conservation of energy, momentum and angular momentum. These conservation laws have more broad application than Newton's law. However I would not pursuit here in this anymore and rather leave the proof to the study of theoretical mechanics, you may read the references given in the footnote in chapter 1.

Finally, what kind of reference system can be treated as inertial frame empirically (since completely isolate a body is not very practical)? It is really depend on the measurement or accuracy of the measurement. For our daily experience, the earth is a pretty good inertial frame, though we know it is spinning and revolving around the sun, which make it actually a non-inertial frame. But in many cases, the acceleration of our earth is small enough, and the effect of inertial force we have to include to make the calculation correct can be neglected, and the earth frame can be approximated as an inertial frame. However in some cases, the effect of earth acceleration has to be corrected, for example battleships shooting each other over distances of 20km (a typical shooting range for battleships in WW2). If you forget to account for that earth is actually a non-inertial frame, your projectile will land in some position tens to a hundred meters away from where you intended. We will see this when we discuss about non-inertial frame and inertial forces. Finally if you really insist on the best inertial frame, the cosmic background radiation offers a standard. If in a reference frame, you measure this radiation background, and find it is homogeneous and isotropic, you are in an inertial frame.

(2) Force

This concept is also discussed in chapter 1. The force is one description of the interaction. It offers a way to calculate and measure how strong the interaction is. Besides this, I do not know what the force is. I only know the effect of the force. It changes the acceleration of the motion from the 2nd law.

Someone may state that the 2nd law defines the force, this is not true. The equation of (4-2) or (4-3) is a casual equation: It tells us cause and effect. The cause is the force, and its effect is to change acceleration. The force on the left side of equation has its own formula, such as the Hook's law for elastic force, gravitational force, electrostatic force and Lorentz force etc. The study of the force, coming from the study of interaction, is one important part in physics and chemistry (this is also stated at the end of section 1.2). Such forces has its independent definition and most important of all, its measurement. For example, it can be measured from the extension of a spring (Hook's law); and using Newton's third law, you can hook the interaction parties to the spring and thus study the force of other interactions (Cavendish's measurement of gravitational force, and Coulomb's measurement of electrostatic force)[34]. In such way, the formula for the forces can be determined *independently* from (4-3). Then you apply this force to some party and observe its motion obeys the 2nd

[34] A detailed account for these famous experiments is given in the book edited by H. Shamos: 'Great Experiments in Physics'

law. Of course, the change of a moving party's acceleration reveals existence of force, it can also tell you how big the party 'feels' the force. But this is not a definition of force, it does not reveal the origin of the force, only show you the effect. The common misconception that F=ma defines a force, may arise from the fact that the unit of force is indeed coming from the formula. That is how Newton (here the unit of force) is defined by $kg.m^2/s^2$.

In short, the force is a measure of interaction. It is defined independently from that of 2^{nd} law. 2^{nd} law does show us the effect of force.

In the KK, there is a section 2.5 on the common forces of different types, such as gravitation, electrostatic, friction, tension etc. Please read it yourself. Among different types of forces discussed in the book, fundamental forces are the gravitation and electrostatic. Other types forces are in principle be able to derive from these fundamental ones, but due to the complication of calculation(astronomical numbers of particles involved), approximate empirical formula are needed, such as F=-kx for elastic tension; F=uN for friction and F=-cv for viscous resistance etc.

(3) Mass, Inertial and Gravitational Mass

We have discussed that the force form can be independently determined, such as reading the scale of your spring 'forcemeter'. The force and the acceleration are related by a physical quantity: mass. We discussed above that the F=ma is not a definition for the force. Here we will see that F=ma is a definition of mass.

Well let's see how you define the mass of some object from relation (4-3). You say, hey put it on the balance and read the result. How about we are in deep space where there is no gravitation and will the balance still work (I entrust you know the difference between mass and weight)? In such cases, F=ma indeed offers a way to measure and define mass. It is essential that the force is independently determined, such as using a elastic force of spring F=-kx. From the extension of the spring, we know the force (we do not know the unit yet, since the unit of force is defined from the mass), the most important is that as long as same x, the extension, the force will be same. We can also measure the acceleration with measurement of length and time. Now comes the definition of mass. We have to use a chunk of matter as our standard, say 1 liter of water, and *define* this liter of water as 1 kilogram (1kg). You may also choose a different matter and assign a different unit (this is the weird English unit ounce and pound came from). Once you set the standard mass, F=ma allows you to measure the mass of other objects.

Suppose two matters, one is our standard unit mass, called 1kg, the other is a chunk whose mass is unknown. We apply the same force to these two objects (same spring and same extension, or you may hook the two with one spring with negligible mass, thus make sure the forces on the two parties are same in magnitude) and measure their acceleration, this will result: $\frac{m_2}{m_1} = \frac{a_1}{a_2}$, and the mass of the unknown can be determined relative to the standard. The mass determined this way is called **inertial mass**, with notation m_{iner}.

There is another mass you have seen, and actually when you measure the mass on earth with a balance, you are not measuring inertial mass, but the gravitational mass: m_{gra}. This mass comes from a different source, which is the Law of Gravitation Attraction:

$$\vec{F} = -G\frac{m_1 m_2}{|r_{12}|^2}\hat{r}_{12} \qquad (4\text{-}7)$$

In this famous formula, there are also masses in it. Actually this formula also defines a mass (i.e. the mass you measured from the balance on earth), the m_{gra}, **gravitational mass**. These two masses (inertial and gravitation) come from two different definitions and there is no obvious reason that these two are always equal. But they are. Actually strictly speaking, the ratio between $\frac{m_{iner}}{m_{gra}}$ is not necessarily 1, but is a constant ***for all matters***. This is actually a fundamental postulate in general relativity and had been subject to rigorous test in the experiments (the Eotvos experiment and its modern version). Can you give an example from daily experience or simple lab setup, to test that the two masses have proportional constant for all matters? As long as $\frac{m_{iner}}{m_{gra}}$ is same for all matters, we can set that they are equal by choosing the corresponding constant G in (4-7). The two masses are equal implies the relation between the gravitation and inertial force which was explored by Einstein, resulting in general relativity.

As a summary of this section, we discussed concept and definition of inertial frame, force and mass. The relations of these to Newton's laws are: The 1^{st} law offers an empirical way to determine an inertial reference frame; 2^{nd} law defines mass, and it relates the force with motion. 3^{rd} law as we shall see will give us the conservation of momentum.

4.3 Application of Newton's Laws and Examples

Using the Newton's law solving the dynamical problems, i.e. the evolution of the system over time, is basically starting from the initial condition and predict the motion (position, velocity) at later time. The basic equation is (4-2) and (4-3), however there are many tricks because there are so many variations. The problems in mechanics sometimes are quite tricky and hard. There are general steps guiding you tackle these problems. The KK's book gives you a guideline (section 2.4) and I can add little, but just restate it with a little bit reorganization.

The general steps in solving the mechanical problems are:

(1) Choose your system.

There are may be many-bodies involved in interaction. Isolate the party of interest. If possible treat the party as mass point.

(2) Analyze the force on the party of interest.

Include all the forces but no more. All the forces you are going to encounter in this class will be force *in contact* except gravitation. This means if party A feels force by party B, A and B have to be in contact, except the force by the gravitation field. We are not going to include inertial force due to the non-inertial frame yet, because we will work in the inertial frame at present.

(3) Choose the coordinate system

A coordinate must be in an inertial frame here. Though choice of coordinate is man-made and arbitrary, a wise choice may simplify the calculation tremendously. There are two choices in 2-D, Cartesian and Polar. Polar will be convenient if the problem involves rotation or has circular symmetry.

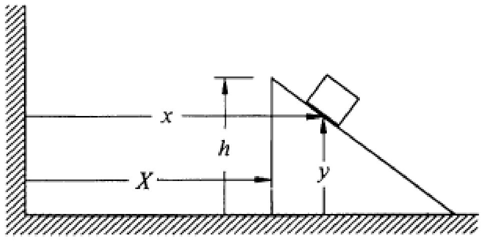
(4) List the equations of motion

Because equation of motion (4-2) or (4-3) are vector equations, so it means more equations, one equation for each components (in 2-D, this will result in 2 equations for x and y components). Sometimes these equations may be enough to solve for answer. Beware that since F=ma is essentially a differential equation. It could be 2^{nd} order differential equation for position or first order equation for velocity, so you may need some technique in solving the ordinary differential equations.

(5) The condition of constraint

There are cases where the number of unknowns exceeds the number of equation of motion. This is because these unknowns are not independent. There are constraint conditions imposed by the problem, these constraints add more equations on the relation between the unknowns.

Example 1.(KK's example 2.4 a) Wedge and block



(1) Choose the system.

The force between the block and wedge (normal force to the slope) will make the wedge accelerate toward the left when the block sliding down. You may choose the block only as your system and try to find the motion of it. However you may have to resort to non-inertial frame because the wedge is accelerating. So in this example, it is more appropriate to choose both wedge and block as our system

(2) Analyze the force (I entrust you can do this)

(3) Choose the coordinate

This is also straight forward here. A Cartesian based on the ground.

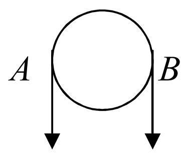
(4) List the equation of motion

There are 3 equation of motion in this problem (x, y component for the block and X component for wedge), however there are 4 unknowns (x,y), X and normal force N.

(5) Apply the constraint.

This could become the trickiest part. In this problem, it is relative simple. The motion of the block has to stay on the slope, this puts a limit on how its x and y components changes. The Δx and Δy relative to the wedge (note: not to the ground) has to satisfy the geometry condition of the slope. The detail is in the book. You may also use that the velocity of the block relative to the wedge has to satisfy: $(\vec{v}_x - \vec{v}_X) / \vec{v}_y = -\cot\theta$.

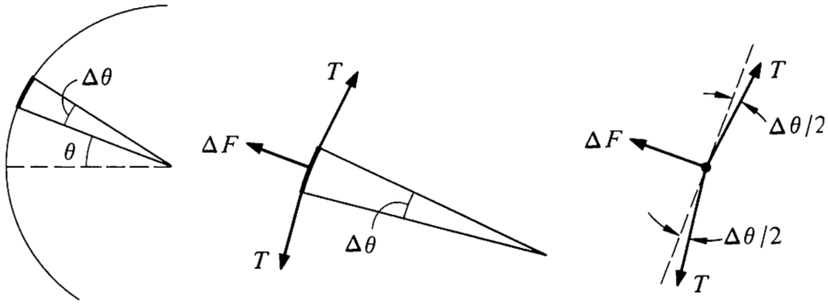
Example 2. Force analysis on pulley



As the figure shows a rope (weightless here for simplification) tight on pulley, and a force T is applied downwards to both ends of the rope (the force on both ends has to be equal if the pulley is frictionless because the rope is weightless). The question is what is the force on the pulley?

Most of you can give the answer quickly; the force on the pulley by the rope is 2T downward. But how you get this?

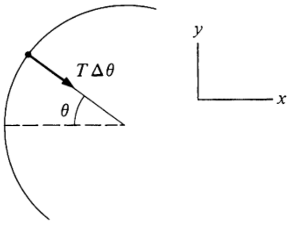
As we stated that the forces are contact force, we have to investigate a small segment of rope that in contact with the pulley and see how it exerts force to the pulley.



The figure shows the force on a small segment of rope that is in contact with the pulley. ΔF is the support force from pulley to the rope, T is tangent with the arc at the two ends. From the balance of forces on the rope (since the rope is weightless, the forces on it has to be balanced all the time). Clearly the value of the ΔF is:

$$\Delta F = 2T\sin\frac{\Delta\theta}{2} \approx T\Delta\theta$$

The force from the rope to the pulley would be same magnitude as ΔF, but reversed in direction, i.e. $\Delta F_{r\to p}$ would point along the radial direction inward:



From the symmetry, the X component of these $\Delta F_{r\to p}$ will add up to zero. For the Y component, they will add up to a force point downward, the value is:

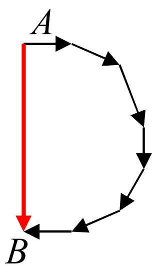
$$\Delta F_{y(r\to p)} = T\Delta\theta\sin\theta$$

Add all the contributions from the arc A to B, as $\Delta\theta \to 0$ this becomes an integral:

$$F_y = \int_0^{\pi} T \sin\theta d\theta = -T\cos\theta |_0^{\pi} = 2T$$

This is how you get simple 2T form the detailed force analysis (this derivation is a little different from the KK because I choose a different orientation)

There is another interesting method called "phasor" method (a geometric method to get the summation of vectors).



Each vector with length $T\Delta\theta$, is the contribution of the force from the small segment centered around θ. As $\Delta\theta \to 0$, the vectors will approach a circle with radius T and the results of summation will be 2T as the red arrow shows.

Example 3 Solving the 1st order ordinary differential equation with **method of separation of variables**. (KK, 2.16)

The problem is solving the velocity change over time for a particle traveling in a viscous media, with viscous resistance force $\vec{F} = -c\vec{v}$.

We shall choose the direction of velocity as direction of our coordinate axis, so the problem is reduced to a 1-dimension problem, and vector equation would becomes scalar equation.

$$m\frac{dv}{dt} = -cv$$

And we want to solve for v(t) provided with initial condition $v(0) = v_0$. This is a 1st order differential equation, and it is also a simple type because we can separate the variables, i.e. group the different variables to the different side of the equations, so we have a relation of the differentials of the variables, here is how:

$$\frac{dv}{v} = -\frac{C}{m}dt$$

Then proceed with definite integration on both sides of the differential equation. The time is from $0 \to t$, and velocity from $v_0 \to v(t)$

$$\int_{v_0}^{v(t)} \frac{dv}{v} = \int_0^t -\frac{C}{m}dt$$ [35]

$$\ln v |_{v_0}^{v(t)} = -\frac{C}{m}t \to \ln\frac{v(t)}{v_0} = -\frac{C}{m}t$$

$$v(t) = v_0 e^{-\frac{C}{m}t}$$

Such basic technique of separation of variables is required in this course.

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[35] If you really stick to math, you may worry about whether the two sides are equal or not, because the two integral are integrated with different lower and upper bound. From physical point of view, the integral make perfect sense, at t=0, v=v_0,etc. There always a one to one relation between t and v during the summation (integral is a summation). If this still not convincing, I shall rewrite the integral on the left hand side as: $\int_0^t (\frac{dv}{vdt})dt = \int_0^t \frac{dv}{dx}dt$. This will equal to the right hand side. Now play the trick of substitution of integrand, make the variable v instead of t. $\int_0^t (\frac{dv}{dx})dt = \int_{v_0}^{v(t)} (\frac{dt}{dx})dv = \int_{v_o}^{v(t)} \frac{dv}{v}$

Example 4. 1-D free falling: an object is at a distance from the earth and falling towards the earth center. The object has mass m and is at a distance of h (h could be very far, say infinity) from the center of earth, initial velocity is 0.

This is seemingly a simple problem:

The distance is x (earth center is origin), and the velocity is v(x), and the force is:

$F = -G\dfrac{M_e m}{x^2}$ and Newton equation will be:

$$m\frac{dv}{dt} = -G\frac{M_e m}{x^2}$$

The trouble is the x is a function of time that we do not know yet x(t), and we cannot solve this equation as it is presented in this form. If the right hand side is an explicit function of t, then it may be solved as a 1st order differential equation (also it is possible that it may not have an analytical solution at all and has to be solved numerically)

You may express the differential equation in forms of variable x instead of v, but that will become a second order differential equation:

$m\dfrac{d^2x}{dt^2} = -G\dfrac{M_e m}{x^2}$ and no simple solutions for this too.

We will see that we can use energy conservation (later) to reduce the above equation to first order differential equation (only involves dx/dt) .

The detailed calculation won't be presented here[36]. The point is to show you that even for this seemingly simple problem, directly using Newton's equation of motion may not give us solution. Yes, the equation will be some form of 1st order or 2nd order O.D.E (ordinary differential equations). But it can be a beast to be solved. The advanced method to solve O.D.E is not required in this course.

As to this example, though direct solving above 2nd order ODE on x may not be easy, there is a useful "trick" to apply:

$$Use\ \frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx} \qquad \text{(4-8)}$$

$$m\frac{dv}{dt}=mv\frac{dv}{dx}=-G\frac{M_e m}{x^2}$$

Then we can solve the relation between v and x using separation of variables to get v(x). The using v(x)=dx/dt to get x(t). The details calculation is straightforward but a bit messy due to integration, you may try it yourself.

We will need other tools to evaluate the motion besides the equation of motion. This will be the conservation relations we shall discuss next. Though at the beginning, these conservation laws may seem to be a corollary from Newton's laws, it turns out that they are more robust and have wide and profound applications in all branches of physics[37]. We

[36] You may find solution in: *American Journal of Physics* Vol.**74** pg1115-1119 (2006)

[37] We discussed but without prove that these conservation laws arise from homogeneous and isotropic of the space and time in Chapter 1.

shall first discuss the momentum and its conservation law; then energy and its conservation; followed by discussion of rotational motion and angular momentum and its conservation law. These compose the contents of the next few chapters of this note (as well as the KK's book).

Chapter 5 Momentum

5.1 Mechanical Momentum

As we stated in the equation of motion (4-1), the Newton's equation is: $\vec{F}=\frac{d}{dt}(m\vec{v})$. In the case that mass is independent of time the equation is reduced to the familiar from of F=ma. From the equation, we see that it's the combination of mass and velocity plays the important role in determination of motion, so this combination deserves a name for itself, and this is the mechanical momentum:

$$\vec{P}\equiv m\vec{v} \qquad (5\text{-}1)$$

The mechanical momentum is a vector along the direction of velocity. The reason it is called mechanical momentum is because it is a special case in the more general definition of momentum[38].

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[38] The more general definition of momentum (the canonical momentum) will be given in the theoretical mechanics: $p_i=\frac{\partial L}{\partial \dot{q}_i}$, where L is the Lagrangian of the system and the $\dot{q}_i$ is the generalized coordinate change with time (the generalized velocity)

In the first chapter, I said that the state of motion of a system can be described by position and velocity (x, v) of the particle (in 1-D). In the light of momentum, that statement should be modified that the state of motion is best described by position and momentum (x, p) in 1-D, or ($\vec{r}, \vec{p}$) in higher dimensions.

5.2 Multi-particle System and Center of Mass

Let's consider a general system consisting of N particles (it could be N separate particles or the particles congregate to form one object), each particle is subjected to a total force 'felt' by the particle: F_i, and this force can further be separated into two parts (only consider the inertial frame here): one is due to the mutual interactions within the system, i.e. the force exerted on the *i*th particle by other particles in the system; the other force is attributed to the external force. Then the total force felt by the particle *i* is:

$F_i = F_i^{\text{int}} + F_i^{ext}$ and for the *i*th particle, we have:

$$F_i = \frac{dP_i}{dt}$$

For each particle we have a similar equation of motion. Now take a summation of all the individual equation of motion:

$$\sum_i F_i = \sum_i F_i^{ext} + \sum_i F_i^{\text{int}} = \sum_i \frac{dP_i}{dt}$$

From Newton's 3rd law, the internal forces will always be in pair, and every pair would be equal in magnitude but reverse in direction, so $\sum_i F_i^{\text{int}} = 0$. The above equation reduces to:

$$\sum_i F_i^{ext} \equiv F_{total}^{ext} = \sum_i \frac{dP_i}{dt} = \frac{d(\sum_i P_i)}{dt} = \frac{dP_{total}}{dt} \qquad (5\text{-}2)$$

This is the most important equation in this chapter. It shows the relation between the force and the change of **total momentum** $P_{total} \equiv \sum_i P_i$ over time. In a multi-particle system, though the individual particle may have its own motion, there is a relation between the total external force and the total momentum change. The relation (5-2) has the same form of equation of motion of one particle, this becomes obvious if we introduce a concept of center of mass, a fictitious particle which has the total mass of the system, its motion will obey the relation (5-2).

5.2-1 Center of Mass

The position of the **center of mass** of a system is a point which is defined as:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M_{total}} \qquad (5\text{-}3)$$

From this definition, it is easy to see that:

$$\dot{\vec{R}} \equiv \frac{d\vec{R}}{dt} = \frac{\sum_i m_i v_i}{M_{total}} = \frac{\vec{P}_{total}}{M_{total}} \qquad (5\text{-}4)$$

The (5-2) will become:

$$\vec{F}_{total}^{ext} = \frac{dP_{total}}{dt} = \frac{dM\dot{\vec{R}}}{dt} \qquad (5\text{-}5)$$

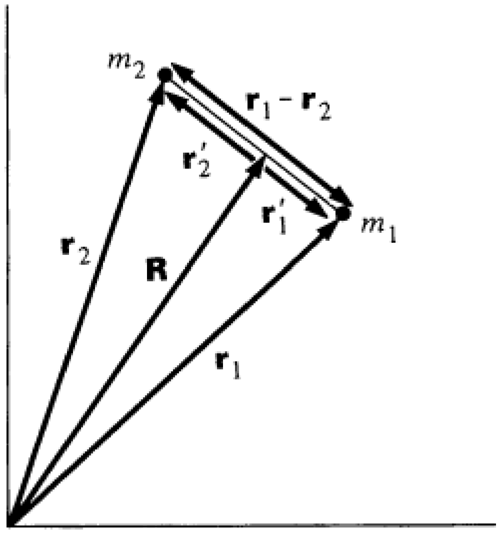
If the total mass do not change over time, the (5-5) would be like F=Ma, the equation of motion of a fictitious particle with the total mass under the total external force.

In some cases, such as a cannon ball exploded during the flight, though each individual piece may fly in different direction, the center of mass will still follow the trajectory of the projectile. Please also see Example 3.5 in KK.

5.2-2 Determine the Center of Mass

From the definition (5-3), the location of center of mass can be determined. The (5-3) is a general vector form and in real calculation, it is best expressed into formula of components. We shall take a look of the simple case where only two particles involved first, then investigate the more general case of continuous distribution.

(1) Center of Mass in a Two-Particle Case



As the figure shows, the center of mass is represented by a position vector **R**. It is from definition:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad (5\text{-}6)$$

It may not be obvious from (5-6) where the R locates. So let's calculate the displacement vector from the C.M(center of mass) to particles 1 and 2, labeled as r_1', r_2':

$$\vec{r}_1' = \vec{r}_1 - \vec{R} = \vec{r}_1 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_2(\vec{r}_1 - \vec{r}_2)}{m_1 + m_2}$$

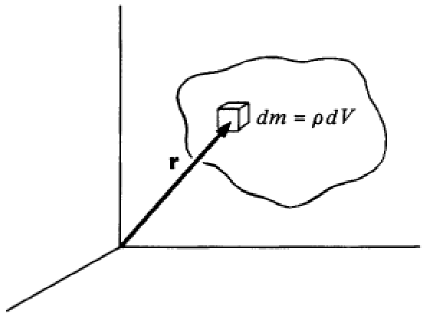
$$\vec{r}_2' = \vec{r}_2 - \vec{R} = \vec{r}_2 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = -\frac{m_1(\vec{r}_1 - \vec{r}_2)}{m_1 + m_2} \qquad (5\text{-}7)$$

r_1', r_2' are both in the direction of $\vec{r}_1 - \vec{r}_2$, so the R is located in the line joining the two particle ($\vec{r}_1 - \vec{r}_2$), distance to each end is given in (5-7). If the two particles are joined by a weightless lever and you put a support right beneath the C.M, it will stay at this balanced position, this is obvious from (5-2), as well as the lever equilibrium you learned in high school.

In real application, we usually choose a coordinate to put the two-particle along a coordinate axis and set a convenient origin (say m_1 is at x=0…), you do the calculation for (5-6) and (5-7) for this choice of coordinate.

(2) Center of Mass for Many Particles

If the particles in the system are more than 2, (5-3) is the formula of C.M for discrete case. If the object has continuous mass distribution:



$$\vec{R} = \iiint \vec{r}\rho dV \,/\, \iiint \rho dV \qquad (5\text{-}8)$$

ρ is the density, and the triple integral is over the volume of the object. If in 2-D case, it will be a double integral over area.

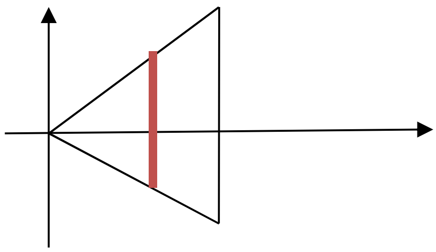
The vector form of C.M. in (5-3) and (5-8) though compact, is not very convenient in calculation, so their components formulas are:

$$X = \frac{\sum_i m_i x_i}{\sum_i m_i} \qquad \text{(5-9) for discrete case}$$

$$X = \frac{\iiint x\rho dV}{\iiint \rho dV} \qquad \text{(5-10) for continuous case}$$

The formulas for the Y, Z components are similar. In the evaluation of C.M., always first try to deduce it from symmetry before plunge into (5-9) or (5-10). KK's has examples 3.3, 3.4, here is another one.

Example: find the center of mass of an equilateral triangle with uniform density, say density=1, the length of each side=L



From symmetry, the C.M must be on the x-axis (actually for equilateral triangle, it also has to be on the other central line, so really you can determine the C.M from geometry), no need to compute Y, Y=0. Let's calculate X (this is also used as a simple example to show you how to setup the double integral, even I know where the C.M is from symmetry):

$$X = \frac{\iint\limits_{Area} xdA}{M} = \frac{\iint\limits_{Area} xdxdy}{M}$$

M is just the area of the triangle. The double integral is over the area (the bound) of the triangle. Of all the double (or triple) integral, it is critical how you divide (also called slice) the region. Here I choose to slice the area into stripes as shown in the red stripe in figure. At certain x, the stripe is extended (the range of y) from $x/\sqrt{3}$ to $-x/\sqrt{3}$, and x is from 0 to $\frac{\sqrt{3}}{2}L$. So the double integral would be:

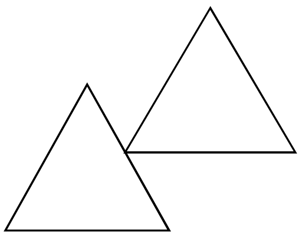
$$\iint\limits_{Area} xdxdy = \int_0^{\frac{\sqrt{3}}{2}L} \int_{-\frac{x}{\sqrt{3}}}^{\frac{x}{\sqrt{3}}} xdydx = \int_0^{\frac{\sqrt{3}}{2}L} \left(\int_{-\frac{x}{\sqrt{3}}}^{\frac{x}{\sqrt{3}}} dy \right) xdx = \int_0^{\frac{\sqrt{3}}{2}L} \frac{2}{\sqrt{3}} x^2 dx = \frac{2}{3\sqrt{3}} \left(\frac{\sqrt{3}}{2}L\right)^3 = \frac{L^3}{4}$$

$$M = \frac{1}{2}L(\frac{\sqrt{3}}{2}L) = \frac{\sqrt{3}}{4}L^2$$

$$X = \frac{L}{\sqrt{3}} = \frac{2}{3}(\frac{\sqrt{3}}{2}L)$$

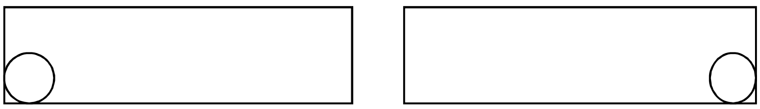
The C.M would be at two-thirds of the height, you may check this result purely from geometry.

The C.M could also be calculated if you divide your system into subsystems, and find out the C.M of each subsystem, and treat these C.M's of subsystem as a mass point and compute the C.M from these mass points. The proof of the theory is straightforward and is left for you. Using this theory, you may find out the C.M. of the object below:



5.2-3 Application of C.M and Characteristics of C.M Frame

Example:



A freight car (M) with length L is sitting on a frictionless surface. There is a ball (mass=m, radius negligible) at the beginning on the left side of the car, and rolls to the right side of the car and stop (there are friction between the ball and the car). The question is what the distance travelled by the car?

This problem is not easy to solve using equation of motion, but with help of C.M, the solution is obvious:

Initially the C.M. is at: $X_1 = \dfrac{M(\frac{L}{2})}{M+m}$

The car will travel distance x to the left and when the ball stopped, the car will stop too, because under no external force, the C.M will not move. The ball will be at (L-x), and the car's C.M will be at (L/2-x). The C.M at this final moment will be:

$$X_2 = \frac{m(L-x) + M(\frac{L}{2} - x)}{M+m}$$

And $X_2 = X_1$, the x can be easily solved then.

If there is no external force, the C.M. will obey the Newton's 1st law. We can choose a coordinate system that travels along with the C.M., and set the C.M. as the origin of this coordinate system. This is what we called center of mass frame. There are a few advantages to choose the C.M as origin, there will be a couple simplifications:

Let $\vec{R}, \vec{r}$ be the vectors representing the C.M and particles in one reference frame, $\vec{R}_c, \vec{r}_c$ the vectors in the C.M frame, clearly since we choose C.M. as origin, $\vec{R}_c = 0$, write it explicitly:

(a) $\vec{R}_c = \frac{\sum_i m_i \vec{r}_c}{\sum_i m_i} = 0$ (5-11)

This can also be directly tested using (5-7)

(b) $\dot{\vec{R}}_c = \frac{\sum_i m_i v_{ic}}{\sum_i m_i} = 0$ (5-12)

$\sum_i m_i v_{ic} = 0$

This shows that in the C.M. frame, the summation of momentum (Note: measured relative to the C.M. frame) is zero. So the C.M frame is also called zero total momentum frame (of course the total momentum of the system as we see from (5-5) is carried by the center of mass). Relation (5-11), (5-12) are important properties of C.M. frame and will play important roles later when we discuss scattering and other applications involving C.M.

5.3 Conservation of Momentum

From (5-2) $\vec{F}_{total}^{ext} = \frac{dP_{total}}{dt}$, if the total external force is zero, then the total momentum will be a constant of motion, i.e. do not change over time. This is conservation of momentum. Noticed the (5-2) is a vector equation, it can be decomposed into components:

$$F_x = \frac{dP_x}{dt}, F_y = \frac{dP_y}{dt} \ and \ F_z = \frac{dP_z}{dt}$$

So if the total external force is not zero, but one of its component is zero(say $f_x=0$), the total momentum component (p_x) will be conserved. (Example 3.6)

Here are some more examples:

1. Consider the free fall object towards the earth. The object and the earth form a closed system, i.e. neglect external forces outside. What is the momentum change of the object?

   It is a free fall and close to the earth surface, the velocity change over time is: $v = gt$, so $p = mgt$. It is increasing over time and clearly not conserved. This because under the influence of the object, the earth will move towards the object too (strictly speaking, the earth is not an inertial frame here), with a velocity so small (because of the large mass of earth) that is negligible, but the momentum of the earth Mv is not small and the total momentum will be zero.

2. Earth revolves around the sun.

   Let's neglect other influences (such as moon, Jupiter etc), so the sun-earth forms a closed system and no external force. The earth revolves the sun in a circular orbit (an ellipse very close to circle). The velocity of the earth changes over time (the direction) so will the momentum of earth. From the conservation of momentum, this requires that the sun will also move in the counter direction of the earth, so that the total momentum of sun+earth will not change. Both sun and earth revolves around a common point, you probably can guess where is that point. It is the C.M., which do not rotate with no external force. Because the sun is so massive that the C.M. of sun+earth almost overlaps with the center of sun, and the motion of sun is not easy to observe.

In the derivation of (5-2) at the beginning of this chapter, it is essential that the Newton's 3rd law makes the contribution of the total internal forces disappear. Thus we have the conservation of momentum under no external forces. That is the reason I stated in Chapter 4 that the 3rd law will result in the conservation of momentum (which is true from Newtonian point of view). However, it turns out that the conservation momentum which is the result of translational invariance of space, has wider applications than the 3rd law. It is more appropriate to say that the 3rd law of mechanics is the result (or special case) of the conservation of momentum.

5.4 Momentum Change and Impulse Theorem

This is just the integral of the most important relation of this chapter (5-2). We integrate the left and right hand side from some initial time t_0 (it could be set as 0) to some final time t:

$$\vec{F} = \frac{dP}{dt} \rightarrow \int_0^t \vec{F} dt = \int_0^t (\frac{dP}{dt}) dt = P(t) |_0^t$$

$$\int_0^t \vec{F}dt = P(t) - P(0) \qquad (5\text{-}13)$$

The left hand side $\int_0^t \vec{F}dt$ is called impulse. The change of the momentum of the particle or the system equals to the impulse (the force has to be the total force on the particle or the system). The (5-13) is called impulse theorem and is useful to evaluate the momentum change given the known force (read the equation from left to right); or from the momentum change to derive the force (from right to left). For the second application (from momentum change to know force), there are generally two strategies.

(1) Evaluate the average force over time interval

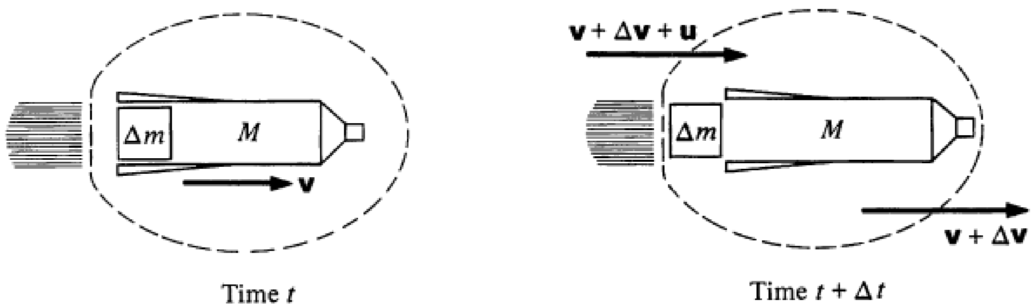
$$\begin{aligned} &\vec{F}_{ave} \equiv \overline{F} \\ &\overline{F}\Delta t = \int_0^{\Delta t} \vec{F}dt = P(\Delta t) - P(0) \equiv \Delta P \end{aligned} \qquad (5\text{-}14)$$

Examples are given in 3.9 and 3.10 in the K&K.

(2) Evaluate the instantaneous force

This is to take a very small time interval, and from the momentum change, the instantaneous F can be evaluated by taking the limit $\Delta t \to 0$. The examples are 3.11 and 3.12.

Here we are going to work out the problem of rocket propulsion from this strategy: At time t, the total mass is $M + \Delta m$, M is the mass of rocket and Δm is the fuel that is going to be ejected. At later time $t + \Delta t$, the fuel will be ejected with a constant velocity $\vec{u}$ *relative to* *the rocket* (note not to the ground or inertial system of your choice), and the velocity relative to the inertial frame is shown in the figure:



The initial momentum is:

$P(t) = (M + \Delta m)\vec{v}$

At later time, the velocity of the fuel in the inertial frame is:

$\vec{v} + \Delta\vec{v} + \vec{u}$ and the momentum of the whole system becomes:

$P(t + \Delta t) = M(\vec{v} + \Delta\vec{v}) + \Delta m(\vec{v} + \Delta\vec{v} + \vec{u})$

The change of momentum (neglect higher order $\Delta m\Delta\vec{v}$):

$\Delta P = M\Delta\vec{v} + \Delta m\vec{u}$

$$F_{ext} = \lim_{\Delta t \to 0}\frac{\Delta P}{\Delta t} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt} \qquad (5\text{-}15)$$

This uses that fact that the exhaust mass rate of change equals decrease rate of mass of the rocket, i.e. $\frac{dm}{dt} = -\frac{dM}{dt}$. (5-15) is the fundamental equation of rocket propulsion.

In the case of zero external force $F_{ext} = 0$, we have:

$M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt} = 0$ or:

$\vec{u}\frac{dM}{dt} = M\frac{d\vec{v}}{dt}$

This looks like F=ma, where $\vec{u}\dfrac{dM}{dt}$ is the propulsion force (recoil force) on the rocket by the fuel ejection. ($\dfrac{dM}{dt}$<0, the force is along the reverse direction of $\vec{u}$). The final velocity can be solved by integrating both sides:

$$\vec{u}\frac{dM}{M} = d\vec{v}$$

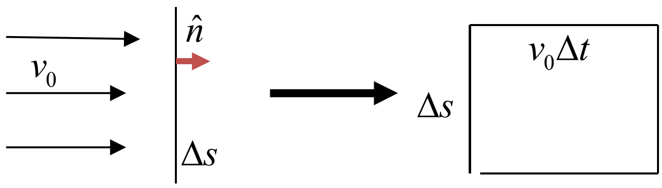
$$\vec{v}(t) - \vec{v}(0) = \vec{u}\ln\frac{M(t)}{M(0)}$$

This is the result of example 3.14 in KK. For the case of rocket under gravitation force, refer to example 3.15.

Another important application of this strategy is momentum transfer (section 3.6 in KK). A flow of particles are hitting the surface and being scattered. Because of the change of momentum of the particles, they must 'feel' a force exerted on them by the surface, and the surface will experience a force too by the 3rd law. This is why we have air or water pressure. How we evaluate such pressure?

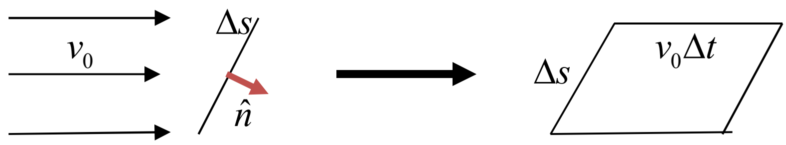
Let's consider the following model, the particles are flowing with certain velocity v_0. It could be a constant, then we will have a simple velocity distribution, also called constant velocity field. Field is nothing but a distribution of physical quantity, if the quantity is scalar (such as temperature), it is a scalar field; if the quantity is vector (such as velocity in this case), it a vector field. If it is not constant, it is generally a function of space and time $\vec{v}(r,t)$. For simplicity, we consider the constant velocity field here.

The strategy of momentum change and impulse is to take a small time interval and work out the momentum change. The first question is within small time interval, how many particles hitting the surface? Equivalent question is how many particles will flow through a surface. This is a question of estimate the 'flux' of a vector field (here the velocity field):



The flux[39] of the vector field is just the volume of the rectangle in the right of the figure above, which is $v_0 \Delta t \Delta s$, the number of particles passing through the surface in the time interval would be just $\Delta m = \rho v_0 \Delta t \Delta s$, ρ is the density of the fluid. $\rho v_0 \Delta t \Delta s$ would be the mass passing through the surface during the time interval Δt. What I draw above is actually a special case where the surface is perpendicular to the velocity field, or equivalently the normal direction is parallel with the v. The general case would be:

[39] Strictly speaking, flux is defined as flow across an area per unit time, i.e. $\Delta t = 1$.



The surface normal $\hat{n}$ (unit vector) will have an angle θ with the velocity. The volume representing the flux will be:

$v_0 \Delta t \Delta s \cos\theta = \vec{v}_0 \cdot \hat{n} \Delta s \Delta t$ (5-16)

$\Delta m = \rho \Delta t \vec{v}_0 \cdot \hat{n} \Delta s = \rho \Delta t v_0 \cdot \Delta \vec{s}$ (5-17)

$\Delta \vec{s} \equiv \hat{n} \Delta s$

The choice of $\hat{n}$ is arbitrary (it may point to the right or left in the figure above, I chose it pointing sort of along the velocity), it won't affect the results as long as you keep it consistent.

Now we can evaluate the change of momentum over the time interval:

$\Delta P = \Delta m (\vec{v}_f - \vec{v}_0)$

And the force on the surface is the –F felt by the particles, from the impulse theorem:

$$F_{surface} = -\frac{\Delta P}{\Delta t} = -\rho \vec{v}_0 \cdot \hat{n} \Delta s (\vec{v}_f - \vec{v}_0) \qquad (5\text{-}18)$$

$$F_{presure} = \frac{F}{\Delta s} = -\rho \vec{v}_0 \cdot \hat{n} (\vec{v}_f - \vec{v}_0) \qquad (5\text{-}19)$$

Two special cases are $\vec{v}_f = 0$ (completely inelastic), and $\vec{v}_f = -\vec{v}_0$ (elastic), you should be able to work out the expression of the pressure felt by the surface from (5-19). Also in the most general case where the surface may not be even a plane, then the total force felt by the surface would be a surface integral (a double integral) of equation (5-18), where $\rho, \hat{n}, \vec{v}$ may depend on the position on the surface. Fortunately we won't consider such beast in this course, and you will learn how to calculate flux of vector field in calculus and divergence theorem etc[40]. Please also refer to examples 3.17 and 3.18 to see the application.

(Important concept: mechanical momentum, center of mass and center of mass frame; conservation of total momentum; impulse and momentum change)

Chapter 6 Work and Energy

In this chapter we shall discuss two important concepts in physics: work and energy. We will first derive the work –kinetic energy from Newton's law, and thus give a definition to work and kinetic energy. For some particular forces satisfying certain requirement, the work done by these forces will have an interesting property: It is independent of the path. This path independence will define a conservative force and a conservative potential associated with the force. We shall discuss in detail the property and criteria for conservative force. The potential also defines energy: the potential energy. The sum of kinetic and potential energy is

[40] It is called Gauss theorem in electrostatics and you will learn it in electro-magnetism.

called mechanical energy of the system under study. It is conserved if only subject to conservative forces and this leads to the powerful concept of conservation of energy, once we acknowledge that the energy can take many different forms, and mechanical energy discussed in this chapter is only part of it. Finally we will discuss an important physical process, scattering between particles, as an example to apply the conservation laws we learned so far.

6.1 Work-Energy Theorem in 1-D

6.1-1 Work-Energy Theorem for a Single Particle

We shall start from this simplest case and extend it later to systems of more particles and higher dimensions. Single particle here means I choose the system containing only one mass point, treat everything else as influence from outside world. In 1-D, 2^{nd} law is:

$$F(x) = \frac{dP(t)}{dt} = m\frac{dv(t)}{dt} = m\frac{d^2x(t)}{dt^2} \quad (6\text{-}1)$$

Notice here in the above equation, I specify the force only explicitly depends on position. This is not a necessary condition here, the general force may depend on velocity and time explicitly. This is related to what we had discussed in chapter 1, though the fundamental forces are only position dependent, the forces ‘felt’ by an open system may depend on time or velocity explicitly. This dependence will have no effect on the following discussion on work-energy theorem, but will affect the definition of conservative force. For simplicity, I shall first assume the force is only position dependent and will include the discussion if the force has other dependence later.

Now I play a 'trick' on the right hand side of equation. We already see two usual forms on the right hand side to express the motion of particle, i.e.: $\frac{dv}{dt}=\frac{d^2x}{dt^2}$, now I will show you another form:

$$\frac{dv}{dt}=\frac{dv}{dt}\frac{dt}{dx}\frac{dx}{dt}=v\frac{dv}{dx} \qquad (6\text{-}2)$$

This is legal operation, if you worry about the math, replace the differential symbol $dt, dx...$ with small change $\Delta x.\Delta t...$[41] This from is also worth to remember and it may become handy in applying 2nd law sometimes, such as what follows:

$$mv\frac{dv}{dx}=F(x)$$
$$mvdv=F(x)dx$$

Integrate both sides from initial position to some final position (see the K&K for detail or refer to footnote 36 on pg 92 of this note):

[41] This is legal for total differential because we can view it as small change of numbers. It is **not** generally legal for partial derivatives, because of the constraints attached to the partial derivatives, see the supplement on partial derivatives. Here as an example: let's say $x=x(p,q), y=y(p,q); p=p(x,y), q=q(x,y)$. $\frac{\partial p}{\partial x}\frac{\partial x}{\partial p}\equiv(\frac{\partial p}{\partial x})_y(\frac{\partial x}{\partial p})_q\neq 1$. Noticed the constraint on the two partial derivatives, one requires hold y constant, the other requires hold q constant. To further illustrate this point, try the relation between Cartesian coordinate and polar coordinate: $\frac{\partial r}{\partial x}\frac{\partial x}{\partial r}=\cos^2\theta$

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F(x)dx \qquad (6\text{-}3)$$

What this relation tells us? It actually defines two things! It tells us two things are equal. We shall define **kinetic energy** as:

$$T = \frac{1}{2}mv^2 \qquad (6\text{-}4)$$

So the left hand side is change of kinetic energy: $\Delta T = T_b - T_a$. The right hand side defines **work** by the force**:**

$$W = \int_{x_a}^{x_b} Fdx \qquad (6\text{-}5)$$

So relation (6-3) can be written with these definitions as:

$$\Delta T = W \qquad (6\text{-}6)$$

It says the kinetic energy change of the system equals to the work done to the system. This work-KE relation and its form in higher dimension is the most fundamental relation in this chapter. If the work done to the system is positive (the force pushes the system), kinetic energy will increase. If the work done to the system is negative (system pushes the outside world, the system will do positive work to the outside world, easy to see this from 3rd law and definition of work), system will lose kinetic energy.

Notice that in definition (6-5), I intentionally only write F, instead of F(x), because this applies to all kinds of forces. However, if F in this 1-D case only depends on x, we will get some interesting result. Let's suppose F is only a function of x, i.e. it is defined uniquely for every x. Then the work is just a definite integral with integrand F(x):

$$W = \int_{x_a}^{x_b} F(x)dx \qquad (6\text{-}7)$$

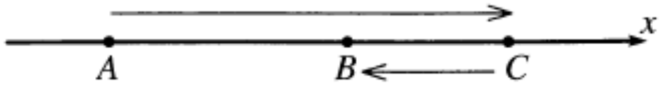
From the fundamental theorem of calculus, we know that this integral equals to the difference of the antiderivative of F(x), evaluated at the end points, i.e.

if $\dfrac{dG(x)}{dx} = F(x) \qquad (6\text{-}8)$

then:

$$W = \int_{x_a}^{x_b} F(x)dx = G(x_b) - G(x_a) \qquad (6\text{-}9)$$

G(x) is called mathematical potential associated with the force. Noticed that there is a striking fact lies under the simple relation (6-9): **path-independence**. It is determined by the starting and ending point only, do not care how the particle moves between x_a, x_b. If it moves directly from x_a to x_b, or from x_a to some other point x_c, then back to x_b, the force will do the same amount of work, ignoring the details of how particle travels between the locations.



Such force is called conservative force and we have relation (6-9) in 1-D case. We will discuss the conservative force in general situation of higher dimension later. In 1-D, the only requirement for a force to be conservative is that it only depends on position, i.e. position alone uniquely defines the value of the force.

We shall define physical potential for conservative force as:

$U(x) = -G(x)$ (6-10)

(6-8) will become:

$$F(x) = -\frac{dU}{dx} \qquad (6\text{-}10)$$

And (6-9) becomes:

$$W = \int_{x_a}^{x_b} F(x)dx = U(x_a) - U(x_b) = U_a - U_b = -\Delta U \qquad (6\text{-}11)$$

The advantage of this definition is when combined with work-energy theorem (6-6), we have:

$$\begin{aligned} \Delta T = W = -\Delta U \\ \Delta(T+U) = 0 \end{aligned} \qquad (6\text{-}12)$$

(6-12) tells us there is a physical quantity that is not change over time in the system, we define this as mechanical energy:

$$E_{mech} = T + U = \frac{1}{2}mv^2 + U \qquad (6\text{-}13)$$

I specify it as mechanical energy because there are other energy forms. There is still one ambiguity in the definition (6-12), namely the U. From (6-11) we see that only the difference between the potential is defined, or from (6-10), the function U(x) can be subject to an arbitrary constant C. This ambiguity on the value of U is removed by our choice of zero potential. i.e. we shall choose a reference point, and specify its potential equals zero: $U(x_{ref.}) = 0$ Because only the difference of potential has physical significance, so the choice of zero potential reference is made by convenience. If you choose a different zero potential reference, every potential may shift by a constant, but this will not affect (6-11) or (6-12), because the constant cancels. The usual choice of potential zero are: for a spring, the equilibrium point of the spring with no force on it; for gravity, we usually choose infinity as zero potential point; sometime we also choose sea-level (or ground level) on earth, etc. With a chosen potential zero, U(x) can be defined from (6-11):

$$\int_{x_{ref}}^{x} F(x)dx = U(x_{ref}) - U(x) = 0 - U(x) = -U(x)$$

$$U(x) = -\int_{x_{ref}}^{x} F(x)dx = \int_{x}^{x_{ref}} F(x)dx \qquad (6\text{-}14)$$

Now you have a formula to evaluate the kinetic, potential and total energy for a single particle system.

Let's take a look again to the example given by the end of Chap.4:

i.e.: 1-D free falling: an object is at a distance from the earth and falling towards the earth center. The object has mass m and is at a distance of h (h could be very far, say infinity) from the center of earth, initial velocity is 0.

This is seemingly a simple problem:

The distance is x (earth center is origin), and the velocity is v(x), and the force is:

$$F = -G\frac{M_e m}{x^2}$$

$$m\frac{dv}{dt} = -G\frac{M_e m}{x^2}$$

This is not easy to directly solve the differential equation as we stated there. Now from energy conservation (or work-kinetic energy theorem, or with the trick I told you here $\frac{dv}{dt} = \frac{dv}{dt}\frac{dt}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$), basically we can first find out the relation between velocity and position v(x), then $\frac{dx}{dt} = v(x)$ will give relation between x and time. I shall briefly workout from conservation of T+U (because the force only depends on position in this case), please work out from work-energy theorem (which also applies to forces that depends on time or velocity) yourself.

I shall choose infinity as my potential zero.

$$U(h) = \int_h^\infty -\frac{GM_e m}{x^2}dx = \frac{GM_e m}{x}\Big|_h^\infty = -\frac{GM_e m}{h} \qquad (6\text{-}15)$$
$$U(x) = -\frac{GM_e m}{x}$$

$$\frac{1}{2}mv^2 - \frac{GM_e m}{x} = -\frac{GM_e m}{h}$$
$$v = -\sqrt{2GM_e(\frac{1}{x} - \frac{1}{h})}$$

The reason I put minus sign is because I define the positive x direction is from 0 to infinity.

$$\frac{dx}{dt} = -\sqrt{2GM_e(\frac{1}{x} - \frac{1}{h})}$$

$$dt=\frac{dx}{-\sqrt{2GM_e(\frac{1}{x}-\frac{1}{h})}}$$

$$t=\int_h^x\frac{dx}{-\sqrt{2GM_e(\frac{1}{x}-\frac{1}{h})}}$$

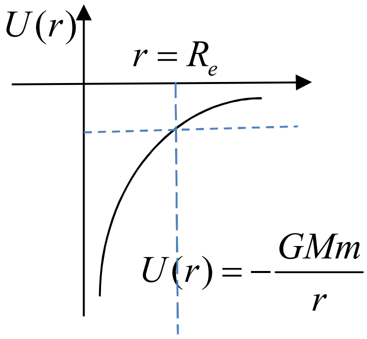
It is not a pleasant integral to evaluate analytically, but impose no problem for numerical method for computer. This is generally true in real problems in mechanics, instead of having wanted x(t), you often end with t(x) (be a analytical or just in integral form) and the inverse x(t) may not be solvable. This example is in the same spirit as example 4.2 in KK, but that one is easier to solve.

It may occur to you that the potential due to gravity is mgx close to the surface of earth, where x is the height here. It looks different than potential given in the (6-15). First the mgx is really the potential difference between the height R_e+x and that of at R_e, let's calculate this difference using (6-15):

$$\Delta U=U(R_e+x)-U(R_e)=-GM_em(\frac{1}{R_e+x}-\frac{1}{R_e})$$

$$=-m\frac{GM_e}{R_e}(\frac{1}{1+x/R_e}-1)\approx-m\frac{GM_e}{R_e}[(1-\frac{x}{R_e})-1)=m\frac{GM_e}{R_e^2}x$$

If we invoke the definition of $g=\frac{GM_e}{R_e^2}$, the above is mgx, so the two are consistent.



It can also be viewed from the figure depicting the potential given (6-15). If you choose the ground level as zero potential, it just shift the whole curve vertically up (or the axis down) so that the $U(R_e)=0$, as the blue horizontal dashed line indicates. Close to R_e, the hyperbolic potential can be approximated by a straight line, whose slope is mg.

For the forces that depend on time or velocity, the argument leads to (6-9) won't apply, that requires the force is uniquely defined from position. This is apparently untrue if forces explicitly depend on time or velocity. For such force, F(x,v,t), the work defined by (6-7) has to be computed with the technique of line integral[42], it generally depends how the particle travels along the line even for this 1-D case[43]. We will not get simple relation as (6-9) and we do not have conservation energy (mechanical) for systems under such forces, these forces are called non-conservative forces, such as friction (it depends on direction of velocity, not defined with only

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[42] The basic technique is called parameterization and is discussed in the supplement 2, section 7-1 under line integral, it is part of multi-variable calculus.

[43] We need more information to evaluate this integral. Not only the staring and ending point, but also other information, such as direction of the particle travel (for friction force) or velocity at certain point (resistance force) etc. This requires details for the path. We shall say that for these non-conservative forces, the work done is path dependent, in contrast to the path independence of work done by conservative forces.

x), air resistance etc.

Someone may argue that even for a force that depends explicitly on time or velocity, for example: $F(x,t) = -k(x - at)$, because of motion obeys 2nd law, we could solve the relation between x and t, say t=t(x) like in the above example. Then throw everything back to F, $F(x,t) = -k[x - at(x)]$, the t(x) is a function of x. It is seemingly that the work will be back in the form of (6-7) so (6-9) will follow. The problem of this approach is:

1) The reason we introduce potential is to solving the problem without solving the equation of motion directly. If you already find out relation of x(t) or t(x), that loses the part of the point of introducing potential.
2) The above method is essentially doing line integral with parametric method. You can always do this to compute the line integral. The parametric function, the t(x) above, is path dependent. i.e. it depends on how the particle's path and how it is moving along it. With different path or different initial state, the parametric function would be different, so will the result of the work computed from (6-7). The "potential" defined this way will have different function forms for different paths and this loses the whole point to introduce potential.

When there are not one but a number of forces acting on the particle, the total work done by the forces is the work done by the total force for this single particle system:

$$W_{total}=\sum_i\int_{x_a}^{x_b}F_i dx=\int_{x_a}^{x_b}(\sum_i F_i)dx \qquad (6\text{-}16)$$

This is true because the path of the work done by different forces is same for the single particle case (it is not true for the many particle system, the forces may exert on different particles and have different paths. The total work will still be sum of individual work, but not equivalent to the work by total force).

The work energy theorem still holds for this case:

$$W_{total}=\Delta T=\frac{1}{2}m(v_b^2-v_a^2) \qquad (6\text{-}17)$$

If the forces are all conservative forces:

$$W_{total}=\sum_i W_{ic}=\sum_i -\Delta U_{ic}=\sum_i -(U_{ic}(x_b)-U_{ic}(x_a)) \qquad (6\text{-}18)$$

Then the total mechanical energy defined as $T+\sum_i U_{ic}$ is conserved:

$$\Delta(T+\sum_i U_{ic})=0 \qquad (6\text{-}19)$$

If the force contains non-conservative force:

$$\begin{aligned} W_{total}&=W_{cons.}+W_{non-c}=\Delta T \\ W_{non-c}&=\Delta(T+\sum_i U_{ic}) \end{aligned} \qquad (6\text{-}20)$$

The mechanical energy change would equal to the work by the non-conservative force.

6.1-2 Work-Energy Theorem for Multi-particle System

Last section, we studied the simplest system, single particle system and use it to introduce some very important concepts and definitions: work, energy, conservative force and potential. Now take a look for multi-particle system, I shall choose two-particle system, it is sufficient and can be generalized to any number of particles (or parties).

(1) Internal Force and External Force

When you choose the system, it is natural to group the force into internal ones (the interaction among the particles inside the system) and external ones (the interaction of particles with outside world). Such grouping not only for convenience, but has following important effect as well (in single particle case all forces are external).

The internal force if any exists, is due to interaction between the particles and always appear in pair obeying 3rd law. This gives an important character of work done by the internal force. Let the force F_{12} be the force act on particle 2 by 1; F_{21} be the force on 1 by 2. We have $F_{21} = -F_{12}$, both forces are depending on the relative positions of the two particles: $F_{21}(x_1 - x_2) = -F_{12}(x_1 - x_2)$ as we discussed in chapter 1 (the symbols used mean F is a function of x_1-x_2). The work-energy theorem can be applied to each particle separately:

$$F_{21}dx_1 = dT_1 = \frac{1}{2}m_1 d(v_1^2)$$

$$F_{12}dx_2 = dT_2 = \frac{1}{2}m_2 d(v_2^2)$$

Add the two relations and apply the 3rd law, we get:

$$F_{21}(x_1 - x_2)dx_1 - F_{21}(x_1 - x_2)dx_2 = d(T_1 + T_2)$$
$$F_{21}(x_1 - x_2)d(x_1 - x_2) = d(T_1 + T_2)$$

If we introduce a new variable $x_1 - x_2$, then the left hand side is just the line integral as in the single particle case but with $x_1 - x_2$ as variable:

$$\int_{x_{1a}-x_{2a}}^{x_{1b}-x_{2b}} F_{21}(x_1 - x_2)d(x_1 - x_2) = (T_{1b} + T_{2b}) - (T_{1a} + T_{2a}) = \Delta(T_1 + T_2) \qquad (6\text{-}21)$$

The work that done by the pair of internal force on the particles thus changes the total kinetic energy of the two particles involved. What is more important, is because the left hand integral only depends on relative positions to the particles, it is same even if the coordinate system is changed. Restate it as: *The work done by a pair of internal force is independent of coordinate of choice.*

If the internal force is conservative as the one written in (6-21), then we could define a potential same as (6-14):

$$\int_{x_{1a}-x_{2a}}^{x_{1b}-x_{2b}} F_{21}(x_1 - x_2)d(x_1 - x_2) = -[U(x_{1b} - x_{2b}) - U(x_{1a} - x_{2a})] = -\Delta U \qquad (6\text{-}22)$$

Combined with (6-21):

$$\Delta(U + T_1 + T_2) = \Delta(U + T) = 0 \qquad (6\text{-}23)$$

Again the total mechanical energy is conserved if only under conservative force. The T here is the **total** kinetic energy of all particles; U is the potential energy between the particles. If there are more than one interaction that are conservative then U could be a summation of all individual potentials associated with conservative forces (for example massive charged particles have both gravitation and electric interactions and potential associated with them). Also noticed that the (6-23) is not the simple summation of individual particles' mechanical energy $(U+T_1)+(U+T_2)$ as one may naively assumed, because the U here already related to the change of the total kinetic energy, you do not count it twice. For the non-conservative internal force (such as the friction, it also appears in pair but depends on the relative velocity between the parties, more strictly it actually depends on the direction of relative velocity), we cannot define potential but (6-21) work-kinetic energy relation still applies:

$$\begin{aligned} W_{total}^{\text{int}} &= W_{cons.}^{\text{int}} + W_{non-c}^{\text{int}} = \Delta T_{total} \\ W_{non-c}^{\text{int}} &= \Delta(U + T_{total}) \end{aligned} \qquad \text{(6-24)}$$

Now let's include the external force. If the system also interacts with the outside world, the force resulting from such interactions is external force for the system. The total work done by the external force here may not be computed from the total external forces like in single particle case, we do not have (6-16), and instead the work by the external force has to be computed individually:

$$W^{ext} = \int_{x_{1a}}^{x_{1b}} F_1^{ext} dx_1 + \int_{x_{2a}}^{x_{2b}} F_2^{ext} dx_2 \qquad \text{(6-25)}$$

If the external forces also have conservative and non-conservative parts, we can define the potential due to the external forces too and include it in the mechanical energy of the system, then you do not need to consider the work by the conservative forces (already included in the potential), (6-24) would become:

$W_{non-c}^{\text{int}+ext} = \Delta(U_{\text{int}} + U_1^{ext} + U_2^{ext} + T)$ (6-26)

$U_1^{ext} + U_2^{ext}$ is the potential on particle 1 and 2 due to the external conservative forces (if any).

Very often, we also choose to compute the work by the external forces and do not care whether it is conservative or not, then:

$W_{total}^{ext} + W_{non-c}^{\text{int}} = \Delta(U_{\text{int}} + T)$ (6-27)

The reason that we have all these different variations of the fundamental work-energy theorem, is because in reality we have a choice to define our system and outside world (thus internal and external), also we have a choice to define the potential for conservative forces. If in doubt, go back to the fundamental work-energy theorem.

(2) Reexamination of Potential Energy

In the single particle case, we define potential for conservative forces, the premise is that the force is only depending explicitly on the particles position, i.e. the particle's position uniquely determines the force. And I also worked example of gravitation potential of a particle in free fall. Let's take a look of this gravitation potential now since we learned multi-particle case. You will find the discussion on the single particle case for gravity is not rigorous. Because the gravity depends on relative positions, so only knowing the position of the one party is not sufficient to determine it. The argument in the single particle case relies on the fact that I choose earth as my origin, so that its position is fixed, then the position of particle can determine the force.

Now we can take a look of this from two-particle point of view: a system of both earth and the object. From (6-21) and (6-22) we see that the change of potential should strictly speaking involves change of kinetic energy of both earth and the object:

$-\Delta U = \Delta(T_{earth} + T_o)$ (6-28)

Only if we choose our inertial frame as *earth is stationary or the earth moves with constant velocity* (both are approximately to inertial), then $\Delta T_{earth} = 0$, and the potential change can be treated as only affect the kinetic energy of the object as in the examples of single particle case. If the earth is not a good inertial frame or we choose another frame in which the earth is not moving at constant velocity, we have to include the change of kinetic energy of the earth. This is because even though the change of velocity may be very small (negligible), the change of kinetic energy may not be small in percentage in (6-28) and has to be taken into consideration.

Let's suppose the earth+object system, the original velocity of earth is V_0, the object is v_0; in the final stage, the velocity changes to V, v respectively for earth and object. So the kinetic energy changes are:

$$\Delta T_e = \frac{1}{2} M_e (V^2 - V_0{}^2)$$

$$\Delta T_o = \frac{1}{2} m (v^2 - v_o{}^2)$$

Let's assume only internal force between object and earth (or the external forces are negligible during the process), then we have conservation of momentum:

$$M_e V_0 + m v_0 = M_e V + m v \rightarrow -M_e (V - V_0) = m(v - v_0)$$

$$\frac{\Delta T_e}{\Delta T_o} = \frac{M_e (V^2 - V_0^2)}{m(v^2 - v_0^2)} = -\frac{V + V_0}{v + v_0} \approx -\frac{2V_0}{v + v_0}$$

The approximation of small change on earth velocity V is used above. We see that this ratio is not necessarily small (it is 0 if $V_0 = 0$). So care may be needed when we deal with potentials arising between the interaction parties, the choice of coordinate frame will decide what kind of equations to use (whether need to include the kinetic energy of both or just one of them). The example to illustrate this would be the calculation of third escape velocity (the velocity of the rocket launched from earth that goes beyond the gravitation field of our solar system, the sun). The detailed calculation can be found in many textbooks[44] and won't be given here. I only discuss the common mistake that one easily made (including me at first time):

The potential energy of the rocket has two parts: one due to earth and the other due to sun:

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[44] For example: 李复 '力学教程'(上), p201; 郑永令 等 '力学' p177.

On the surface of earth, the rocket's potential would be:

$$U = -GM_e m\frac{1}{R_e} - GM_s m\frac{1}{R_{e-s}}$$

By the time the rocket escape out of solar system, its potential would be zero. This increase of potential would be achieved from decrease of kinetic energy of the rocket, so the rocket final velocity would be zero at infinity. Then:

$$\Delta U + \Delta T = 0 \to \frac{1}{2}mv_0^2 = GM_e m\frac{1}{R_e} + GM_s m\frac{1}{R_{e-s}}$$

And you will get incorrect answer, but why? Following our discussion I hope you see what is wrong here (before you look my reasoning below). The problem is either the sun or the earth will move, and the its kinetic energy change cannot be neglected. The general choice is choosing an inertial frame where the sun is stationary (which is a better inertial frame than the earth), then the earth will move around the sun with velocity almost 30km/s. In the above calculation the potential change on the earth-object part would have to include the kinetic energy change of the earth during the process. Actually it is probably safest to start from a system including the sun+earth+rocket and list out potential energy and kinetic energy ($E = \frac{1}{2}mv^2 + \frac{1}{2}M_e V^2 - \frac{GM_e m}{r_{e-o}} - \frac{GM_s m}{r_{s-o}} - \frac{GM_s M_e}{R_{s-e}}$) and using the conservation of total mechanical energy and momentum (also be careful there, because the earth-object are subject to force from sun so to apply the conservation of momentum between earth and object may require some thoughts) to solve the problem, instead of treating the rocket only as the mistake above. Please find out the solution yourself or read the reference given above.

(3) Energy and Choice of System and Coordinate System

From the above discussion we have seen that the fundamental theorem on mechanical energy is work-energy theorem. In the application of this theorem, it usually requires we define our system of interest and a choice coordinate system (inertial frame). We now take a detailed look on this.

First let's see that if the work-energy theorem is correct in one inertial frame, it applies to all inertial frames. This is expected because work-energy theorem is derived from 2nd law, which applies to all inertial frames (of course neglecting relativity here). But I shall work it out anyway.

Suppose we have two inertial frame, x and x' (still in 1-D), the x is moving with constant velocity v_0 with respect to the x' system. So:

$$x' = x + v_0 t, t = t' (Galileo\ Transformation)$$

$$v' = v + v_0$$

In the x system, we have the work-energy theorem as in (6-3), in the x':

$$\int_{x'_a}^{x'_b} F dx' = \frac{1}{2} m v'^2_b - \frac{1}{2} m v'^2_a \qquad (6\text{-}29)$$

I will show that (6-29) is equivalent to (6-3):

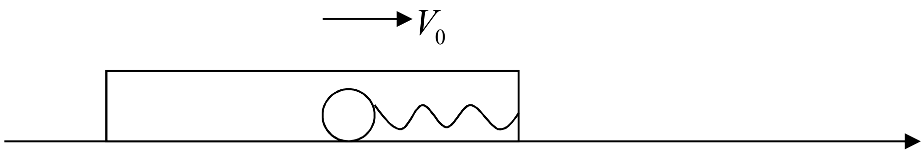
$$\frac{1}{2}mv_b'^2 - \frac{1}{2}mv_a'^2 = \frac{1}{2}m[(v_b + v_0)^2 - (v_a + v_0)^2$$
$$= \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 + mv_0(v_b - v_a)$$

The kinetic energy change has an extra term if expressed in velocity in the x. This is because the displacement in x' is different from that in x. The left hand side of (6-29) can be evaluated as:

$$\int_{x_a'}^{x_b'} F dx' = \int_{t_a}^{t_b} F v' dt = \int_{t_a}^{t_b} F(v + v_0) dt = \int_{t_a}^{t_b} F v dt + v_0 \int_{t_a}^{t_b} F dt$$

The first part on the right hand side will give the kinetic energy change in the x coordinates (the first part of kinetic energy change in the x'), the second part using impulse-momentum relation, will give the second half of kinetic energy change in the x'. So work-energy theorem works in both inertial frames, even though the form of kinetic energy and work will be different in the two coordinates.

This brings a subtle point in computing work and kinetic energy. Their forms depend on coordinate system of choice though W-E theorem always holds. We have seen that for a pair of mutual internal force, the work done by it is same for all coordinate systems. But the work by the external forces may depend on coordinate systems, so will the kinetic energy. These may best be illustrated by working an example. This will bring out the importance of our choice of system of interest and coordinate system:



A cart is moving at constant velocity V_0; The mass of cart is M>>m the mass of the ball; The spring is massless and its spring constant is k, and its equilibrium point under no stress is at x_0 initially. At t=0, the ball with initial velocity v_0 (w.r.t cart) hits the relaxed spring, and final time is when the ball stops from the point of view for a local observer travelling along the cart. Neglect friction forces. Now please analysis the work-energy relation from two points of view: the observer on cart (the x coordinate system) and a ground observer (the x' coordinate system).

(a) Only include the ball as system of interest:

For the cart observer, the kinetic energy change is:

$$\Delta T = -\frac{1}{2}mv_0^2$$

$$\Delta U = \frac{1}{2}k(x_b - x_0)^2$$

In the cart frame, the x_0 is a constant, the force on the ball is determined by the ball's position x and is conservative and the potential is given above. The mechanical energy of the ball is conserved, i.e.:

$$\frac{1}{2}k(x_b - x_0)^2 = \frac{1}{2}mv_0^2$$

The x_b can be computed from the relation.

For the ground observer:

The change of kinetic energy is:

$$\Delta T' = \frac{1}{2}mV_0^2 - \frac{1}{2}m(V_0 + v_0)^2 = -\frac{1}{2}mv_0^2 - mV_0 v_0$$

The work done by the force is:

$$\int_{x_a'}^{x_b'} -k(x' - x_0')dx' = \int_{x_a'}^{x_b'} -k(x' - V_0 t)dx'$$

This work cannot be reduced to a potential energy for the explicit time dependence of force, and you do not have a well defined potential for the ground observer. The computation has to carry out through work-energy theorem and is much harder.

(b) Include the ball and spring as system

The difficulty for the ground observer above is because the work done on the ball by the external force is depending on the coordinate. By including the ball and spring in the system, the work by the elastic force of the spring becomes internal force and its work on both ball and spring would be same for the cart and ground observer. The work done by the spring for both observers then would be:

$$W_{elastic} = -\Delta U = -\frac{1}{2}k(x_b - x_0)^2$$

This work (or potential change) would be equal to the kinetic energy change of both spring and ball, but since the spring is massless, then the total kinetic energy only consists that of the ball. So for the cart observer, same as before:

$$W_{elastic} = -\Delta U = -\frac{1}{2}k(x_b - x_0)^2 = -\frac{1}{2}mv_0^2$$

The mechanical energy is conserved.

For the ground observer: The kinetic energy change is still:

$$\Delta T' = \frac{1}{2}mV_0^2 - \frac{1}{2}m(V_0 + v_0)^2 = -\frac{1}{2}mv_0^2 - mV_0v_0$$

The work done by the elastic force between the spring and ball is:

$$W_{elastic} = -\Delta U = -\frac{1}{2}k(x_b - x_0)^2$$

$$W_{elastic} = -\frac{1}{2}mv_0^2 \neq \Delta T'$$

$$\Delta T' + \Delta U = -mV_0v_0 \neq 0$$

The work-energy relation is not correct and the mechanical energy is not conserved from the ground observer's point of view. Something is missing here. The thing missing here is there is actually an extra force, an external force to the spring-ball system, This force is from the cart wall to the spring. The force does not do any work for the cart observer because there is no displacement of this force here in the cart frame; but will do work from point of view of the ground observer. If you include the work done by this force, then the work-energy theorem would be correct. The work by this force is exactly:

$$W' = \int_{t_a}^{t_b} FV_0 dt = V_0 \int_{t_a}^{t_b} F dt = V_0 \Delta p = -mV_0v_0$$

The problem is still harder for the ground observer to solve.

(c) Include the cart, spring and ball in the system

For the cart observer, the situation is almost same as before. The internal force total work would still be $W_{elastic} = -\Delta U = -\frac{1}{2}k(x_b - x_0)^2$ (actually two pair of forces, the other pair between spring and cart does not do any work). This work would include all kinetic energy changes, but for him the cart is stationary and won't contribute to the kinetic energy. So the relation is same as before.

For the ground observer:

Since the forces for this system are all internal forces and the work would same as: $W_{elastic} = -\Delta U = -\frac{1}{2}k(x_b - x_0)^2$, and the kinetic energy change have to include that of cart:

$$\Delta T' = [\frac{1}{2}mV_0^2 - \frac{1}{2}m(V_0 + v_0)^2] + \frac{1}{2}M(V^2 - V_0^2) \approx -\frac{1}{2}mv_0^2 - mV_0v_0 + MV_0(V - V_0)$$

$$MV_0 + m(v_0 + V_0) = MV + mV \quad V \approx V_0$$

Then you see that the last two terms in the kinetic energy part just cancels, and:

$$\Delta T' = -\frac{1}{2}mv_0^2$$

And $-\Delta U = \Delta T'$

The mechanical energy is conserved for the ground observer too in this system. Some of you may worry about the approximation signs above, in one coordinate the mechanical energy is conserved, while in the other I have to made approximation to show that. Actually, in this example, the cart is not 100% inertial frame, it is a pretty good one since M>>m, but just not perfect. So the energy conservation derived for the cart observer is indeed an approximation; while the ground inertial observer has the exact formula from work-energy theorem, include all works and all kinetic energy and conservation of momentum. This exact computation may be slightly off from the results obtained by the cart observer, so approximation had to be made from exact to approximate value.

In summary the above example showed that: The work-energy theorem always work for all system in all coordinate. But the computation of kinetic energy and work can be quite different for different choice of system of interest and in different coordinate, so choose wisely.

(4) Work-Energy in Center of Mass Frame

For the system of many particles, we have seen that sometimes it is convenient to work under the center of mass frame, i.e. choose the coordinate travel with C.M, with the C.M as origin, a zero total momentum frame. We shall see what the work-energy theorem in such frame is.

First the work by the external force changes the kinetic energy of C.M. The C.M. is defined as a fictitious point that with the total mass of the system and with the total external force acting on it (5-5):

$$F_{total}^{ext} = M\frac{dV_{CM}}{dt}$$

Following the same argument leading to (6-3), we have:

$$W_{total}^{ext} = \sum_i \int F_i^{ext} dx_i = \sum_i \int F_i^{ext}(dX_{CM} + dx_{ic}) = \int (\sum_i F_i^{ext}) dX_{CM} + \sum_i \int F_i^{ext} dx_{ic}$$
$$= \int F_{tot}^{ext} dX_{CM} + \sum_i \int F_i^{ext} dx_{ic} = W_{CM}^{ext} + W_c^{ext}$$

$$W_{CM}^{ext} == \int F_{tot}^{ext} dX_{CM} = \Delta T_{CM} \qquad (6\text{-}30)$$

This is only the first half of story. There are relative motions among particles besides the translation of C.M. The total kinetic energy can be proved to have two components, one is the kinetic energy of the C.M.; the other is the kinetic energy of relative motions of the particles (This is also called Konig theorem):

The relation between the velocity in an inertial frame and that observed in a C.M frame is related by (derivable form (5-7)):

$$v = V_{CM} + v' \qquad (6\text{-}31)$$

v is the velocity in one inertial frame, v' is the velocity observed in the C.M. frame.

$$T_{total} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (v_i' + V_{CM})^2 = \frac{1}{2}\sum_i (m_i V_{CM}^2 + 2 m_i V_{CM} v_i' + m_i v_i'^2)$$
$$= \frac{1}{2} V_{CM}^2 \sum_i m_i + V_{CM} \sum_i m_i v_i' + \frac{1}{2}\sum_i m_i v_i'^2 = \frac{1}{2} M V_{CM}^2 + \frac{1}{2}\sum_i m_i v_i'^2 \qquad (6\text{-}32)$$
$$= T_{CM} + T'_{total}$$

From the work-energy theorem in the inertial frame, we have:

$$W_{total} = \Delta T_{total}$$
$$W_{total}^{ext} + W_{total}^{\text{int}} = \Delta T_{CM} + \Delta T'_{total}$$

From (6-30), we have right away:

$$W_c^{ext} + W_{total}^{\text{int}} = \Delta T'_{total} \quad (6\text{-}33)$$

The kinetic energy change in the C.M. frame is due to the work by the internal forces and the external forces in the CM frame.

But there is a catch in the above argument, is the C.M. frame inertial? It is under external force, it will have non-zero acceleration. Then how can we apply work-energy theorem at all in this non-inertial frame under external force (recall that work-energy is derived from 2nd law which requires inertial frame)? You will "feel" an inertial force (also called fictitious force) due to the acceleration of the frame. It turns out (easy to prove once we learned non-inertial frame) that for this fictitious force, the total work is zero. That's why we have (6-33).

6.2 Work-Energy in Higher Dimension

We have discussed thoroughly the work-energy theorem in the simple 1-D case, and introduced definition of work, kinetic energy, conservative force and potential energy. All these would apply to the higher dimension, the math and the form of equations would be a little different due to the vector nature of force, displacement and velocity, etc. However, the physics remains the same.

6.2-1 Work-Energy for Single Particle

The kinetic energy in higher dimension is defined as:

$$T = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2} m \vec{v} \cdot \vec{v} \qquad (6\text{-}34)$$

The change of it over time is:

$$\frac{dT}{dt} = \frac{1}{2} m \frac{d(\vec{v} \cdot \vec{v})}{dt} = m \frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v} \qquad (6\text{-}35)$$

$\vec{F} \cdot \vec{v}$ is also called **power** of the force, i.e.:

$$P = \vec{F} \cdot \vec{v} \qquad (6\text{-}36)$$

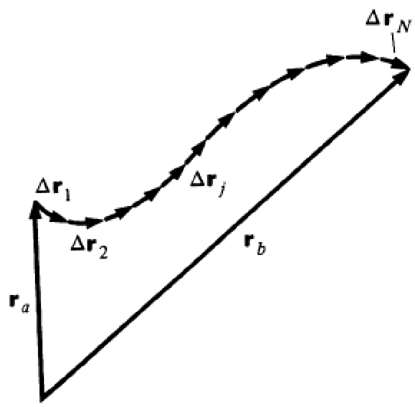
It is the work done by the force in unit time (related to the kinetic energy change of the system in 6-35).

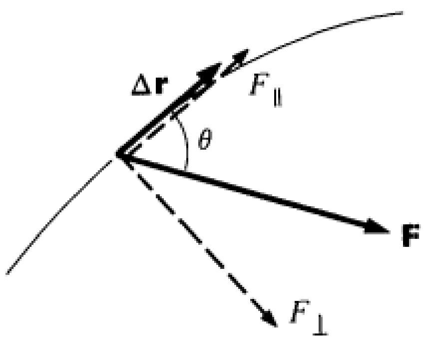
$$dT = \vec{F} \cdot \vec{v} dt = F \cdot d\vec{r}$$

$d\vec{r}$ is the infinitesimal displacement vector along the trajectory (the path)

$$\Delta T = T_b - T_a = \int_{C, r_a}^{r_b} \vec{F} \cdot d\vec{r} \qquad (6\text{-}37)$$

The symbol $\int_{C, r_a}^{r_b}$ is a line integral, along a path C (curve) connecting the initial and final position a and b, as shown in the figure below.





For the line integral as in 6-36, we can view it as cut the curve into small line segment Δr_i, take the dot product of force with small displacement vector and take the summation. $d\vec{r}$ is a vector along the direction of tangent line at certain point on the curve (also along the direction of velocity), with magnitude of the arc length ds, i.e:

$$d\vec{r} = \hat{T}ds \qquad (6\text{-}38)$$

Then the work will be:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T}ds \qquad (6\text{-}39)$$

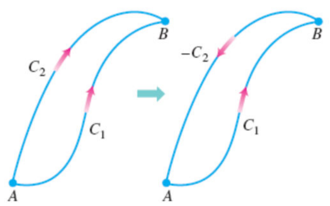
We see that it's the force component that parallel with the tangent direction of the trajectory (parallel with the velocity, also clear from power from 6-36) that contribute to the work and change of kinetic energy. This is expected from the fact that force perpendicular to the motion only changes the direction of velocity but will not affect its magnitude (centripetal force in circular motion and Lorentz force in E-M).

To compute the line integral as in (6-37) is a little more complicated than the 1-D case. The strategy is to reduce it to some sort of definite integral, and the general method is the parameterization of the curve (curve is essentially 1-D in geometry, that can be specified by single variable, this variable is called parameter of the curve). A simple discussion is in KK section 4.6; a little more detailed discussion on how to compute the line integral is also given in math supplementary 2, section 7.1 and 7.2[45]. KK example 4.4, 4.5, 4.6 also gives evaluation of work in some simple cases.

6.2-2 Path Independence, Conservative Force and Potential Energy

What we discussed here is just an extension of 1-D case. Below is a brief summary, the details is in section 7 of the supplementary 2. The line integral in (6-37) is generally depending in the specific path connecting the initial and final position. But for a special group of forces, the work is path independent.

(1) Path Independence of Work



In the figure, for the arbitrary paths connecting A,B, the work along any path would be same, this is path independence. A corollary is that for any arbitrary close curve (also called a loop), the work done by the force along the loop in one cycle is zero. The two statements are equivalent, both can be used as definition on path independence of line integral.

Path Independence expressed in formula:

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[45] Section 7 in supplementary 2 is where I discussed about line integral, path independence, conservative force and potential, and Green Theorem. These stuffs are closely related (the math background as well as physical modeling) to the work-energy in this chapter, so please read the whole section in the supplementary. In the notes here, if there are materials overlapping with the supplementary, I shall just say please refer to…

$$\int_{C_1,A}^{B} \vec{F}\cdot d\vec{r} = \int_{C_2,A}^{B} \vec{F}\cdot d\vec{r} \qquad (6\text{-}40)$$

C_1,C_2 are two arbitrary curve connecting A,B. Or:

$$\oint_C \vec{F}\cdot d\vec{r} = 0 \qquad (6\text{-}41)$$

C is an arbitrary loop, starting from and ending to A.

(2) Conservative Force and Its Criteria

Conservative force: If the work done by a force is path independent, the force is conservative. (We have already discussed this in 1-D, but path independence is more dramatic in higher dimension)

KK Example 4.7.4.8, 4.9 give you examples of path independence of work done by force. Please also refer to supplementary 2, section 7.3.

Our next question is what kind of force is conservative in higher dimension, we see that in 1-D, the requirement is that the force only explicitly depends on position. In 2 and 3-D, the requirement is a little more than mere dependence on position.

Still 1) the force should be only depends on position explicitly. If the force explicitly depends on time or velocity, then the force cannot be conservative, i.e. the work will be path dependent (this is discussed in the comment by the end of section 7.6 in supplementary two). 2) The vector force is a *gradient* vector, i.e. the force can be written as the gradient of a scalar function:

$$\vec{F}\left(x,y,z\right) = \nabla f(x,y,z) \qquad (6\text{-}42)$$

∇ is a differential *operator* defined as:

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad (6\text{-}43)$$

Please see the section 5 in supplementary 2 for details on gradient.

So if the force vector is expressed in its component form in Cartesian:

$$\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$$

Then the conservative force would have:

$$M(x,y,z) = \frac{\partial f}{\partial x}, N(x,y,z) = \frac{\partial f}{\partial y}, P(x,y,z) = \frac{\partial f}{\partial z} \qquad (6\text{-}44)$$

(6-44) is still not convenient in seeing whether a force is conservative or not. Given a force, i.e. knowing its component M, N, P, we can use property of the 2nd order partial derivative to test the conservative:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial N}{\partial x}, \text{ or } M_x = N_y$$

$$\begin{aligned} &\frac{\partial M}{\partial z} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial P}{\partial x} \text{ or } M_z = P_x \\ &\frac{\partial N}{\partial z} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial P}{\partial y} \text{ or } N_z = P_y \end{aligned} \qquad (6\text{-}45)$$

This test can be put in another compact form by introducing *curl* of a vector. The curl of F is (also written as *Curl(F)*) defined as:

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = (P_y - N_z)\hat{i} + (M_z - P_x)\hat{j} + (N_x - M_y)\hat{k} \qquad (6\text{-}46)$$

We see right away (6-45) means:

$$\nabla \times \vec{F} = 0 \qquad (6\text{-}47)$$

This says, if the force is conservative, its curl is zero everywhere. This is also consistent with (6-42), that the force is a gradient vector, because we have relation:

$$\nabla \times \nabla f = 0 \qquad (6\text{-}48)$$

(Throw in the definition of gradient and curl and do the proof yourself)

In summary, I have told you that for a position dependent-only force, if it is a gradient vector (6-42) or equivalently its curl is zero everywhere (6-47), the force is conservative.

This gives us a tool to use (6-45) or (6-47) to test the conservative. But still one question remains: If the force satisfies (6-42) or (6-47), I told you the force is conservative, that means the work done by this force need to be path independent by definition of conservative force. I have to show you that indeed this is the case, i.e. (6-42) and (6-47) indeed lead to path independence of work.

The proof lies in the fundamental theorem of gradient and Green theorem (in 2-D; Stokes theorem in 3-D; see 7.4, 7.5 and 7.6 in supplementary 2)

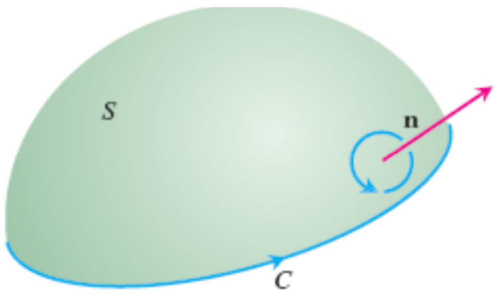
The fundamental theorem of gradient tells us the line integral of a gradient vector along a curve connecting two positions in forms of (6-37) is path independent, only determined by the function difference between the starting and ending point:

$$\int_{any\ \mathrm{C},r_a}^{r_b} \nabla f \cdot d\vec{r} = f(r_b) - f(r_a) = f(x_b, y_b, z_b) - f(x_a, y_a, z_a) \qquad (6\text{-}49)$$

Green (or Stokes in 3-D) theorem offers a method to evaluate the line integral along a loop with surface integral (the surface is *any* bounded surface enclosed by the loop)[46]:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}ds \qquad (6\text{-}50)$$

The left hand side is the line integral (work) along a loop C, the right hand side is the 'flux' of the curl of vector field ($\nabla\times\vec{F}$) through the surface bounded by C, see the figure below.



Then if we have (6-47), $\nabla\times\vec{F}=0$ everywhere, from (6-50) we have:

$\oint_C \vec{F}\cdot d\vec{r}=0$ for any close curve. This is equivalent to path independence.

So both (6-42) or (6-47) ensures that the force is conservative.

Now we can make a summary for conservative force, the following statements are equivalent for the conservative force (Don't forget that the premise is that the force only explicitly depends on position):

1) The work done is path independent

2) The work done along an arbitrary loop is zero

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[46] There are some subtleties in Green and Stokes theorem, the vector field has to be defined and differentiable not only along the closed curve (as required by the line integral), but also defined and differentiable in the surface enclosed by the loop. The loop and the region enclosed need to be simply connected. The orientation of the surface element and the direction of line integral need to be defined consistently. Please refer to Calculus textbook for detail.

3) The force is a gradient of a scalar function

4) The curl of the force is zero.

1 And 2 are definitions and corollary of conservative force, they are equivalent. The previous discussion show that 3 will lead to 1 and 4 will lead to 2. 3 and 4 are related by (6-48).

(3) Potential Functions Associated with Conservative Force

For a conservative force we have $\vec{F}\left(x,y,z\right)=\nabla f(x,y,z)$ (6-42), the force is the gradient of a scalar function f. This scalar function is called potential function (associated with the force). The work done by the conservative force along any curve can be evaluated by the potential difference at the end points (6-49). The potential f defined here is called mathematical potential, because physical potential U has a minus sign:

$U \equiv -f$ (6-51)

And corresponding (6-42) and (6-49) becomes:

$\vec{F} = -\nabla U$ (6-52)

$$\int_{r_a}^{r_b} \vec{F}\cdot d\vec{r} = -[U(r_b)-U(r_a)] = U(r_a)-U(r_b) = U_a - U_b \qquad (6\text{-}53)$$

These two relations give us tools to compute the force knowing the potential (using 6-52); or vice versa. There are generally two ways from force to potential, either from (6-52) or (6-53), the example and methods are illustrated in supplementary, 7.5, as well as in KK's example 4.11, 4.12.

Here I shall first discuss the potential for constant force (a variation of example 4-10 in K&K)

Example 1. Potential for a constant force

$$\int_{C,r_a}^{r_b} F\cdot d\vec{r} = F\cdot\int_{C,r_a}^{r_b} d\vec{r} = F\cdot(\vec{r}_b-\vec{r}_a) = U_a - U_b$$

If we define U=0 at origin (r=0), then

$$U(x,y,z) = -\vec{F}\cdot\vec{r} = -Mx - Ny - Pz$$

M,N,P are just components of this force. This constant force is certainly conservative. $\nabla\times\text{Costant}=0$, or from $\vec{F} = -\nabla U = M\hat{i} + N\hat{j} + P\hat{k}$

Example 2. I want to have a discussion on central force, i.e. the force only depend on the radius from the origin and not on direction. KK's example 4.8 gives one proof that for this kind of force, the work only depends on initial and final position vector. Here I want to use what we learned above to prove the same thing: The force is conservative.

Method 1: Geometric.

From the definition of work is the dot product and displacement vector, we can prove the work is path independent for central force. Please do this yourself (the trick is draw radius and circular arc, and the work done by the force along the arc would be 0 because radial force is always perpendicular to the arc. Try to divide and approximate any curve with this radial+arc segment)

Method 2: Test using (6-45)

To make thing simple, I would assume the force is 2-D here. The test for the conservative force is just $M_y=N_x$ from (6-45) in 2-D, M,N are the force component in Cartesian. I know the force in polar form, only depend on r, I have to transform it into Cartesian to use (6-45)

$$\vec{F}(r) = F(|r|)\hat{r}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$$

$$\vec{F}(r) = F(\sqrt{x^2 + y^2})\frac{x}{\sqrt{x^2 + y^2}}\hat{i} + F(\sqrt{x^2 + y^2})\frac{y}{\sqrt{x^2 + y^2}}\hat{j}$$

$$M_y = \frac{\partial}{\partial y}[F(\sqrt{x^2 + y^2})\frac{x}{\sqrt{x^2 + y^2}}] = x\frac{\partial}{\partial y}(\frac{F(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}})$$

The partial derivative can be evaluated with chain rule directly, but a substitution may be easier:

$$\sqrt{x^2 + y^2} = r$$

$$M_y = x\frac{\partial}{\partial y}[\frac{F(r)}{r}] = x\frac{d}{dr}[\frac{F(r)}{r}]\frac{\partial r}{\partial y} = x[\frac{1}{r}\frac{dF}{dr} - \frac{F}{r^2}]\frac{y}{\sqrt{x^2 + y^2}}$$

A similar calculation will have:

$$N_x = y\frac{\partial}{\partial x}[\frac{F(r)}{r}] = y\frac{d}{dr}[\frac{F(r)}{r}]\frac{\partial r}{\partial x} = y[\frac{1}{r}\frac{dF}{dr} - \frac{F}{r^2}]\frac{x}{\sqrt{x^2 + y^2}}$$

Indeed $M_y=N_x$. This appears complicated because we are using Cartesian, which is not the best choice here.

Method 3: Calculate curl of F in polar coordinates

$$\vec{F} = P(r,\theta)\hat{r} + Q(r,\theta)\hat{\theta} \qquad \text{(6-54)}$$

The curl of F in Cartesian in 3-D is defined in 6-46, and in 2-D here we only have the k component. To transform the 6-46 in polar coordinate is not easy (it can be done by finding relation between M,N and P,Q, and express M_y, N_x in forms of partial derivatives P.Q with respect to r,θ, it is quite messy, see the supplementary for the expression of gradient in polar for example, what follows can also be derived from relation (100) there), so I will just give you the results of curl in polar coordinate (I also cheated here too, instead of doing the transformation as outlined above, I just copy the formula for $\nabla\times\vec{F}$ in cylindrical coordinate [47] and only keep its z component in our 2-D case)

The curl of vector is 2-D polar coordinate is:

$$Curl(\vec{F}) = \frac{1}{r}[\frac{\partial}{\partial r}(rQ) - \frac{\partial}{\partial\theta}(P)] \qquad (6\text{-}55)$$

P, Q are defined in 6-54, the radial and angular component of the vector in polar. In our example of central force, the radial component is only a function of radius r, independent of θ; the angular component is zero. i.e. $P = F(r); Q = 0$. So the curl(F) is zero and the force is conservative.

The discussion above involves quite a lot of math. The math consists of a heavy portion in the course of multi-variable calculus (the supplementary 2 can be seen as multi-variable calculus in a nutshell). I hope the supplementary and the brief discussion here will give you a clear

[47] The formula can be found in many math textbooks on vector analysis. Or in physics books such as Greiner's Chap.11. It is always necessary for you to derive these formula once in your lifetime. You do not memorize them but at least you will understand how these beasts come from. So please read my example in supplementary by the end of section 7.6 and try to work out the formula of curl in cylindrical and spherical coordinate yourself. (This is not required for this course)

guideline, though not rigorous proof and extensive examples. So it may be heavy in math but far from a big mess.

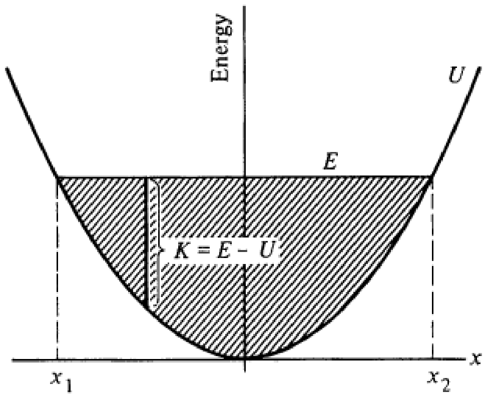
The physics on the work-energy theorem in higher dimension is actually quite similar to that in 1-D. For the conservative force, the work equals to potential energy differences (6-53), and applying the work-energy theorem (6-37), we have (for conservative force):

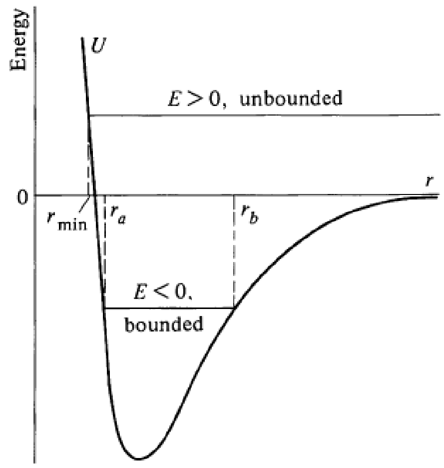
$$W = \int_{r_a}^{r_b} F \cdot d\vec{r} = -(U_b - U_a) = -\Delta U = \Delta T \qquad (6\text{-}56)$$

We will have $\Delta T + \Delta U = \Delta(T+U) = 0$ like in 1-D. We will define that the total mechanical energy as: E=T+U too. In fact all the formula (such as including non-conservative force, relation (6-20); internal-external force in multi-particle system, etc.) in 1-D case equally applied well here, just replace Fdx with $\vec{F} \cdot d\vec{r}$.

6.3 Energy Diagram and Harmonic Approximation

As we have discussed that all fundamental forces (gravity, electrostatic, nuclear) only depend on the relative position of particles and thus are conservative forces (like radial force only depends on distance but not direction). For the conservative forces, we can define a potential associated with them. The energy diagram is a plot of the potential with respect to the relative position between the particles.



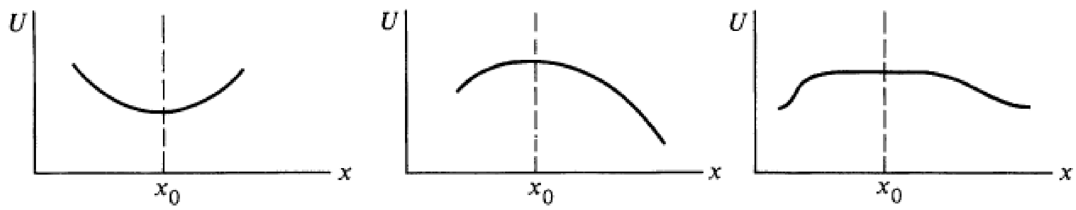


The figure on the left above shows a typical potential energy diagram (a ideal parabolic for harmonic potential, i.e. $U = Ax^2$). For the particle only subject to this potential (under the conservative force $\vec{F} = -\frac{dU}{dx} = -2Ax$ for this 1-D model), the mechanical energy E=T+U (or K+U as in the figure, K for kinetic energy) is conserved, and is a constant (indicated by a horizontal line in the figure). The kinetic energy can be estimated given the location (indicated by the vertical distance from the E line to the potential).

The figure on the right shows a more realistic Leonard-Jones Potential for two atoms to form a diatomic molecule. The potential has zero reference point at infinity (two atoms are separated far away). If the total energy is<0, we have a bound state, the atoms forming the molecule are trapped inside the potential and doing oscillation. If the energy is >0, the molecule becomes unstable, it will dissociate into atoms. This diagram also shows how two atoms form a molecule. The two atoms approach each other (r becomes smaller) and make collision, then form molecule. However, if only two atoms exist, no molecule can be formed because of conservation of energy. When the two atoms approach each other from far away (U=0), they have positive kinetic energy, so the total E>0. They cannot form a stable molecule. It requires a third party (another atom, or catalyst surface etc.) to carry away the excess energy and make the total mechanical energy of two atoms <0 after collision.

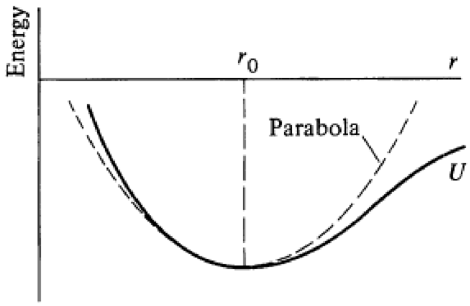
So knowing the potential diagram is important for many analyses. The potential diagram also shows us the force 'felt' by the particle. It is just to the reversed direction of the slope of the potential curve or to the reverse of the gradient of U in higher dimension.

So for a concave upward potential, the forces are trapping force, pointing to the local minimum point of potential (which is called equilibrium point, because force =0 there). The minimum is a stable equilibrium due to the fact of trapping force. For a concave downward potential, the force are repulsive, pointing away from the maximum point of potential, the maximum point at which the force is zero, is called unstable equilibrium.



$\frac{d^2U}{dx^2} > 0$ stable | $\frac{d^2U}{dx^2} < 0$ unstable | $\frac{d^2U}{dx^2} = 0$ neutral

For any trapping potential(the one with local minimum, concave up), very close to the local minimum, it can be approximated by a harmonic potential $U=Ax^2$.



The geometry is clear from figure above, the proof lies in Taylor Expansion of the potential around the equilibrium point:

$$U(x) = U(x_0) + \frac{dU}{dx}|_{x_0} (x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2}|_{x_0} (x - x_0)^2 + ... \qquad (6\text{-}57)$$

If x is close to the x_0, we can neglect the higher order terms. The first term $U(x_0)$ can be set as zero, it is the lowest potential. The second term is zero because the first order derivative is zero at minimum, the second order term will give us a parabolic potential---the harmonic potential, and the particle's motion will be harmonic oscillation around the equilibrium point. This is called harmonic approximation, and is the fundamental for phenomena such as molecular vibration.

6.4 General Law of Conservation of Energy

We have only discussed one special type of energy, the mechanical energy. Actually even for mechanical energy, it consists of two forms of energy, the kinetic and potential. The mechanical energy is conserved (unchanged with time) if our system only subjected to conservative force. Under the force of non-conservative type, the mechanical energy will change according to the work-energy theorem.

The question is what the mechanical energy changes into? It turns out it changes into other forms of energy, such as heat, electricity or light etc. It is Joule who first measured and determined the energy transfer between mechanical and heat in 1840's. He basically used wheel-paddle apparatus, doing mechanical work to a tank of water[48]. The work done by the wheel can be calculated and the temperature rise of water can be measured. What Joule proved is so called Work-Heat Equivalence, he showed that:

1 calorie=4.18 Newton.meter (which is define as joule in his honor).

Joule also did the experiment showing the electric can generate heat, the heating rate is proportional to the square of the current. This implies that all heat and electricity can be treated like mechanical energy as other forms of energy. This is formally postulated as general laws of conservation of energy, that total energy (general form) is conserved, it just changes from one type to another.

This general energy conservation is among the most robust physical principles as the conservation of momentum (both linear and angular). It is applied everywhere in daily life, such as hydro power plant. The fall of

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[48] The details and original paper is in Shamos ed. "Great Experiment in Physics", pg 169

water from high ground, the potential energy changes into kinetic, and this drives the turbine generator to make electricity (kinetic to electrical), and this electricity heats up the wire in a light bulb (electricity to heat) and light bulb emits light (heat to light). The conservation of energy also played important role in the discovery of neutrino, a mysterious fundamental neutral particle proposed from β decay. The protons in the nuclei may decay into a neutron and an electron which was discovered in 1930's. $p \leftrightarrow n + e$. The puzzling part is that since the proton and neutrons energy only can take certain discrete values(a quantization effect in quantum mechanics), so from conservation of energy, it was expected that the electron's energy had to be also in discrete values. But the measurement of the electron's energy is continuous. At the time Niles Bohr proposed that this may demonstrate that the energy is not conserved in quantum world. But Wolfgang Pauli put the money in the basket of energy conservation (whether he actually made the bet with Bohr, I am not sure) and made a bold proposal that there is some other particle involved in the process and that is termed as neutrino. It took almost 30 years for experimentalist (Cowan and Reines) to prove the existence of neutrino because these mysterious particles do not involve in electro-magnetic or even strong interaction in atoms or nuclei. They are subjected to the so called weak interaction[49]. The beta decay should be

[49] For a introductory of beta decay and neutrino, please refer to Thornton and Rex: "Modern Physics for Scientists

written as: $\upsilon + p \leftrightarrow n + e$, where υ stands for neutrino, and its energy is not discrete and takes continuous distribution. So the conservation of energy is saved and new particles discovered.

6.5 Scattering Problem (Collision between Particles)

Scattering problem is just particles under the mutual interaction, the initial state (defined as the positions and velocity of the particle at time=0) will change into other state at later time. This is the most general definition of scattering, and many problems in mechanics can be treated as scattering. The scattering in the narrower sense is that the two particles approach each other in a collision course, collide with each other and the states of particle changes due to the mutual interaction (you may switch scattering and collision in this narrow sense). Such process is common in physics and chemistry; for instance, the formation of molecules by atoms (the interaction is electro-magnetic); the creation or annihilation of particles in high energy physics where energetic (>GeV) particles are brought close together; and in daily life such pool game, the billiard balls collide and bounce away.

Though the details of calculation involves details of interaction, i.e. we need to know how particles interact to determine the final states after

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and Engineers" section 12.7. Or go WIKI.

scattering from the initial conditions[50], conservation of momentum and energy will offer guidelines. That's why we take a look on this process as an example to apply what we have learned in the previous two chapters.

6.5-1 Scattering in 1-D

Again we first look into this simple case. The two particles approach each other in 1-D and collide, the initial conditions of particles are usually provided, then what is the final states after collision (here, the collision happened at certain spot in space, so the final states of particles are just their velocity)? Well we have conservation laws on momentum and energy, the question is whether they apply here?

For the particles collide in a short time, the force between them will be much larger than the external force (such as gravity, friction etc), so we can treat that only internal forces play important role in scattering. This means the total momentum of the system is conserved. As to the total energy, it is still conserved, but besides mechanical energy it may transform into other forms, such as heat. So generally the mechanical energy may or may not be conserved, that is really depending on the specific process. So giving the initial conditions (particles and their initial velocity), we only have one equation from conservation of momentum.

[50] Actually the process in science is reverse in many cases. That is from the experimental results of scattering process, measuring the states of particles after scattering, people can learn what kind of interaction involved in the scattering.

That is insufficient to solve for the velocities (two velocities here for particle 1 and 2). There are two simple cases, however, that the final velocities can be determined.

(1) Elastic Collision

In this case, the mechanical energy is conserved, no loss to other energy forms during the scattering process. Then for particles 1 and 2 we have:

$$m_1 v_{10} + m_2 v_{20} = m_1 v_{1t} + m_2 v_{2t}$$

$$\frac{1}{2} m_1 v_{10}^2 + \frac{1}{2} m_2 v_{20}^2 = \frac{1}{2} m_1 v_{1t}^2 + \frac{1}{2} m_2 v_{2t}^2$$

That is just conservation of momentum and energy. The potential energy before and after the collision is taken to be the same. So it does not appear in the energy equation. $v_{10}, v_{20}, v_{1t}, v_{2t}$ are the initial and final velocities of particle 1 and 2, they can be positive or negative numbers; positive if along the positive direction defined, and negative if otherwise.

The equations above can be used to solve the two unknowns (try it yourself, solving the final velocities without reading the following). I shall rearrange the equations to make relations more clear:

$$m_1(v_{1t} - v_{10}) = m_2(v_{20} - v_{2t}) = -m_2(v_{2t} - v_{20})$$

Or $m_1 \Delta v_1 = -m_2 \Delta v_2$; $\Delta v_1 = v_{1t} - v_{10}$ etc.

$$m_1(v_{1t}^2 - v_{10}^2) = -m_2(v_{2t}^2 - v_{20}^2)$$

Divide the two equations (provided that $v_{1t} - v_{10} \neq 0$; if $v_{1t} = v_{10}$, then $v_{2t} = v_{20}$, these of course satisfies the conservation relations but it is a trivial solution that there is no physical interaction between the two particles):

$$\begin{aligned} &v_{1t} + v_{10} = v_{2t} + v_{20} \\ &or\ v_{1t} - v_{2t} = -(v_{10} - v_{20}) \end{aligned} \qquad (6\text{-}58)$$

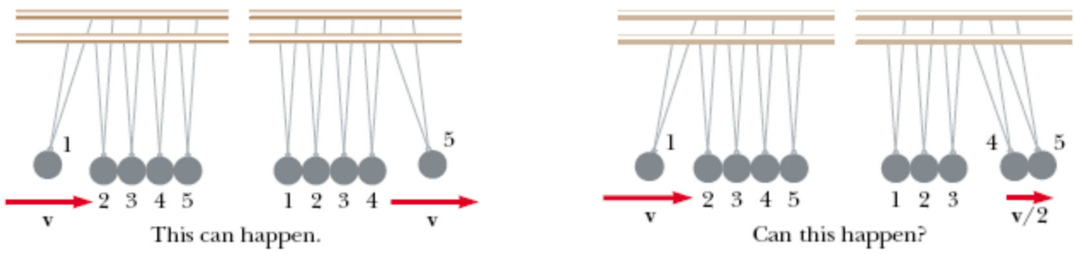
It shows that the relative velocity between the two particles would be same in value before and after the collision, but the direction is reversed. With this relation and conservation of momentum, the individual velocities can be solved:

$$\begin{aligned} v_{1t} &= \frac{m_1 v_{10} - m_2 v_{10} + 2m_2 v_{20}}{m_1 + m_2} \\ v_{2t} &= \frac{m_2 v_{20} - m_1 v_{20} + 2m_1 v_{10}}{m_1 + m_2} \end{aligned} \qquad (6\text{-}59)$$

The formula is symmetric with respect to the switch the label 1 and 2 (can you think about a reason for this?).

(a) If $m_1 = m_2, v_{20} = 0$, two equal mass, one is stationary:

$v_{1t} = 0, v_{2t} = v_{10}$, after collision the original moving party will come to stop, and the original stationary one will travel with same velocity. This is what you see in the classical tick-tock toy made of steel balls[51].



(b) $m_1 >> m_2$

Divide both numerator and denominator in (6-59) with m_1, and neglect

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[51] Picture taken from Serway and Jewett "Physics for Scientists and Engineers" 6th ed. Chap 9, Figure 9.10.

the terms containing $\frac{m_2}{m_1}$, we will get approximately:

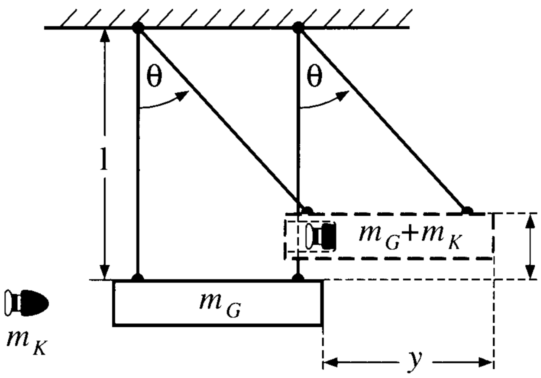
$v_{1t} \approx v_{10}; \quad v_{2t} \approx 2v_{10} - v_{20}$

This is the formula explains that if ping-pong ball hits the wall, the ping-pong ball will have a reversed velocity with same magnitude; as well as the demo in problem 4.23 in KK.

(2) Completely Inelastic Collision

This is when the two particles stick together after the collision. Now there is only one final velocity $v_t = v_{1t} = v_{2t}$, and it can be solved from conservation of momentum. The energy loss can also be computed (details won't be given here).

Examples: one example is the old craftsman's method of measuring the speed of bullets:



Ballistic pendulum and bullet.

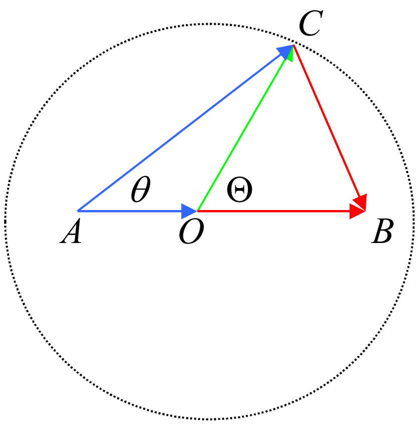
If you measured angle of swing of the pendulum, the speed of bullet can be calculated.

Another is shown in the figure above. A small sandbag is dropped vertically on a moving cart. The sandbag does not have initial horizontal velocity, the cart is moving horizontally with v_0. At later time, the sandbag will travel with same horizontal velocity v' as the cart due to interaction between them (friction). Neglect the friction between cart with ground, you can calculate the v' easily, as well as the loss the kinetic energy (what is the cause of this loss? And can you express it in work formula?).

Another example would be during the inelastic collision, very often we want to have most kinetic energy transforming into other energies, i.e. the loss of kinetic energy after the scattering needs to be the largest. If our facility can provide certain amount of energy initially (the power of the accelerator is fixed), how we arrange the two particles to collide to get maximum loss of kinetic energy? (this will be left as a homework, hint: center of mass)

For other collision in 1-D, as we stated above we need to know relations between kinetic energies before and after the scattering. If the ratio of loss of kinetic energy is given, $\frac{T_t}{T_0} = A$, or the relation between change of relative velocity: $\frac{v_{1t} - v_{2t}}{v_{10} - v_{20}} = B$, then the final velocities can be computed similarly as elastic case.

6.5-2 Elastic Scattering in 2-D and Center of Mass Frame



As the title suggests, I shall only discuss the elastic scattering in 2-D. The complete inelastic can be solved similarly as in 1-D with the conservation of momentum. The general inelastic would be a little too complicated. So I shall focus on elastics scattering here for 2-D case. You may wonder what the figure above means? I will give you the meaning of each line and show that this figure would be very useful in solving the 2-D *elastic* scattering. Here the figure is just an advertisement of what is coming up.

(1) General Discussion

In 2-D collision[52], we have four unknowns (the velocities of the final states are vectors with 4 undetermined components for the two particles,

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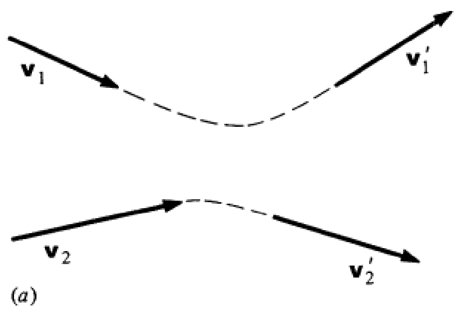
[52] Generally if the two particles' initial velocities and the line connecting their center of mass of each particle are in one plane, and the force between them also along the line of connecting the center, the collision will be remain in the plane (a 2-D case). This could be argued from the symmetry point of view, i.e. the initial conditions and interactions are invariant with reflection with respect to the plane; the results would be also reflection invariant with respect to the plane.

or magnitude and direction need to be determined). The conservation of momentum will give us two relations in 2-D (along each axis of coordinate system) and the conservation of mechanical energy (here only kinetic energy) will give another. So the problem cannot be fully solved (4 unknowns, 3 relations from conservation laws) by just considering the conservation laws. Extra information is required from experiments or through detail analysis of interaction. In this section we assume that such information is available, for example if we know the direction of particle 1 after scattering (only direction, magnitude still need to be determined), then the final velocities can be calculated.

(2) Treatment in the Lab Frame



As the figure shows the collision viewed in a lab frame (with reference to some fixed coordinate system in the lab, or for the ground observer stationary with respect to lab), the $\vec{v}_1, \vec{v}_2$ are known, and the direction of v_1' is also known. Then in this coordinate system, we have:

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x}$$
$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}$$
$$\tan\theta = \frac{v'_{1y}}{v'_{1x}}$$
$$\frac{1}{2} m_1 \vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2} m_2 \vec{v}_2 \cdot \vec{v}_2 = \frac{1}{2} m_1 \vec{v}'_1 \cdot \vec{v}'_1 + \frac{1}{2} m_2 \vec{v}'_2 \cdot \vec{v}'_2$$

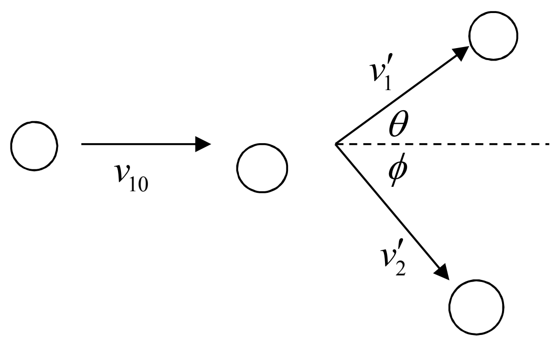
The last energy relation can also be expressed as:

$$\frac{1}{2} m_1 (v_{1x}^2 + v_{1y}^2) + \frac{1}{2} m_2 (v_{2x}^2 + v_{2y}^2) = \frac{1}{2} m_1 (v_{1x}'^2 + v_{1y}'^2) + \frac{1}{2} m_2 (v_{2x}'^2 + v_{2y}'^2)$$

These relations can be used to solve for the v's.

Example: Considering the collision between two billiard balls with equal mass. Just like in pool game, with one ball (the color ball) is stationary and the other (the white cue ball) with initial velocity v_{10}; After collision the white ball will fly to a direction with angle θ relative to the original v_{10}. Find the velocities of the color ball and white cue ball after collision. A very interesting result is that the direction of the color ball travels will be perpendicular to the travel direction of the white ball after collision. A fact used often by pool players (if you do not know this before, I hope this will improve your pool performance☺)

To work this in the lab frame is:



$$mv_{10} = mv_1' \cos\theta + mv_2' \cos\phi$$
$$0 = mv_1' \sin\theta - mv_2' \sin\phi$$
$$\frac{1}{2}mv_{10}^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2$$

You can solve for v_1', v_2', ϕ for this problem; You can also prove that the two balls will travel perpendicularly after collision, also prove that the angle θ cannot exceed 90 degree, i.e. the ball 1 cannot be scattered backward. Do it yourself. It is straightforward but a bit messy.

If you only want to prove the v_1', v_2' are perpendicular, it is easier and can be proved by momentum and energy conservation:

$$\vec{P}_0 = \vec{P}_1' + \vec{P}_2'$$
$$\frac{\vec{P}_0 \cdot \vec{P}_0}{2m} = \frac{\vec{P}_1' \cdot \vec{P}_1'}{2m} + \frac{\vec{P}_2' \cdot \vec{P}_2'}{2m}$$

Then by take the scalar product of the first relation, you get $\vec{P}_1' \cdot \vec{P}_2' = 0$.

(3) Treatment in the Center-of-Mass Frame

For the scattering problem, it is almost always easier if we work in the Center-of-Mass frame (C.M. frame), especially for the elastic collisions, because the kinetic energy of each particle will be same before and after the collision in C.M. frame (Recall that the kinetic energy depends on the coordinate system). Here is the proof along with some important relations in C.M. frame.

The notations I shall use are: $\vec{R}, \vec{V}$ are position and velocity vectors for the center of mass, because there is no external force, so $\vec{V}$ will be a constant. $\vec{r}_{1L}, \vec{r}_{2L}$ are position vectors for particles 1 and 2 in the lab frame; $\vec{r}_{1c}, \vec{r}_{2c}$ are for particles 1 and 2 in C.M. frame; $\vec{v}_{1L}, \vec{v}_{2L}$ are velocities for 1 and 2 before collision in Lab frame; $\vec{v}'_{1L}, \vec{v}'_{2L}$ are velocities for 1 and 2 after collision in lab frame; $\vec{v}_{1c}, \vec{v}_{2c}; \vec{v}'_{1c}, \vec{v}'_{2c}$ are velocities for 1 and 2 before and after the collision in C.M. frame.

From definition of center of mass, we have:

$$\vec{R} = \frac{m_1\vec{r}_{1L} + m_2\vec{r}_{2L}}{m_1 + m_2}$$

$$\vec{V} = \frac{m_1\vec{v}_{1L} + m_2\vec{v}_{2L}}{m_1 + m_2}$$

$$\vec{P} = m_1\vec{v}_{1L} + m_2\vec{v}_{2L} = M\vec{V}$$

$M=m_1+m_2$. These relations are just (5-3), (5-4) previously. We also have the relation between position vectors in the C.M. with that in lab frame (5-7):

$$\vec{r}_{1c} = \vec{r}_{1L} - \vec{R} = \vec{r}_{1L} - \frac{m_1\vec{r}_{1L} + m_2\vec{r}_{2L}}{m_1 + m_2} = \frac{m_2(\vec{r}_{1L} - \vec{r}_{2L})}{m_1 + m_2}$$

$$\vec{r}_{2c} = \vec{r}_{2L} - \vec{R} = \vec{r}_{2L} - \frac{m_1\vec{r}_{1L} + m_2\vec{r}_{2L}}{m_1 + m_2} = -\frac{m_1(\vec{r}_{1L} - \vec{r}_{2L})}{m_1 + m_2}$$

From there we have the velocity relations:

$$\begin{aligned} \vec{v}_{1c} &= \vec{v}_{1L} - \vec{V} = \vec{v}_{1L} - \frac{m_1\vec{v}_{1L} + m_2\vec{v}_{2L}}{m_1 + m_2} = \frac{m_2(\vec{v}_{1L} - \vec{v}_{2L})}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}\Delta\vec{v}_L \\ \vec{v}_{2c} &= \vec{v}_{2L} - \vec{V} = \vec{v}_{2L} - \frac{m_1\vec{v}_{1L} + m_2\vec{v}_{2L}}{m_1 + m_2} = -\frac{m_1(\vec{v}_{1L} - \vec{v}_{2L})}{m_1 + m_2} = -\frac{m_1}{m_1 + m_2}\Delta\vec{v}_L \end{aligned} \quad (6\text{-}60)$$

The relations for the $\vec{v}'$s are similar because these relations are from definition of center of mass and vector summation. (6-60) tells us in the C.M. frame the velocities of the two particles will be along same line and reversed in direction, this is expected because we know that in C.M.

frame the total momentum is zero (5-12). The $\Delta\vec{v}_L$ is the relative velocity between particle 1 and 2 in lab frame. Since this is relative velocity, it does not depend on choice of coordinates, i.e. :

$$\Delta\vec{v}_L = \Delta\vec{v}_c = \Delta\vec{v} \qquad (6\text{-}61)$$

This is also clear from (6-60), the momentum is:

$$\begin{aligned} \vec{p}_{1c} &= m_1\vec{v}_{1c} = \frac{m_1 m_2(\vec{v}_{1L}-\vec{v}_{2L})}{m_1+m_2} = \mu\Delta v \\ \vec{p}_{2c} &= m_1\vec{v}_{2c} = \frac{m_1 m_2(\vec{v}_{1L}-\vec{v}_{2L})}{m_1+m_2} = -\mu\Delta v \end{aligned} \qquad (6\text{-}62)$$

(6-62) μ is the reduced mass and is defined as:

$$\mu = \frac{m_1 m_2}{m_1+m_2} \qquad (6\text{-}63)$$

In an elastic collision, we have:

$$\vec{p}_{1c} + \vec{p}_{2c} = \vec{p}'_{1c} + \vec{p}'_{2c} = 0 \qquad (6\text{-}64)$$

This is conservation of momentum in C.M. frame.

$$\frac{1}{2}MV^2 + \frac{|\vec{p}_{1c}|^2}{2m_1} + \frac{|\vec{p}_{2c}|^2}{2m_2} = \frac{1}{2}MV^2 + \frac{|\vec{p}'_{1c}|^2}{2m_1} + \frac{|\vec{p}'_{2c}|^2}{2m_2}$$

This is conservation of energy written with Konig theorem (if you forget it, just write the normal form and use substitution $\vec{v}_{1L} = V + \vec{v}_{1c}$). Since V is constant and we have:

$$\frac{|\vec{p}_{1c}|^2}{2m_1} + \frac{|\vec{p}_{2c}|^2}{2m_2} = \frac{|\vec{p}'_{1c}|^2}{2m_1} + \frac{|\vec{p}'_{2c}|^2}{2m_2} \qquad (6\text{-}65)$$

Combine with (6-64), we have:

$$\begin{aligned} &|\vec{p}'_{1c}| = |\vec{p}_{1c}|;\ |\vec{p}'_{2c}| = |\vec{p}_{2c}| \\ &|\vec{v}'_{1c}| = |\vec{v}_{1c}|;\ |\vec{v}'_{2c}| = |\vec{v}_{2c}| \end{aligned} \qquad (6\text{-}66)$$

So the *magnitude* of velocity and momentum is unchanged during the scattering, what changed is the directions. The $\vec{v}'_{1c}, \vec{v}'_{2c}$ are still along one line, reversed in direction, and this line has an angle to that before the collision. We can get $\vec{v}'_{1c}, \vec{v}'_{2c}$ from (6-66) and get the velocities in the lab frame using:

$$\vec{v}'_{1L} = V + \vec{v}'_{1c} \; ; \; \vec{v}'_{2L} = V + \vec{v}'_{2c} \qquad (6\text{-}67)$$

This can be done relative easy with geometric method as the example shows.

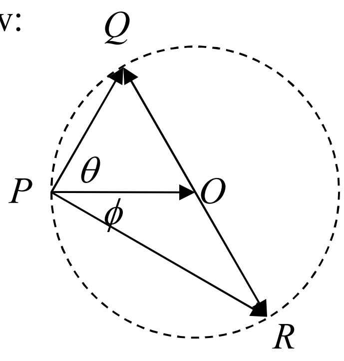
Consider the previous example of collision between billiard balls with equal mass, one is stationary and the other moves with v_0. After collision one travels with angle θ with respect to v_0. Now let's work this in C.M. frame:

$$\vec{V} = \frac{1}{2}\vec{v}_0$$

$$\vec{v}_{1c} = \vec{v}_{1L} - \vec{V} = \frac{1}{2}\vec{v}_0$$

$$\vec{v}_{2c} = \vec{v}_{2L} - \vec{V} = -\frac{1}{2}\vec{v}_0$$

After collision, we know that the velocity will change direction but not magnitude in C.M. frame, i.e. $|\vec{v}'_{1c}| = |\vec{v}'_{2c}| = \frac{1}{2}|\vec{v}_0|$, we can draw a figure below:



$\vec{PO} = \vec{V}$, $\vec{OQ} = \vec{v}'_{1c}$, $\vec{OR} = \vec{v}'_{2c}$, $\vec{PQ} = \vec{V} + \vec{v}'_{1c} = \vec{v}'_{1L}$. $\vec{PR} = \vec{V} + \vec{v}'_{2c} = \vec{v}'_{2L}$

The θ is the angle formed between PQ and PO from the problem, OQ and OR are along same line from (6-64) and in this example PO, OQ, OR have same length (magnitude) and lies on the circle with radius of $\frac{1}{2}|v_0|$

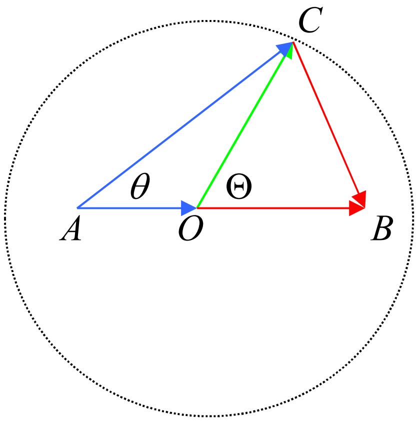
From simple geometry, we see right away: $\theta + \phi = \frac{\pi}{2}$, $|PQ| = v_0 \cos\theta$ and $|PR| = v_0 \sin\theta$ which are the velocities of 1 and 2 after collision in the lab frame. Also we see that no matter how the direction OQ changes, the θ has the largest value of $\frac{\pi}{2}$, when Q overlaps with P (this is just 1-D scattering where cue ball stops and color ball flies with v_0 in lab frame). So there is no back scattering in lab frame. You see how easy to find the answer, provided we set up the problem correctly in C.M. frame, which takes some work but straightforward. Please also read the example 4.19 in K&K on limitation of scattering angle in lab frame.

The geometric method using velocity is nice but there is a small catch, the $\vec{v}'_{1c}, \vec{v}'_{2c}$ will always be along same line, but are not equal in magnitude in general (the billiard example is a simple case). Of course you may draw two circles with radius of r_{1c} and r_{2c} respective for the calculation of velocities for particle 1 and 2, but there is a better way: Using momentum. The main property of C.M. frame is that it is a zero total momentum frame as in (6-64), so $\vec{p}'_{1c}, \vec{p}'_{2c}$ are always same in magnitude and reverse in direction. i.e. they will lie on the same circle with radius of $|\vec{p}'_{1c}| = |\vec{p}_{1c}|$.

To compute the momentum and velocity in lab frame:

$$\begin{aligned}
\vec{p}'_{1L} &= m_1\vec{v}'_{1L} = m_1(\vec{V} + \vec{v}'_{1c}) = m_1\vec{V} + \vec{p}'_{1c} = \frac{m_1}{M}\vec{P} + \vec{p}'_{1c} \\
\vec{p}'_{2L} &= m_2\vec{v}'_{2L} = m_2(\vec{V} + \vec{v}'_{2c}) = m_2\vec{V} + \vec{p}'_{1c} = \frac{m_2}{M}\vec{P} + \vec{p}'_{2c}
\end{aligned} \tag{6-68}$$

(6-68) is the basis of geometric method and that is the graph I am showing you at the beginning of this section.



$\vec{OC} = \vec{p}'_{1c}$, $-\vec{OC} = \vec{CO} = \vec{p}'_{2c}$, $\vec{AO} = \frac{m_1}{M}\vec{P} = m_1\vec{V}$, $\vec{OB} = \frac{m_2}{M}\vec{P} = m_2\vec{V}$, $\vec{AC} = \vec{p}'_{1L}$, $\vec{CB} = \vec{p}'_{2L}$. θ is the scattering angle of particle 1in lab frame relative to the total momentum (the direction that center of mass travels); Θ is the scattering angle of 1 in C.M. frame. Knowing the masses, the total momentum $\vec{P}$, magnitude of the momentum in C.M.(whose magnitude can be computed from initial conditions, since it does not change during elastic scattering) $|\vec{p}'_{1c}|$ and one angle (equivalent of knowing AO,OB,OC and one of the angle θ or Θ), the rest of calculation will be solving geometric problems using this graph. For example:

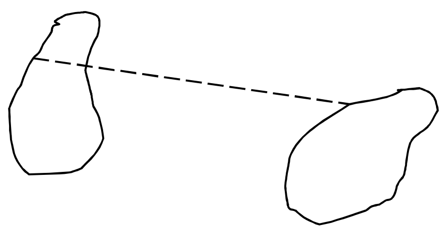
$$\tan\theta = \frac{|OC|\sin\Theta}{|AO| + |OC|\cos\Theta} = \frac{|\vec{v}'_{1c}|\sin\Theta}{|V| + |\vec{v}'_{1c}|\cos\Theta}$$

The angle of CB and the magnitude of |AC|, |CB| can be computed similarly (using Pythagoras theorem or cosine laws of triangle etc.), Please try them yourself. This is a general method solving scattering problems in 2-D.

(Important concept: work (line integral of force dot displacement), mechanical energy, work-energy theorem. Path independent work and conservative force, conservative force and potential; Given a potential how to find force; and given a force how to know it is conservative or not (zero curl), and from a conservative force to find out its potential. Conservation of energy, solving scattering with conservation relations and in C.M. frame)

Chapter 7 Rotation, Angular Momentum and Motion of Rigid Body

In the previous chapters, we focused on the motion of particles or even for extended body, treating it as a particle (the mass concentrated at the C.M. etc), and only studied the **translational motion**. In this Chapter, I will discuss the **rotational motion**, another important type of motion. For a rigid body, the motion can always be *decomposed as the translational* *motion of a certain point (almost always chose the C.M), and a rotational motion around that point.* Here the **rigid body** is a special class of objects which is defined as: *the distance between any two points on the body does not change over time.* Of course rigid body is also a physical model (an idealization), but a useful one. The decomposition of motion into translation by a point and a rotation around that point sounds certainly possible and intuitive.



For the object drawn in the figure (the two objects are identical, forgive my drawing if they appear otherwise), the rigid body does not change shape in the motion. For any point on the body, its position at later time could be found out by first doing a translation of one particular point (the one connected by line in the figure) and then doing a rotation around that point (also called pivot point for this reason for that particular point). The rigorous proof that the motion of rigid body is equivalent to translation + rotation is the Euler or Chasles theorem, which you can find in KK's notes 6.1.

However, the study of rotation is not an easy task (at least not as easy as translation). This is probably due to facts: 1) The math is a bit messy. It involves cross product of vectors a lot. Once you get used to it, you will find that it is not that bad. 2) It is a little bit hard to imagine rotational motion in higher dimensions (no problem for 2-D). Our intuitive won't be much help in complicated rotational motion. Quite often, you cannot rely on your intuition to picture the motion as in translation case. Because of these, this chapter would probably the most difficult part so far, and you will work out some hard problems[53]. I will try to explain things nice and clear in this chapter, and KK's book in my opinion may not be the best in this respect (It is still tops many other books). So I will reorganize the materials a little bit, covering both KK's chapter 6 and 7 in one. I shall 1) start from the simplest case in rotation: a **pure rotation** in 2-D, using it to introduce important concepts **angular velocity**, **angular momentum, torque and moment of inertia**. This is the easy part. 2) We shall study the formal vector definition and treatment of angular momentum and torque. Derive the most important equation of rotation (equivalent to F=dP/dt): Torque=change of angular momentum. The importance (or the tricky part) is to understand that in what coordinate system you can apply such equation. Then we shall study the motion of rigid body still in 2-D but with translation+rotation involved. This is still relatively easy. 3) We shall discuss rotation in 3-D for rigid body, a very important relation

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[53] In the preparation of this notes, I worked all the KK's problems in Chap.6 and 7. The 50+ problems took me almost 10 days to finish, though it is not full time but it doubled the time I spent on the equivalent number problems in previous chapters.

(besides that Torque=change of angular momentum) is the relation between angular velocity and angular momentum, introducing the inertia tensor. See how we work out problems with these two important relations, such as understanding the gyroscope. I hope through discussion on the first two parts, this one would appear natural and acceptable. I will not intend to cover Euler equations in depth which is often used to solve the general rotational motion (KK section 7.7 and beyond). Actually I will derive the Euler equation in the discussion of some examples, but I would leave the formal treatment to analytical mechanics. You will certainly ‘suffer’ this in that course, but I hope the stuffs you learned here will alleviate suffering much.

Here let me introduce the concept of **degree of freedom**: that is how many independent unconstrained (free) variables needed to describe a system.

This is best illustrated by examples: S=degree of freedom

a) Single particle in 3-D: clearly 3 variables are needed (x,y,z; or other coordinate such as spherical…), so S=3.

b) N particles in 3-D, no constraint between any of them: S=3N. If there are m constraints, say there are relations between the distance of particles, etc. Then S=3N-m

c) Rigid body in 3-D: due to the constraint put on the rigid body, there are now only 6 degrees of freedom. This is easiest to see from Chasles theorem: 3 degree of freedom due to position of the fixed point (say CM, its X,Y,Z); another 3 degree of freedom due to rotation: the direction of axis around which rotates (2 degree of freedom, a direction of an unit vector in 3-D) and the rotation angle around this axis. (any rotation can also equivalently be specified by 3 Euler angles, but won't be covered here)

d) Rigid body in 2-D: S=3: 2 due to the position of the fixed point; 1 due to rotation angle (since here in 2-D, the direction of axis is fixed, i.e. the direction perpendicular to plane)

7.1 Pure Rotation in 2-D[54]

7.1-1 Angular Velocity, Moment of Inertia, Angular Momentum and Torque

If we nail down a point of the rigid body, then the only motion possible for that body would be rotation to the fixed point (pivot). What is the variable that can change then? It is the angular displacement as shown in the figure left (degree of freedom=1), thus the treatment below carries great analogy to 1-D translational motion.

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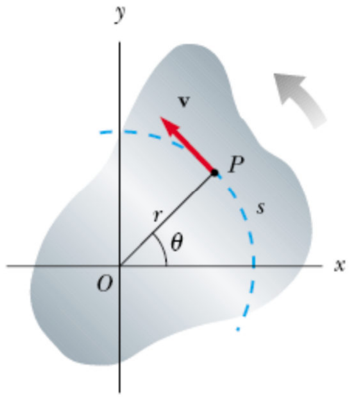
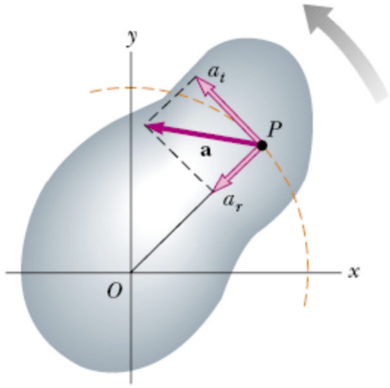
[54] This part you can find detailed accounts in physical textbooks for the course of University Physics. The popular ones are: Halliday et al "Fundamental of Physics"; Serway and Jewett's "Physics for Scientists and Engineers". These are textbooks easier than KK's, used in regular courses in university physics or honor courses in high school. The figures I used in this section mostly come from the Serway's 6^{th} edition. (Halliday's and Serway's are basically equivalent, so pick one to read if you need to)





Even for a 3-D object, we can project $\vec{r}$ to the plane perpendicular to the rotation axis (z-direction in above) and find the displacement angle similarly[55]. For simplicity we only consider 2-D object (like a thin piece of hard paper) in this section[56].

Let the θ be the displacement angle defined above (analogous to the x displacement in translation), we can find its rate of change over time and define angular velocity as:

$$\omega = \frac{d\theta}{dt} \qquad (7\text{-}1)$$

The θ is in unit of radians and is defined as increasing in counter clockwise (c.c.w) rotation, so the ω is positive for c.c.w rotation and negative for c.w (clockwise) rotation. In 3-D, ω would be represented as a vector with the positive direction defined as above[57], which is also the

[55] This is equivalent to use cylindrical coordinate (ρ,θ,z) in the 3-D case, where z is along the rotation axis, and ρ,θ are the distance and displacement angle with respect to the rotation axis.

[56] There is an even simpler case that the uniform circular motion of one particle revolves around an axis. I reckon that you are all familiar to that. What I discussed here would also apply to this simpler case.

[57] There is a subtlety here that the finite angular displacement is not a vector like translational displacement, but the infinitesimal angular displacement and the angular velocity are vectors. (KK, notes 7.1)

right hand rule (curl your four fingers towards the rotation direction and thumb will give you the direction of positive ω). Another important fact is that ω would be same for every points on the rigid body.

From the figure above left, it is easy to see that the relation between the translational velocity (in the later parts, I shall just use velocity for the translation) and angular velocity is:

$$|v| = |\omega| r \qquad (7\text{-}2)$$

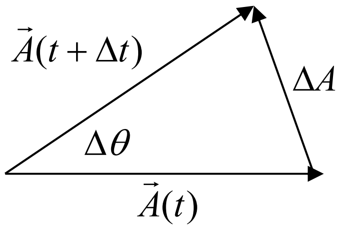
Its direction is along the tangent line and since it is rigid body, there is no radial velocity. You should be able to prove (7-2) before you check my answer in the footnotes below.[58] For the acceleration part, refer to the figure above right, it has tangential and radial components:

$$\begin{aligned} \vec{a}_t &= r\dot{\omega}\hat{\theta} = r\alpha\hat{\theta} \\ \vec{a}_r &= -\frac{v^2}{r}\hat{r} = -\omega^2 r\hat{r} \end{aligned} \qquad (7\text{-}3)$$

The α is called angular acceleration. The radial acceleration (centripetal acceleration)is due to the direction change of the velocity, and its proof is actually given a while ago (section 3.7-2), here I will just use (3-44). i.e. for a directional change of vector due to rotation:

---

58 $\vec{v}=\hat{T}\frac{ds}{dt}$, $\hat{T}$ is unit vector along tangent direction, s is the arc length. $ds=rd\theta$, this is the reason to measure angle in radians. $\frac{ds}{dt}=\frac{rd\theta}{dt}=r\omega$.



$(\frac{d\vec{A}}{dt})_{\perp} = |A|\frac{d\theta}{dt}\hat{a}_{\perp} = \omega \times \vec{A}$ (7-4) copied from (3-44)

Here the $\vec{A}$ will be velocity $\vec{v}, \vec{\omega}$ would point to +z direction:

$\vec{\omega} \times \vec{v} = \omega^2 r(\hat{z} \times \hat{\theta}) = -\omega^2 r\hat{r}$

The kinetic energy of the pure rotation is[59]:

$$K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i r_i^2 \omega^2 = \frac{1}{2}(\sum_i m_i r_i^2)\omega^2$$

We use the fact that the ω is same for the whole body in rotation in the above equation. We shall define the moment of inertia of rotation as:

$$I_o = \sum_i m_i r_i^2 \qquad (7\text{-}5)$$

The reason for the subscript o in I_o is to remind you that the moment of inertia depends on the pivot of choice. It is obvious from the definition that the position vector depends on the choice of origin. So for same object with different choice of pivots, the I can be quite different. I shall just use I in the following formula but its dependence on choice of pivot should be remembered.

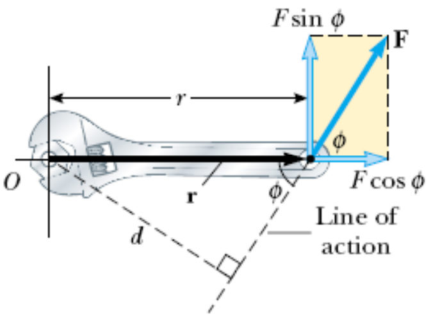
$$K = \frac{1}{2}I\omega^2 \qquad (7\text{-}6)$$

[59] I shall use K to represent the kinetic energy from now on instead of T (which stands for translation) as before. Also in the derivation I shall use $\sum_i m_i$ as if the body consists of discrete particles. It is easily extended to $\int \rho dV$ for continuous distribution of mass, where ρ stands for density.

Continue the analogy, we see that I is analogous to mass m in translation. Just as we define mechanical linear momentum as p=mv, we shall define[60]:

$L = I\omega$ (7-7)

And if you push the analogy further, you cannot help wonder what is the analogous of F=ma? Is it $F = I\alpha = I\frac{d\omega}{dt} = \frac{dL}{dt}$? This guess is in the correct track, we shall see that the left hand of force will be replace by something we call torque.



In the figure above, the effective force for rotation will be perpendicular to r. This is familiar for everyone ever use wrench or just open a heavy door, you would not push or pull the door or wrench in the radial direction. Now back to work energy theorem:

$\Delta W = \Delta K$

$\Delta W = \vec{F} \cdot \Delta\vec{r} = \vec{F} \cdot \hat{T}\Delta s = F \sin\phi r \Delta\theta$

(ϕ in the figure above is the angle between the force and position vectors)

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[60] We shall give a more general and formal definition of angular momentum and torque later when we talk about angular momentum in general sense, not limited to the pure rotation in 2-D. The definition here would reduce to a special case for the general definition.

We shall define torque as:

$\tau = Fr\sin\phi = (F\sin\phi)r = Fd$ (7-8)

The last two terms show the usual way to compute the torque, either take the vertical component of force and times r or take the product of force and vertical distance d. Then the work kinetic energy expressed in torque and angular properties are:

$$\tau\Delta\theta = \frac{1}{2}I\Delta\omega^2 \qquad (7\text{-}9)$$

Or in integral form: $\int_{\theta_i}^{\theta_f} \tau d\theta = \frac{1}{2}I(\omega_f^2 - \omega_i^2)$ (7-10)

From (7-9), we have:

$$\begin{aligned}
&\tau\Delta\theta = \frac{1}{2}I\Delta(\omega^2) = I\omega\Delta\omega = I\frac{\Delta\theta}{\Delta t}\Delta\omega = I\frac{\Delta\omega}{\Delta t}\Delta\theta \\
&\tau = I\frac{\Delta\omega}{\Delta t} \xrightarrow{\Delta t\to 0} = I\alpha = \frac{dL}{dt}
\end{aligned} \qquad (7\text{-}11)$$

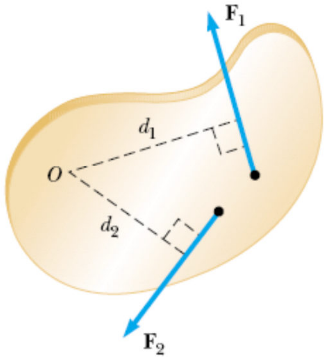
This is the fundamental relation between torque and change of angular momentum in rotation analogous to F=ma=dP/dt. We can also derive the angular impulse-change of angular momentum from (7-11):

$$\tau\Delta t = I\Delta\omega = \Delta L \qquad (7\text{-}12)$$

So you see that the relations in translation motion all have counter parts in rotation. Which is no surprise since they all originate from Newton's laws, and here are just expressed in terms suitable for the study of rotation.

First point need to be stressed here is that the torque and angular momentum are all dependent on the choice of pivot like the moment of inertia, this is clear from their definition of (7-7) and (7-8). (7-9) (or 7-10), (7-11) and (7-12) are used often in the study of rotational motion.

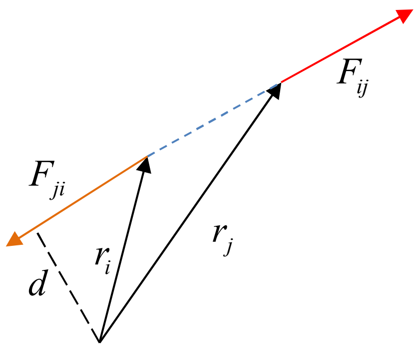
Second point is regarding to the sign of the torque. The convention is that it obeys the right hand rule consistent with our definition of angle and angular velocity. If it creates c.c.w rotation, the torque is positive and if it creates c.w. rotation, it is negative. In the figure below, F_1 would generate a positive torque $\tau_1 = F_1 d_1$ and F_2 would generate a negative torque $\tau_2 = -F_2 d_2$



Third point is that under different forces, what is the form of the above relations? Well using the principle of superposition of forces, it would be straightforward to see that:

$$\tau_{total} = \sum_i F_i r_i \sin\phi_i \qquad (7\text{-}13)$$

And this total torque (with their sign convention) will be in the relations (7-9) to (7-12). In principle, the force should include both external and internal forces, but we shall see that the torque by the internal forces will not play roles in relations like (7-9) to (7-12). The argument is straightforward and you should try it yourself.



The combined torque due to a pair of internal forces will be zero provided that the pair of internal forces are 1) equal in magnitude and reversed in direction (3rd law). 2) the directions of internal forces are parallel to the direction joining the two particles (fundamental forces in classical physics satisfy this). With these two conditions, you can prove the torques due to internal forces is zero. So we shall only consider the torques due to entirely by the external forces from now on.

As a summary for this section, please fill in the corresponding formula for pure rotation in the table below:

Table 7.1. The Comparison between Translation (1-D)and Rotation (2-D)

1-D Translation	Fixed Axis Rotation (2-D)
Position: x	Angle: θ
Velocity: $\frac{dx}{dt}$	
Acceleration: $\frac{d^2x}{dt^2}$	
Kinetic Energy: $\frac{1}{2}mv^2 = \frac{p^2}{2m}$	

Mass: m	
Linear Momentum: $p = mv$	
Equation of motion: $F = ma = \frac{dp}{dt}$	
Work: $dW = Fdx$	
Power: P=Fv	
Impulse: Fdt=dp	

I give you a start and try to fill the others by yourself.

7.1-2 Computation of Moment of Inertia and More on Angular Velocity

In this section we are dealing with some technical issues. It is important to know the moment of inertia of some common shaped object. The formula $I = \sum_i m_i r_i^2$ can be extended for even 3-D object:

$$I_z = \sum_i m_i \rho_i^2 = \sum_i m_i (x_i^2 + y_i^2) \qquad (7\text{-}14)$$

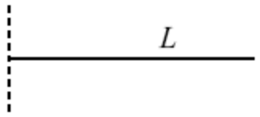
Z stands for the rotation axis along z direction, ρ_i^2 refers to the distance to the axis. The computation of moment of inertia will become calculation of integrals.

For example a thin rod rotates around its C.M.:

The density is even $\rho = M / L$

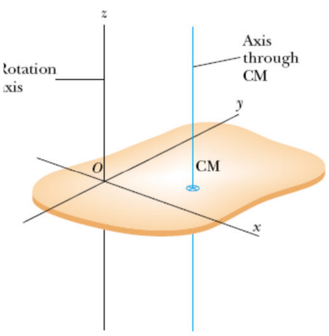
$$I=\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{M}{L}x^2dx=2\frac{M}{L}\int_0^{\frac{L}{2}}x^2dx=2\frac{M}{L}\frac{1}{3}(\frac{L}{2})^3=\frac{ML^2}{12}$$

For rotation around pivot at the edge:



$$I=\int_0^L\frac{M}{L}x^2dx=\frac{M}{L}\frac{1}{3}L^3=\frac{1}{3}ML^2$$

If you compare the two, they are different for the reasons we stressed before, the moment of inertial (as well as torque and angular momentum), the values depend on pivot of choice. You further take the difference between the two I's, and the difference is $\frac{ML^2}{4}=M(\frac{L}{2})^2$. This is no coincidence. It is the result of a general theorem on moment of inertia: **Parallel Axial Theorem**. It related the moment of inertia with respect to C.M. to moment of inertia with respect to a shifted pivot (the rotation axis is shifted to a distance d away from C.M., but the axis is still along the same direction).



$I_o=I_{CM}+Md^2$ (7-15)

(d is distance from o to C.M., and o may not even on the body)

The proof is easy:

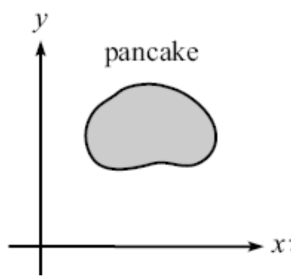
$x_o = X_{CM} + x_{cm}; y_o = Y_{CM} + y_{cm}$

X_{CM},Y_{CM} are the coordinates of C.M. with respect to O, x_{cm} and y_{cm} are the coordinates of points with respect to C.M.

$$I_o = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i[(X_{cm} + x_{icm})^2 + (Y_{cm} + y_{icm})^2]$$
$$= \sum_i m_i(x_{icm}^2 + y_{icm}^2) + \sum_i m_i(X_{cm}^2 + Y_{cm}^2) + 2X_{cm}\sum_i m_i x_{icm} + 2Y_{cm}\sum_i m_i y_{icm}$$

The last two terms are zero because of the definition of C.M., it is the property we invoke a few times before. The first two terms are just the two terms in (7-15).

The parallel axial theorem can be applied to any object, but has to relate to the I_{CM}. There is another theorem only deals with 'pancake' shaped object (thickness can be neglected), this is called perpendicular axial theorem which will work for any pivot point on the 'pancake':



Pick any pivot point that the object will rotate, and if the rotation axis is along z direction (out of the paper):

$$I_z = \sum_i m_i(x_i^2 + y_i^2)$$

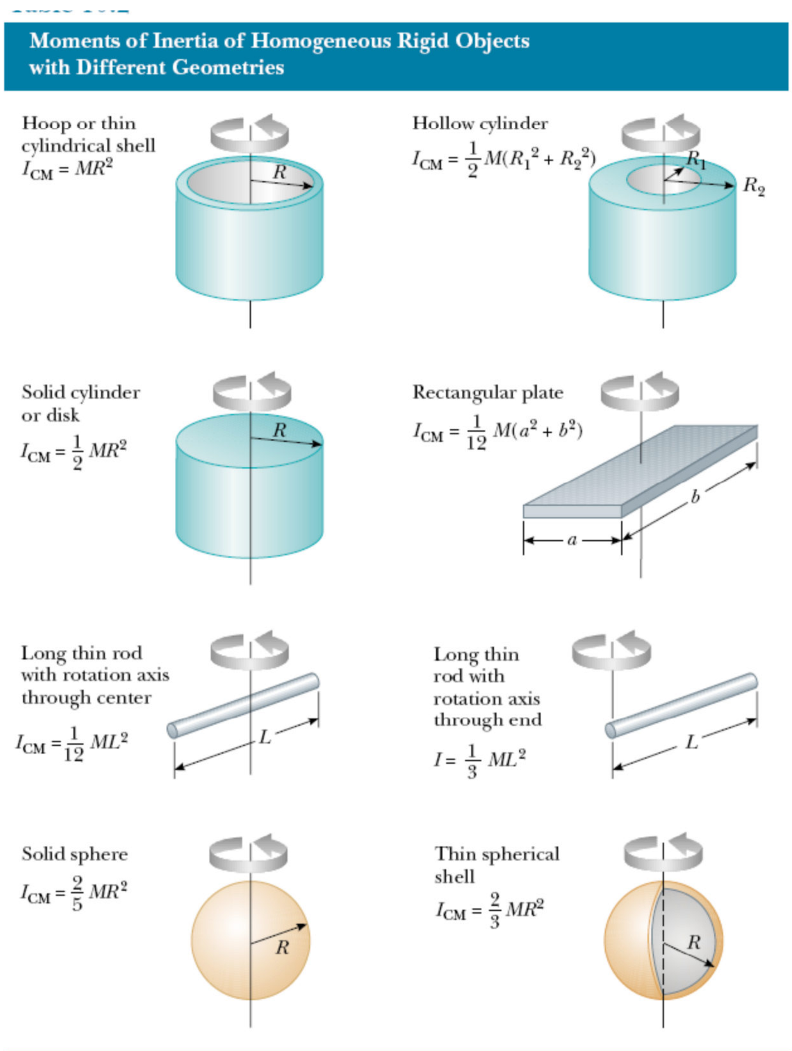
For the **same pivot**, if the rotation axis is x or y axis, we can similarly define moment of inertia around those axes's:

$$I_x = \sum_i m_i(z_i^2 + y_i^2), I_y = \sum_i m_i(x_i^2 + z_i^2)$$

For the pancake shaped, $z_i=0$. We have:

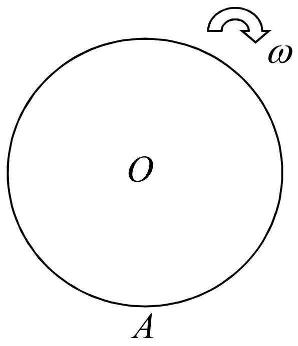
$$I_z = I_x + I_y \qquad (7\text{-}16)$$

The definition and the two theorems would allow you to compute the moment of inertia of different shaped objects with respect to different pivot and axis. Table below list the *I*'s for some of typical objects. You should confirm them yourself, mostly do the integrals using Cartesian, Cylindrical and Spherical coordinates[61].

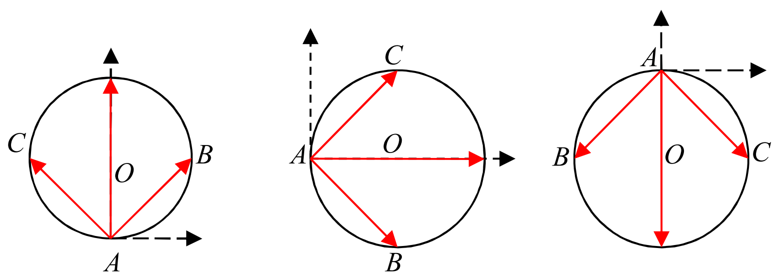


[61] For more examples, please refer to Morin's section 7.3.

We have seen that the moment of inertia depends on the pivot and axis of rotation. How about angular velocity? Consider the following situation:



A circular disk with a fixed pivot at the center O, rotate with angular velocity around z-axis. Imagine we fixed a poor bug at the edge of the disk at A (the bug A will move along with the disk). From the point of view of this bug, what is the motion of the disk? And if it is a kind of rotation what is the angular velocity? You may not give the answer right away, since the rotation is not as intuitive as the translation. The following pictures may help:



The pictures show for 3 instants, the relative positions of the bug A and some points on the disk. Noticed the coordinates are having the bug as origins, i.e. in these coordinates the bug does not move. The direction of the axis is invariant with time (+x always points to the right, and +y always points upward). This frame is called **translational coordinate system**, because the origin may move (from the point of view of somebody in the inertia system) but the direction of axis's or the base vectors do not rotate. In such coordinate system (the bug at origin and the axes do not rotate), the whole disk would appear to the bug as rotating around it as the picture shows.

To further prove this, let's pick $AB = \vec{r}_{AB}$, because of rigid body, its length never changes: $|\vec{r}_{AB}|^2 = \vec{r}_{AB} \cdot \vec{r}_{AB} = cons$. Then the change of this vector could only occur due to directional change, no radial change possible: $\frac{d}{dt}|\vec{r}_{AB}|^2 = 2\vec{r}_{AB} \cdot {d\vec{r}_{AB}}/{dt} = 0$. And from (7-4), the velocity could be expressed as: $\frac{d\vec{r}_{AB}}{dt} = \omega_A \times \vec{r}_{AB}$. So it appears AB does rotate around A with some angular velocity ω_A. How about other points relative to A, well from the rigid body, we know that the relative relations between the points should not change (the distance fixed means the shape is fixed, i.e. if the 3 sides of a triangle is fixed, so will the angles), this means the angles between AB, AO,AC will not change over time. This leads to that the AO,AC will rotate with same angular velocity as AB. Now what is the angular velocity with respect to A, ω_A? It equals the ω_O, the original angular velocity with respect to the fixed pivot. This may not be intuitive to you (at least not to me), we should prove it: Let's pick AO(since all rotate with same angular velocity),the linear velocity of O with respect to

A in the A coordinate is: $|v_{AO}| = R|\omega_A|$. R is the radius of the disk. This relative velocity from the coordinate with o as origin is: $|v_{OA}| = R|\omega_O|$, the two magnitude should be same and this gives $|\omega_A| = |\omega_O|$. As to the direction of rotation, we see from the figure both case the rotations are clock wise, then $\omega_A = \omega_O$.[62]

From the above argument, we see that

(1) For the observer on the body (move along with the body in a *translational* type coordinate), the motion would be a rotation with **same angular velocity.** Even if the A is not on the body, as long as it moves along with the body rotates (as if connected to the body with massless rod), the above argument will also apply.

(2) The A rotates with the body experience acceleration, so the coordinate system (even translational type) built with it as origin is not inertial. Though the kinematic properties discussed above apply, you should worry about what happened to the dynamic rules based on Newton's laws? Generally, it has to be modified to include the effect of non-inertial frame (by introducing the fictitious inertial force). However, we shall show later for a very special point of the body, the C.M., even when it is accelerating, the dynamic rules are just same as inertial frames.

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[62] This can be proved more elegantly later with the relation like $\vec{v}=\vec{\omega}\times\vec{r}$. Then in the O origin, the velocity of any point (say B)with respect to A is: $\vec{v}_{AB} = \vec{\omega}_O \times \vec{r}_{AO} + \vec{\omega}_O \times \vec{r}_{OB} = \vec{\omega}_O \times (\vec{r}_{AO} + \vec{r}_{OB}) = \vec{\omega}_O \times \vec{r}_{AB}$. In the A origin, the relative velocity is: $\vec{v}_{AB}=\vec{\omega}_A\times\vec{r}_{AB}$. Since r_{AB} is arbitrary, this will give us the two angular velocities are same.

If we choose a fixed point in the inertial frame, say still the A, but this time the bug is not attached to the spinning disk, it just stay on the ground watching the disk spins. This would appear a bad choice of origin to describe the rotation motion of the disk. The origin and the disk do not form a rigid body in this case, the AB, AC distance change with time. There is no single angular velocity to describe the rotation from A; AO never moves but AB,AC will and their angular changes are not simple with respect to A. We are going to see that later there is a way to solve this difficulty (if we have to choose such A as origin in the first place) by decomposing the motion as translational motion of C.M. with respect to A, and the rotational motion with respect to C.M. (this will be covered in section 7.2).

There is another issue about choice of coordinate system. We discussed what happened if we shift our origin, how about coordinate axes? The rotation not only depends on the origin but also depends on how we set up the axis. All above, I stressed that the axis of the coordinate will not rotate. i.e. their direction never changes, this is what I called translational type coordinate axis. If we set up axis that rotates, the angular velocity observed will be quite different from that of translational type. In the rotational type axis (I call this **rotational type coordinates**), the angular velocity will depend on the rotation of axis too. If the axis rotates with same angular velocity, then the apparent angular velocity in the rotating frame would be zero. But we have to keep in mind that this rotation type coordinate system is non-inertial. The corrections (in order to apply the Newton's laws for dynamical problems) for this non-inertial would be a bit more nasty than the translational type. It may appear simple for kinematic description of the rotation (i.e. angular velocity or angular momentum appears zero), special cares need to be taken in dynamical problems due to the inertial forces. So in this chapter, I shall avoid such rotational type coordinate as much as possible. If I do not specify the coordinate axis, it is assumed that it is translational type (i.e. the directions of axes do not change over time, though the origin may move). You are encouraged to draw the rotational type coordinate with the 3 disks figure above (for origin O and for origin A) to see the difference between the rotational and translational coordinates.

The reason that I blibber-blubber about the choice of origin and coordinates is because this is the source of most confusions and mistakes arise in solving rotational problems. Because the expressions of moment of inertia, angular momentum, angular velocity and torque all depend on such choices, and indeed the choices can have dramatic effect on solving the problems (The physics involving vectors do not depend on choice of coordinate systems, but the method to get the correct answer does, and naturally we want to use the easiest and safest).

The obvious choice for the 2-D pure rotation considered here would of course be the pivot point O and translation type axis. In general cases, the easiest and safest choice are usually *1) the fixed pivot point on rigid body in inertial frame around which the object rotates; Or 2) the C.M. of the object as origin for rotation part (this is especially true when the motion is not pure rotation, but involves translation and rotation in the most general cases).*

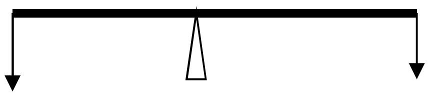
As to the coordinate axes, I shall use translational type as much as possible. We shall see the reason from discussions in later sections and you may need to come back to reread this section later.

7.1-3 Examples for 2-D Pure Rotation and Conservation of Angular Momentum

The most straightforward type that providing torque and object shape and pivot (last two combined will give you moment of inertia; or the last two combined with force distribution would allow you to calculate the torque), finding its angular acceleration and velocity, will not be discussed; it is just like solving the translational motion in 1-D given the force and mass.

Example 1: Equilibrium condition for static object.

Before when we only consider the translation, the object may remain stationary if all the forces add up to zero. Now including the possible rotation, we require besides the *total forces are zero*, an additional condition that the *total external torque need to be zero too* for stationary object. This would be used to solve problems like see-saw type:

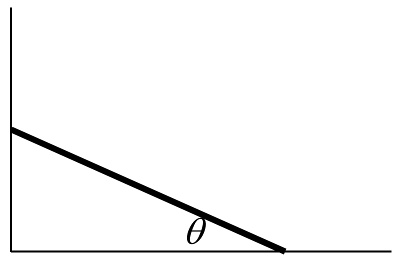


With no external force, where should we put support?

The answer is under the C.M. of course. If you take pivot as origin, this will make the zero torque. Note this argument applies an important fact that needs to be proved: A uniform force proportional to mass (gravity here) applied to an object; the torque with respect to the pivot is equivalent to all the forces applied to the C.M. The proof is in KK example 6.7, it used definition of torque as cross product which I have not formally introduced. An argument following this result is that if the C.M. has acceleration, the inertial force will be in forms of –ma, which is also gravity like for translational type coordinates. So the torque by such inertial force will be acting as if on the point of C.M., and if you take C.M. as origin, then this inertial torque will be zero. The coordinate with C.M. as origin behaves as an inertial frame for the consideration on rotation.

Under the situation of stationary object (object at balance or equilibrium), actually you can choose any point as origin for the analysis, the total external force=0 will guarantee that the total external torque will be zero with respect to any origin (exercise for you to prove, KK problem6.1).

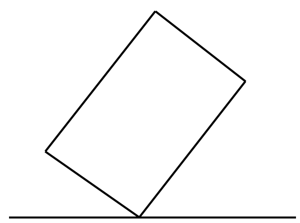
There are more variations for this type of problem, such as putting forces on the ends of see-saw, and find out the forces under balance etc. The strategy is just picking an origin and analyzing both the force and torque. Another similar example would be:



A uniform ladder with mass m leans against wall and ground. The friction force along the vertical is assumed negligible, the friction force on the ground cannot exceed $\mu_{static}N$, then find the minimum angle that the rod can keep balance.

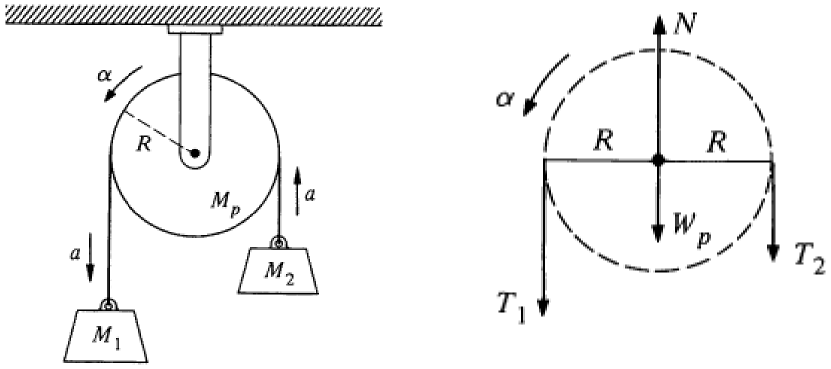
It is probably easiest to choose the end attached to the vertical wall as origin. But you can also choose C.M. as origin. Do it yourself either ways. (the answer would be threshold angle is $\cot\theta = 2\mu_{static}$)

Another typical one is:



Find the maximum tilt angle that the object won't fall over. It is easy for the above case by considering the position of C.M. Picking the support as origin, if C.M. is to the right of the support, the torque of gravity will create c.w. rotation and the object falls over. If the C.M. lies to the left of support, the object will fall back. A little bit nastier version is in problem 6.35.

Example 2: Atwood machine with massive pulley (KK example 6.10)



The details are in the textbook and it basically combine force analysis for M1 and M2, and torque analysis for the pulley. Two points need to be discussed: 1) Noticed that the tensions at two ends of the rope are not equal. This is because the friction force between the rope and pulley. Without this friction, the pulley will not rotate, and you will just have a Atwood machine for a massless pulley. The friction is the source of torque to the pulley, and the torque value can be calculated as (T1-T2)R (T1, T2 is not the direct source of torque on pulley, the difference T1-T2 is the friction force on the pulley). 2) The condition of rolling without slippery. This is saying that the translational distance of the rope would be same as the arc length of the rotation. $\Delta l = \Delta s = R\Delta\theta$, then you will have relations such as: $v = \frac{dl}{dt} = R\frac{d\theta}{dt} = R\omega$, and $a = R\alpha$. Which is the constraint relation imposed by rolling without slippery condition. With these two explanations, you will have no trouble to solve this problem.

I will ask the similar question with slight different known variables. Suppose that initially all are stationary, and then the masses will fall, rise, the pulley will rotate with no slippery. If the mass M1 falls by distance h from the original position, what is the velocity of it?

This is essentially the same problem as before, there you calculate the acceleration and if the mass M1 travels h with the calculated acceleration, the velocity is easy to be determined ($v^2 - v_0^2 = 2ah$). This is indeed the method I want you to use initially. Now I want to use energy conservation to do it: $-\Delta U = \Delta K$, $(M_1 - M_2)gh = \frac{1}{2}(M_1 + M_2)v^2 + \frac{1}{2}I\omega^2$, this will give me the answer quickly. And the striking thing is that this will give the exactly the same answer as you get initially (try both methods to convince you if you have not worked this problem before). I say it is striking because the mechanical energy conservation works with the presence of friction. The friction force between the rope and pulley which as analyzed above is necessary for the pulley to rotate, but this friction force does not create heat loss. This is because we are in a very special condition, namely rolling without slippery. The friction does not generate energy loss in such condition; its work is to transform the translational kinetic energy into rotational kinetic energy. I'd better prove this once for all. For simplicity I only consider the work by the friction force (the gravity is turned off or cancelled by support, or in this case its work is included in the potential change). For the M1 and M2 here, the work by friction force is:

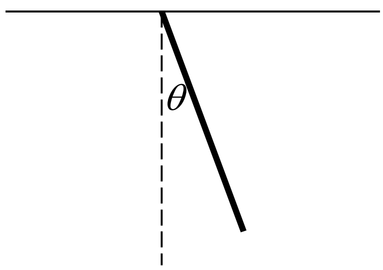
$F_{fri}\Delta l = \Delta E_{M_1,M_2}$ (actually T1, T2 do the work on M1, M2;but combined is equivalent to friction force)

The torque on the pulley is $\tau = F'_{fri}R$, and its work is:

$\tau\Delta\theta = F'_{fri}R\Delta\theta = F'_{fri}\Delta l = \Delta K_{pulley}$

The rolling with no slippery is crucial in the above equation. From the 3[rd] law, $F'_{fri} = -F_{fri}$, you add up the two will get the mechanical energy conservation of the pulley+mass system.

Example 3: Physical Pendulum



I assume you all know for a pendulum by a massive particle connected by the massless string with length L, the angular frequency ϖ and period T of oscillation is (for small angle): $\varpi = \frac{2\pi}{T} = \sqrt{\frac{g}{L}}$

Now we hang a uniform rod and we can neglect any friction at the pivot (so the rod can oscillate forever), what is the angular frequency for this pendulum (the pendulum with certain shape is called physical pendulum in contrast to the idealization of mass point model)?

At first look. It may appeal to you that treating the rod as a mass point with C.M, at L/2, and it is like a mass point m with string of L/2, and the

$\varpi = \frac{2\pi}{T} = \sqrt{\frac{g}{L/2}}$. Well this is on the right track but the answer is off the target. The fact that it cannot simply treated as a mass point is because in the C.M. frame, there is rotation around C.M. So there will be rotational kinetic energy along with the translational kinetic energy of the C.M. (Konig theorem). The potential energy drop will be distributed into translational energy of C.M. and rotational energy around C.M. The result is less translational energy, less speed and so longer period. So the simple model above won't work and we have to consider it from torque-angular momentum change.

Choose the pivot as origin, since I do not want to investigate the forces there. External torque would only have contribution from gravity with this choice of origin:

$$\tau = -mg\frac{L}{2}\sin\theta$$

(the reason of minus sign is this torque creates c.w. rotation as drawn)

The angular momentum is (shoot, I just realized I choose L for length and I have to use non-convention A for angular momentum):

$$A = I\omega = \frac{1}{3}mL^2\omega$$

Apply the fundamental equation of motion for rotation:

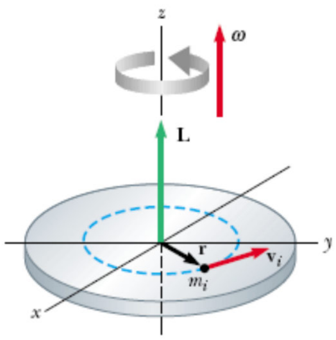
$$\tau = \frac{dA}{dt}$$
$$-mg\frac{L}{2}\sin\theta = \frac{1}{3}mL^2\frac{d\omega}{dt}$$
$$\frac{d^2\theta}{dt^2} + \frac{g}{2L/3}\theta = 0$$

This will give us $\varpi = \frac{2\pi}{T} = \sqrt{\frac{g}{2L/3}}$, so the equivalent mass point pendulum would be with string length of 2L/3.

This example is a special case of physical pendulum discussed in KK, 6.6.

Example 4. Conservation of Angular Momentum

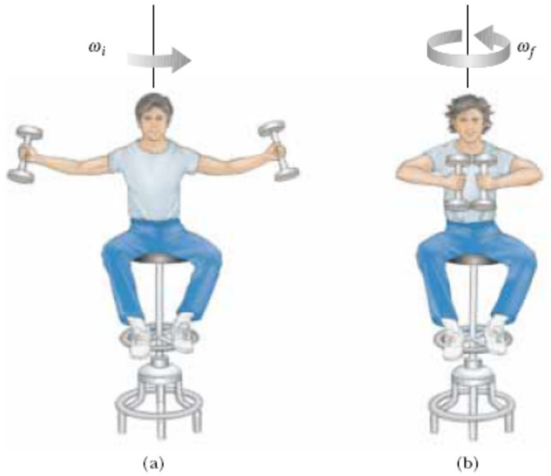
From the fundamental equation of rotation, $\tau_{ext} = \frac{dL}{dt}$, we see that under the situation of no external torque, the angular momentum will be conserved.



Considering a disk rotating around axis as shown with initial angular velocity ω, now if I drop on top of it another identical disk and finally the two disks will rotate together again with another angular velocity ω', find the ω'. Or I spit a gum to the original disk and the gum will rotate with the disk as that little black dot, what is the final angular velocity?

You can solve these easily with conservation of angular momentum.

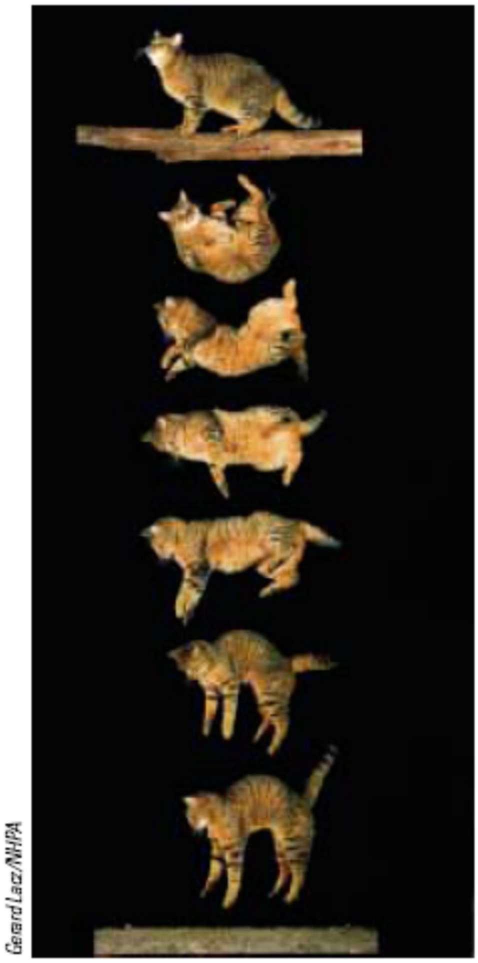
Another favorite example to illustrate the conservation of A.M. is figure skating athlete produces fast rotation by changing moment of inertia:



(I am sorry I did not find a nice picture with pretty girl figure skating, so please instead bear with me with this masculine guy)

It is easy to understand the guy increases the angular velocity by pulling the dumbbell inward, make smaller moment of inertia. Suppose $I_f = \frac{1}{2} I_i$, then from conservation of A.M.: $I_i\omega_i = I_f\omega_f \rightarrow \omega_f = 2\omega_i$. This is nice and clear. Now let's further ask the question about energy: $\frac{1}{2}I\omega^2$. It is clear that the final energy will be twice as big as initial one: $K_f = 2K_i$. Where does the increase of energy come from? An analogous scenario is playing with the swing, in order to get higher and higher on a swing, the person need to stand tall at lowest point and stay low at the highest point. Please give the reasoning yourself.

Another interesting example is the legend cat landing always on 4 feet when falling, as the figure below, I will let you to figure out how the kitty achieve this incredible gymnastic performance.



7.2 Formal Definition of Angular Momentum, Torque and General Motion of Rigid Body in 2-D

In the last sections, I started from the simplest case in rotation, treating pure rotation of rigid body in 2-D. We define the angular velocity and from there introduced moment of inertia and angular momentum, then derive the fundamental dynamic equation for rotation, torque=change rate of angular momentum,

In this section, I shall start from the formal definition of angular momentum, since like linear momentum, angular momentum is a fundamental physical quantity. We shall see that the angular momentum for 2-D rigid body rotation is just a special case of the general definition. We will also re-derive the fundamental equation for rotation with the formal definition of angular momentum and torque, and stress on what kind of choice of origin this equation will apply. We shall also derive a formula in treating motions involving both translation and rotation, a formula similar in spirit to the Konig Theorem in energy (the kinetic energy of a ensemble of particles is a summation of the kinetic energy of C.M., and the kinetic energy of individual particles relative to the C.M, review the discussion leading to equation 6-32)[63]: The total angular momentum of the system is a summation of the angular momentum of the C.M.(as a mass point with total mass), and the angular momentum of individual parties relative to the C.M. (i.e. the angular momentum in C.M. frame). From these we can solve the motion of a rigid body in 2-D with translation and rotation. In next section (7.3), we will treat the general motion involving rotation in 3-D for rigid body.

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[63] Actually there is also in the same spirit, a theorem for linear momentum. The total momentum can be treated as summation of momentum of C.M. as whole and momentum of individual parties relative to the C.M. In this case, the property of C.M. frame makes the second contribution zero.

7.2-1 Angular Momentum

In this part I shall treat the general definition of angular momentum, not limited only to the rigid body. Actually the property of rigid body guarantees one ω, the angular velocity for all parties on the body, so if a relation below does not invoke angular velocity, it is a general case. If it invokes angular velocity ω, it refers to rigid body.

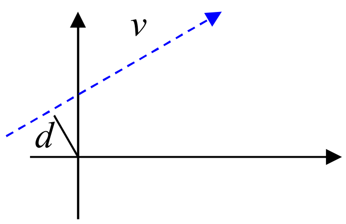
The angular momentum of single particle is defined as:

$\vec{L} = \vec{r} \times \vec{p}$ (7-17)

$\vec{r}$ is the position vector relative to certain origin, $\vec{p}$ is the linear momentum, and in mechanics, $\vec{p} = m\vec{v} = m\dfrac{d\vec{r}}{dt}$, so:

$\vec{L} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$ (7-18)

Angular momentum is the cross product (it is a good time for you to review your cross product if you need to) of two vectors, itself is a vector perpendicular to both $\vec{r}$ and $\vec{p}$ (to the plane formed by $\vec{r}$ and $\vec{p}$). From its definition, it is clear that L depends on origin of choice (same as we talked last section), consider the following simple example:



The particle is traveling with constant velocity v along the dashed line in the figure. If we choose the origin as shown, the angular momentum is:

$\vec{L} = m\vec{r} \times \vec{v} = -mvd\hat{z}$

But if you choose the origin on the line of v, the L would be zero. That is why in the discussion of translations, we do not invoke the concept of angular momentum. KK example 6.2 shows another example that angular momentum depends on choice of origin.

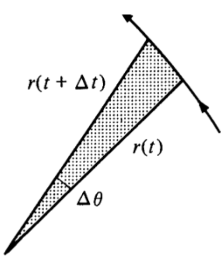
In the simple case of a particle moving around center with uniform velocity (c.c.w), choose the center as origin:

$L = m\vec{r} \times \vec{v} = mr^2\omega\, \hat{z}$

This is just like a rigid body 2-D rotation ($I = mr^2$). Actually you can construct a rigid body, connecting the pivot with mass point with a massless rod. But now consider the general case that the mass point's orbit may not be circular, and the definition of angular momentum still allows us to calculate the angular momentum of the particle with respect the center. It is still in the form as above, i.e. $L = m\vec{r} \times \vec{v} = mr^2\omega\, \hat{z}$.

(prove this yourself or check KK example 6.3)

Another example would be Kepler's 2nd law on motion of planets around the sun: it states that the area velocity is a constant (see KK example 6.3 for detail). This is equivalent to conservation of angular momentum:



The area swept by the planet within a short time interval can be approximated by a slim triangle, with $\vec{r}$ is one side and $\vec{v}\Delta t$ as another (the short one close to the arc length). The area of this triangle is:

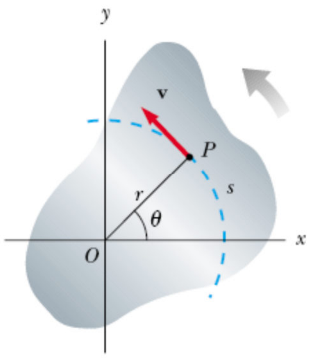
$|A| = \frac{1}{2}|\vec{r}\times\vec{v}\Delta t| = \frac{1}{2}\Delta t\frac{|\vec{L}|}{m}$ (use the geometry meaning of cross product)

$\frac{d|A|}{dt} = \frac{|\vec{L}|}{2m} = cons.$, so the magnitude of L will be same for all time. Or from the conservation of angular momentum under central force, Kepler's 2nd law is a natural result.

For multi-particles or a collection of them, the definition of total angular momentum is:

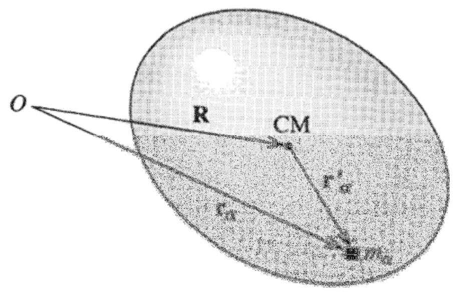
$$\vec{L}_{total} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i m_i\vec{r}_i \times \vec{v}_i \qquad (7\text{-}19)$$

For the 2-D rigid body rotating around a pivot:



The above definition will give exactly $L = I\omega$. (prove it yourself, noticed the rigid body requires the |r| is fixed)

Also let's further study the (7-19), to see what happened to angular momentum if we shift the origin for a collection of particles. Let's assume that the angular momentum of some object (not necessarily rigid) is measured with respect to some fixed origin O, L_O. Now we shift the origin to some other point O', what in particular interest is this new origin overlaps the C.M. of the object. Please refer to the figure below. The question is what the relation between the angular momentum with respect to these different origins?



r_i, v_i are referring to O; r_i', v_i' are referring to O'; R,V are displacement and velocity between O,O'. (Here O' can be any shifted origin, I have not invoked C.M. yet)

$$\vec{L}_O = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i m_i \vec{r}_i \times \frac{d\vec{r}_i}{dt} = \sum_i m_i (R + r_i') \times \frac{d}{dt}(R + r') = \sum_i m_i (R + r_i') \times (V + v_i')$$

$$\vec{L}_O = \sum_i m_i R \times V + \sum_i m_i r_i' \times v_i' + \sum_i R \times m_i v_i' + \sum_i m_i r_i' \times V$$

The above is correct for any shifted O', however, if the O' is the C.M., the relation is much simplified because the last two terms are zero (R,V are independent of the indices and can be taken out of the summation or integration). For O' is the C.M., we have:

$$\vec{L}_O = MR_{CM} \times V_{CM} + \sum_i m_i r_i^{cm} \times v_i^{cm} \qquad (7\text{-}20)$$

Or $\vec{L}_O = \vec{L}$(C.M. as a point to O)+$\vec{L}$(angular momentum in C.M. frame)

(7-20) is the relation similar to the Konig Theorem for the kinetic energy for multi-particle case. It speeds up the evaluation of angular momentum for any choice of origin. For example, when we study the earth rotating around sun and spinning itself, the angular momentum can be decomposed as orbit angular momentum (the C.M. of earth revolves around sun), and spin angular momentum. More important as we shall see below, the changes in both these angular momentums obey the fundamental equation of rotation. For that, we need torque.

7.2-2 Torque and Fundamental Equation for Rotation

The torque of single force acting on some place is defined as:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (7\text{-}21)$$

The force may be decomposed as parallel component and vertical component with respect to the position vector, only the vertical component contribute to the torque as we discussed in previous section. From definition of cross product: $|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\phi$, where ϕ is the angle between **r**,**F**, same as our 2-D case before. Also notice that the torque depends on choice of origin. F should be replaced with total force if many forces exist to get total torque.

For multi-particle system subject to different forces:

$$\vec{\tau}_{total} = \sum_i \vec{r}_i \times \vec{f}_i^{ext} \qquad (7\text{-}22)$$

The forces need to be considered in computation of torque only including the external forces, since the summation of torques by internal forces between the particles will add up to zero as discussed previously (3rd law and $\vec{f}_{ij}$ is parallel with $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$).

The reason to define torque as (7-21) and (7-22) is that this will give us the fundamental equation of rotation, equivalent to F=dp/dt.

(1) Single particle

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = (\frac{d\vec{r}}{dt}) \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad (7\text{-}23)$$

Noticed this applies to the inertial frame because we apply the 2nd law. The coordinate system has to be inertial, and torque and angular momentum have to be measured from a fixed point (the origin of the coordinate system)[64].

(2) Multi-particle system

From (7-19):

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \sum_i \vec{r}_i \times (f_i^{ext} + f_i^{\text{int}}) = \sum_i \vec{r}_i \times f_i^{ext} = \vec{\tau}_{tot} \ (7\text{-}24)$$

This also applies to a fixed point (the origin of the coordinate) in an inertial frame.

We have seen that the angular momentum can also be decomposed:

$$\vec{L}_O = MR_{CM} \times V_{CM} + \sum_i m_i r_i^{cm} \times v_i^{cm} = \vec{L}_{O-CM} + \vec{L}_{CM}$$

$\vec{L}_{O-CM}$ is the angular momentum of the C.M. behaving as massive point with respect to fixed origin O; $\vec{L}_{CM}$ is the angular momentum of particles with respect to C.M.

We can do the similar thing to torque:

[64] You may find an example in 李复'力学教程'(上), example 6.1.4.

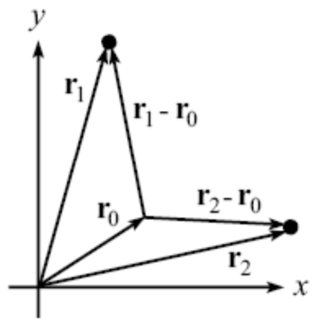
$$\vec{\tau}_{tot} = \sum_i \vec{r}_i \times \vec{f}_i^{ext} = \sum_i (\vec{R}_{cm} + \vec{r}_i^{cm}) \times \vec{f}_i^{ext} = \vec{R}_{cm} \times \sum_i \vec{f}_i^{ext} + \sum_i \vec{r}_i^{cm} \times \vec{f}_i^{ext}$$
$$= \vec{R}_{cm} \times \vec{F}_{tot}^{ext} + \sum_i \vec{r}_i^{cm} \times \vec{f}_i^{ext}$$

$\vec{R}_{cm} \times \vec{F}_{tot}^{ext}$ is the torque by total external force acting on the point mass at C.M., according to (7-23), it should be equal to $\frac{d\vec{L}_{O-CM}}{dt}$. And from (7-24), $\vec{\tau}_{tot} = \frac{d\vec{L}_O}{dt} = \frac{d\vec{L}_{O-CM}}{dt} + \frac{d\vec{L}_{CM}}{dt}$, then we have:

$$\frac{d\vec{L}_{CM}}{dt} = \frac{d}{dt}(\sum_i m_i r_i^{cm} \times v_i^{cm}) = \sum_i \vec{r}_i^{cm} \times \vec{f}_i^{ext} \equiv \vec{\tau}_{cm} \qquad (7\text{-}25)$$

This says the change of angular momentum in the C.M. frame equals to the torque measured in the C.M. frame. Basically the fundamental equation applies to the C.M. frame (with C.M. at origin), in spite of the fact that the C.M. may be moving and possibly accelerating, i.e. non-inertial. This is another remarkable property and advantage of the C.M. frame.

It is worth looking at the same result from another point of view.



The angular momentum is originally defined in the x-y inertial frame with respect to origin. Now consider the angular momentum L’ with respect to a point r_0, where r_0 may change over time:

$$\vec{L}' = \sum_i (\vec{r}_i - \vec{r}_0) \times (\vec{p}_i - \vec{p}_0)$$

With $\vec{r}\,'s, \vec{p}\,'s$ all are defined in the inertial frame.

$$\begin{aligned}
\frac{d\vec{L}'}{dt} &= \sum_i m_i(\dot{\vec{r}}_i - \dot{\vec{r}}_0) \times (\dot{\vec{r}}_i - \dot{\vec{r}}_0) + \sum_i (\vec{r}_i - \vec{r}_0) \times m_i(\ddot{\vec{r}}_i - \ddot{\vec{r}}_0) \\
&= \sum_i (\vec{r}_i - \vec{r}_0) \times m_i(\ddot{\vec{r}}_i - \ddot{\vec{r}}_0) = \sum_i (\vec{r}_i - \vec{r}_0) \times (\vec{f}_i^{ext} + \vec{f}_i^{\text{int}}) - \sum_i (\vec{r}_i - \vec{r}_0) \times m_i \ddot{\vec{r}}_0 \\
&= \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{f}_i^{ext} - (\sum_i m_i \vec{r}_i - M\vec{r}_0) \times \ddot{\vec{r}}_0
\end{aligned}$$

$\sum_i m_i \vec{r}_i$ is just the definition of C.M., it equals to MR_{cm}, so:

$$\frac{d\vec{L}'}{dt} = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{f}_i^{ext} + (MR_{cm} - M\vec{r}_0) \times \ddot{\vec{r}}_0$$

We see that if the possible moving point $\vec{r}_0$ is just C.M., then the second term is always zero and the first term is just the torque with respect to the C.M., which gives us exactly (7-25)[65].

There is one more subtlety in applying the (7-25), the torque-A.M. relation in C.M. frame. We see that we have to choose C.M. as origin in order to use(7-25), but that still leaves the question on the choice of coordinate axis (or the base vectors) for the computation. **It requires translational type coordinate axes**! The reason is (7-24) applies only for inertial frame. As stated before, I will choose ***translation type coordinated system with C.M. at origin*** (i.e. base vectors are unit vectors not changing with time) in application of (7-25). It is also possible to evaluate the change of vectors in the translational type

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[65] There will be other situations that the second term will be zero for some other special r_0 on the object, such as the contact point in a rotation without slippery, sometimes called instantaneous center, where the $(\vec{R}_{CM} - \vec{r}_0) \perp \ddot{\vec{r}}_0$

coordinate with choice of rotational type coordinate (you may wonder why taking trouble doing this, see the footnote 66 below). However, if you choose rotational type coordinate system (base vectors rotate with time), the computation of vector change over time will be a bit tricky, so I will leave that part when we deal with rotational frame when we talk about non-inertial frames next chapter[66].

7.2-3 General Motion in 2-D

Now we can treat the general motion, including both translation and rotation for rigid body. The reason for rigid body and 2-D is mainly because the rotation is easiest in this situation. I will leave the general rotational motion to section 7.3. The basic model is Chasles Theorem we mentioned at the beginning of this chapter, that the motion can be divided into translational motion of a mass point at C.M., and a rotation around C.M. From the discussion above we see that the rotation in C.M. frame

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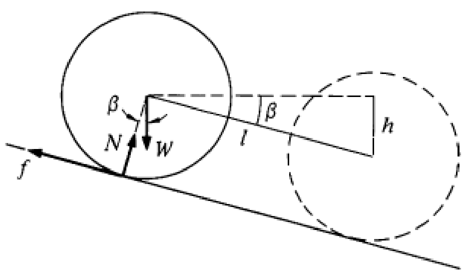
[66] If you read ahead or are just curious now what happened for the expression of time derivative of vectors in a rotational coordinates: There is a difference in the vector change viewed from translational type and rotational type coordinate: $(\frac{d\vec{A}}{dt})_{Lab}=(\frac{d\vec{A}}{dt})_{rot}+\vec{\Omega}\times\vec{A}$. $\vec{A}$ is a vector that can change over time: $(\frac{d\vec{A}}{dt})_{Lab}$ is the change of the vector viewed from a translational type coordinate (called lab frame here), $(\frac{d\vec{A}}{dt})_{rot}$ is the change of the **same vector** but viewed from rotational coordinates. $\vec{\Omega}$ is the angular velocity vector of the rotation of coordinate (viewed from lab). I shall leave the detailed proof later. Just by looking at it, it makes sense. Consider a constant vector $\vec{A}$ in the lab frame, it won't change. However in the rotating frame, it is changing, rotating with $-\vec{\Omega}$. On the contrary, if the $\vec{A}$ appears constant in a rotating frame, it will appear rotating with $\vec{\Omega}$ in lab frame. To avoid such complexity, I will stick to translational type coordinates as much as possible. To say as much as possible instead of absolutely always, is because in the general treatment of rotational motion, such as Euler equation, we have to adopt rotational type coordinates(the reason will be given in section 7.3). Since we are not going too deep on this, it won't be a big issue here.

obeys fundamental equation, thus all formula we derived in section 7.1 (treating the pure rotation of rigid body in 2-D) would apply here in the C.M. frame! (Please review the relations in section 7.1, or the table 7.1 if you finished it yourself). The motion of C.M as a massive point is just the motion of mass point we are already familiar with (with Newton's 2^{nd} law, or apply momentum, energy and angular momentum you learned in this chapter); the rotational motion in C.M. frame is reduced to what we discussed in 7.1 for 2-D (It obeys (7-25) and formula in table 7.1). This is best illustrated by some examples.

In KK example 6.14, 6.15, 6.16, they solved problems with two methods. Method 1 is what I preferred and follows the strategy outline above. Method 2 is choosing a fixed point in lab frame, and solves it from fundamental equations of rotation. Both are equivalent, you pick your preference.

Here are some more examples:

Example 1: Similar to KK 6.17



The disk rolls without slippery down an inclined slope, it drops in height h, what is the velocity of the disk?

You can solve this by C.M. point translation and rotation in C.M. frame, the analysis of forces and torque will give you equations of motion, and with the constraints of rolling without slippery, you can solve for the acceleration of translation of C.M. point, and rotational acceleration, then the velocity after traveling a distance $l = h / \sin\beta$ can be computed.

The easier way is to use energy conservation, we have already discussed that the energy conservation holds under rolling without slippery. The kinetic energy are consisting of two parts (Konig Theorem): Translational energy of C.M. point and rotational kinetic energy of the disk. If everything starts at stationary, then the increase of kinetic energy equals to the decrease of potential:

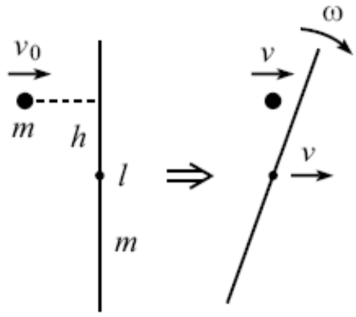
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}mb^2)(\frac{v}{b})^2 = \frac{3}{4}mv^2$$

(note: $v = -\omega b$ here from definition of positive of v and ω, it does not matter in the square)

The translation velocity will be smaller to that of a mass point with no rotation.

Consider another example (similar to KK problem 6.30): If I throw a disk with radius b, mass m with initial velocity v_0 on a horizontal friction surface, due to the friction the disk will reach the stage of rolling with no slippery, find its velocity at that moment (please think about yourself before peek the hint in the footnote)[67].

Example 2: Elastic collision involving rotation:



As the figure shows, a mass point m with initial velocity v_0, hit a uniform rod with mass also m, the hitting happened at h above C.M. of the rod. The collision is elastic and the mass point will not change direction after collision. The whole system is on a surface with no friction. Then what is the motion of the system after collision? i.e. what are the velocity of mass point, the motion of the rod (the translation velocity of C.M., and angular velocity, sorry in the drawing the two velocities are same)?

There will be 3 unknowns the v_m of the mass point, the v_{cm} and ω. We will need to invoke all conservation laws. There is no external force, so the total momentum and angular momentum for the rod+mass point system will be conserved, so is the mechanical energy for the elastic collision.

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[67] You will need impulse theorem for both linear and angular momentum. The energy is not conserved here, because during the skidding process before reach rolling without slippery, the friction will do negative work to the disk. $f\Delta t = m\Delta v$, $\tau_{cm}\Delta t = fb\Delta t = I\Delta\omega$, when rolling with no slippery, $v = -\omega b$ and you will get answer by cancelling Δt . Of course you can also solve it from conservation of angular momentum by choosing a fixed point in lab frame where the torque due to friction is zero, the weight and support force will cancel in torque so the angular momentum with respect to this origin will be conserved. Both methods are equivalent.

Conservation of liner momentum (in a lab fixed frame):

$$mv_0 = mv_m + mv_{cm}$$

Conservation of energy (in lab fixed frame):

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_m^2 + \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

(Konig theorem is applied in the rod's kinetic energy part)

Conservation of angular momentum (you may use impulse theorem for the change of angular momentum and relate to change of linear momentum, the results would be same): This is to find the relation between ω, v_m, v_{cm}, there are quite a few ways to achieve this using A.M. conservation, since you have choices of origin (they will of course all give same answers provided you do it correctly):

(a) Choose the origin overlap and travels with C.M.(along x only) i.e. moving with v_{cm} in the lab frame.

In this choice of origin, the initial angular momentum is (from mass point and the rod. Don't forget the rod, it would appear translating with $-v_{cm}$ before the collision with respect to this choice of origin, but it turns out zero, because of r, v are parallel. I still write 0 below to stress this point):

$L_i = -m(v_0 - v_{cm})h + 0$ (the signs follows right hand rule)

After collision, the mass point appears v_m-v_{cm} to C.M. of rod, and rod angular momentum would be a pure rotation:

$$L_f = -m(v_{cm} - v_m)h - I_{cm}\omega = -m(v_m - v_{cm})h - \frac{ml^2}{12}\omega$$

$$-m(v_0 - v_{cm})h = -m(v_m - v_{cm})h - \frac{ml^2}{12}\omega$$

$$mv_m h + \frac{ml^2}{12}\omega = mv_0 h$$

Combined with conservation of linear momentum and energy, all unknowns can be solved.

(b) Choose the origin a **fixed** point in lab frame, overlap initially with the C.M. of rod:

$$L_i = -mv_0 h$$

$$L_f = -mv_m h + R \times mv_{cm} - \frac{ml^2}{12}\omega = -mv_m h - \frac{ml^2}{12}\omega$$

(the contribution of the C.M. of the rod after collision with respect to this origin is zero, because R is parallel with v_{cm})

Conservation of A.M. will give you exactly same results as in (a).

(c) Choose a **fixed** point in lab frame initially overlaps the lower end of the rod as origin. (this will be left as an exercise for you to finish, you should get same relation, also noticed that you cannot choose the origin as a point moving with the lower end of the rod or if you insist doing so, then the inertial forces has to be taken into consideration)

I hope this will illustrate the subtleties and possible pitfalls in computation of angular momentum, so please specify your choice clearly in solving problems. The fixed points in inertial frames and C.M. of the rigid body are all valid choices (because fundamental equation apply to these choices), but the computation details may vary.

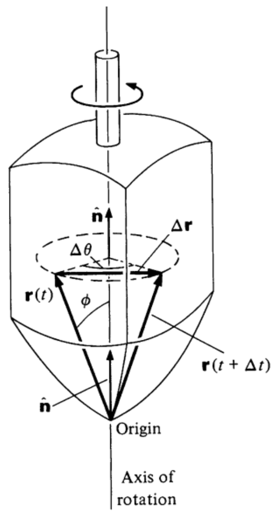
These (section 7.1 and 7.2) conclude our easier parts in the discussion of angular momentum and rotation. We are going to wade into some deeper water in the next section. I hope you are better prepared by what you have learned so far. If not…eh, ^^! drowned?

7.3 General Rotation of Rigid Body

In this section, I shall talk about the rotation of rigid body in most general case, not 2-D object, fixed axis rotation necessary. I shall first give a detailed discussion on angular velocity vector, a new definition of angular velocity and also one of the most important relations in rigid body; Next I shall relate the angular momentum vector to the angular velocity vector, a new 'beast' called moment of inertia tensor will emerge and you will see how we handle it. We are also going to derive the Euler equation that handles the general rotational motion.

7.3-1 Angular Velocity Vector

The note 7.1 in KK proves that though the finite angular displacement is not a vector, the angular velocity is a vector. The angular velocity of rotation with respect to the rotational axis is defined as:



$$\vec{\omega}=\frac{d\theta}{dt}\hat{n} \qquad (7\text{-}26)$$

$\hat{n}$ is the unit vector specifies the direction of rotational axis, θ is defined increasing according to right hand rule. i..e. facing the positive direction of the $\hat{n}$ (looking down in the above figure), if the rotation is counter-clockwise, then θ increase. This is same as we defined angular velocity in 2-D case (there the $\hat{n}$ is always the +z direction, here $\hat{n}$ could be any direction in a frame).

If we choose the origin on the rigid body that rotates with it: then the position vectors will be a vector with fixed length, and the velocity is:

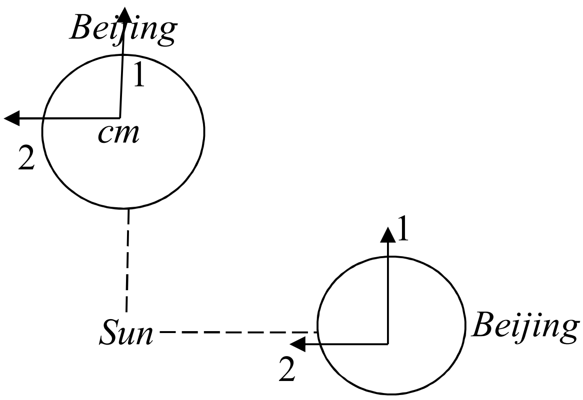
$$\vec{v}=\frac{d\vec{r}}{dt}=\vec{\omega}\times\vec{r} \qquad (7\text{-}27)$$

Here I just use (3-44) where replace the general ***fixed length*** vector **A** with $\vec{r}$ in rigid body. A proof with simple geometry and definition of cross product is also provided in the KK pg 290. Actually in the proof, there is nothing particular about $\vec{r}$, you can replace it with any ***fixed length*** vector **A** and get (3-44).

(7-27) is simple and important, most of formula in this section will originate from it, and I call it the most important relation for rigid body. In principle, we can choose any origin that rotates along with the rigid body (remember the bug moves with the disk?), but to apply the fundamental equation, we generally only choose a point on the body which is also fixed in inertial frame (then the point has to be on the axis of rotation) or the C.M.

(7-27) can also be used as definition of angular velocity, since we know the definition of v and r very well. This is useful when sometimes the rotational axis is not obvious. This definition is also useful when we combine the angular velocities together. The $\vec{\omega}$ is a vector and we expect it can be decomposed into components as vector does, which is true. When we have motions involve a few $\vec{\omega}$'s, we expect they add up as vectors, which is also true. But this last point is far from intuitive, so I'd better explain it more clearly.

Let's take a look of an example to illustrate this not-so-intuitive addition of angular velocities: Consider a very general case like earth rotates around the Sun. Let's take the Sun is motionless and can be the origin of an inertial frame. The earth rotates around the Sun and also spins around North-South pole. How we describe the motion of earth in solar system? We know the answer from the last section: decompose the motion into C.M. point of earth around the Sun and rotation with respect to C.M. of earth (we cannot treat the motion of earth in solar system as pure rotation, since viewing from the sun, the r's are not fixed in length). That is correct. The C.M. point revolves around the Sun with angular velocity $\vec{\Omega}$ ($|\vec{\Omega}|=2\pi$/year). What about rotation with respect to C.M. of earth? I should state this more clearly: If we choose a translational type coordinate whose origin is C.M. of earth, what is the angular velocity of earth in this coordinate system? It is tempting to answer it $\vec{\omega}$, pointing along the N-S pole, with $|\vec{\omega}|=2\pi$/day. But this is **not** the correct answer! $\vec{\omega}$ is not the angular velocity of rotation of earth in the translational type coordinate at C.M., it is the angular velocity in a rotational coordinate centered at C.M. of earth, the axes rotate as the earth moves around the Sun. Even when earth does not spin around N-S pole, the revolution around the Sun will appear as a rotation for the translational C.M. coordinates. In the figure below, I setup a C.M. coordinate (I am tired of put "translational" ahead every time, so assume it if I do not mention otherwise). *Assume no spin of earth* itself, then the point A on earth (say Beijing) will be always facing away from the Sun. As earth revolves, the motion of earth in the C.M coordinate will be a rotation with $\vec{\Omega}$:



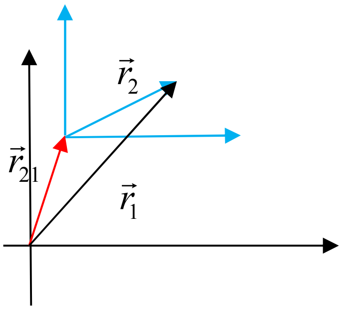
1,2 are directions of axis's (or base vectors). With the inspection of angle change, the rotation of *spinless* earth in the C.M. frame has angular velocity $\vec{\Omega}$. Now if we turn on the spin of the earth, the total angular velocity in the C.M, coordinate will be $\vec{\Omega}+\vec{\omega}$.

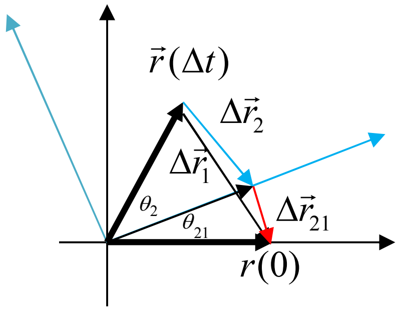
Actually there is a theorem on this kind of addition of angular velocity just like addition of translational velocity between translational frames. It says: *If an object rotates around some axis with angular velocity* $\vec{\omega}_{obj-2}$ *in frame 2 (just a coordinate system I call it 2); and the frame 2 is rotating around some axis with angular velocity* $\vec{\omega}_{21}$ *with respect to frame 1. Then the angular velocity of the object in frame 1 is:*

$$\vec{\omega}_{obj-1} = \vec{\omega}_{obj-2} + \vec{\omega}_{21} \qquad (7\text{-}28)$$

Using this theorem, we can get the earth rotation in C.M. frame quickly. Frame 2 is the rotational C.M. frame, and $\vec{\omega}_{spin} = \vec{\omega}_{obj-2}$, $\vec{\Omega} = \vec{\omega}_{21}$.

I'd better prove 7-28, with the angular velocity is defined as in (7-27), it is quite like the translational velocity case, please refer to the figure below:





For the translational case, the black frame is 1, the blue frame is 2, and it is moving with some velocity v_{21} with respect to 1. We have

$\vec{r}_1 = \vec{r}_2 + \vec{r}_{21}$, $\dfrac{d\vec{r}_1}{dt} = \dfrac{d\vec{r}_2}{dt} + \dfrac{d\vec{r}_{21}}{dt}$,[68] so: $\vec{v}_1 = \vec{v}_2 + \vec{v}_{21}$ (the familiar case)

For rotational case, frame 2 rotates with respect to frame 1 with angular velocity $\vec{\omega}_{21}$. Let's consider the case, initially the two frames overlap at t=0, and the position vector of some point is shown as r(0) (it overlaps with the x and x' axis at t=0, this is just for easy drawing). After a small Δt. The vector will rotate to a position indicated by $\vec{r}(\Delta t)$. The change of vector viewed within frame 1 is $\Delta\vec{r}_1$; viewed within frame 2 is $\Delta\vec{r}_2$, they are related by: $\Delta\vec{r}_1 = \Delta\vec{r}_2 + \Delta\vec{r}_{21}$, so as $\Delta t \to 0$:

$\vec{v}_1(t=0) = \vec{v}_2(t=0) + \vec{v}_{21}(t=0)$, same as translational case. Now we use (7-27), the angular velocity for the vector rotates in frame 1 is $\vec{\omega}_1$, in frame 2 is $\vec{\omega}_2$, and frame 2 rotate with $\vec{\omega}_{21}$ respect to 1, then: $\vec{v}_1 = \vec{\omega}_1 \times \vec{r}(0) = \vec{\omega}_2 \times \vec{r}(0) + \vec{\omega}_{21} \times \vec{r}(0)$, r(0) is arbitrary and this will lead us to (7-28)[69].

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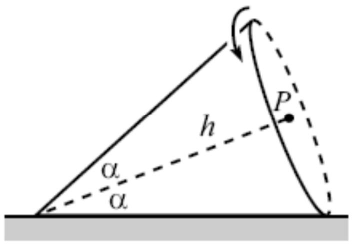
[68] Of course I am using Galileo transformation, neglecting special relativity here.

[69] Noticed due to my poor drawing, I cannot draw the most general case in 3-D, with $\vec{\omega}$'s point to arbitrary directions. So I made the drawing in 2-D, all the rotations are along the z-axis (perpendicular to paper). In this special drawing, you can just use relation between the angles to prove (7-28).However, in the general case in 3-D

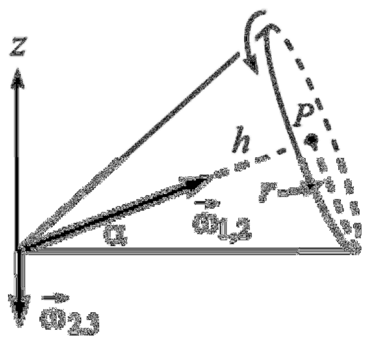
The reason I elaborate on this is because I think it is not very intuitive (at least not to me). If you think the above argument helps you understanding the addition of angular velocity, then that is great. If you think the addition is pretty obvious, then neglect my blibber-blubber.

Now let's take a look on one example to see how well you understand the above (example taken from Morin's book, problem 8.3)



A cone as shown is rolling without slipping on a table. The geometry of cone is shown in the figure. Let the speed of the center of base P is v. What is the angular velocity of the cone with respect to the lab frame at the instant shown?

I will choose the coordinate and view the rotation as rotating around h of cone and the rotation around z-axis in lab frame as the figure below:



$\vec{\omega}_{23}$ is the angular velocity of the cone rotating around z axis, the rotation

and arbitrary rotational axis, the above argument using vectors relations may be easier.

of frame 2 (a rotating frame with origin overlapping of frame 3 the lab frame)with respect to lab frame:

$\vec{\omega}_{23} \times \vec{r}_p = \vec{v}$, so $|\vec{\omega}_{23}| h\cos\alpha = v$, and $\vec{\omega}_{23} = \dfrac{v}{h\cos\alpha}(-\hat{z})$

$\vec{\omega}_{12}$ is the spin of cone within frame 2, the direction is known, the amplitude can be computed from rolling with no slipping:

$$r|\vec{\omega}_{12}| = |\vec{\omega}_{23}| \frac{r}{\sin\alpha}$$

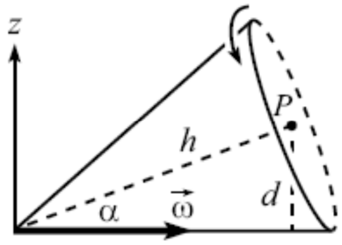
$$|\vec{\omega}_{12}| = \frac{v}{h\cos\alpha\sin\alpha}$$

$$\vec{\omega}_{12} = < \frac{v}{h\sin\alpha}, \frac{v}{h\cos\alpha} >$$

The angular velocity with respect to lab frame is $\vec{\omega}_{13}$:

$\vec{\omega}_{13} = \vec{\omega}_{12} + \vec{\omega}_{23} = < \dfrac{v}{h\sin\alpha}, 0 >$, along the horizontal.

Another quick method is by observing the tip of the cone which is stationary in lab frame always, and there is another point that is stationary at that particular instant in the lab frame, it is the point on the circular base of the cone that touches the table, it is velocity in lab frame is zero from the rolling without slipping (if you do not see it right away, try the rolling disk first). Both of these points are on the axis of rotation so that their speeds are zero, then the instantaneous rotation axis in lab frame is the line joining the tip and the base point as shown:



Using: $\vec{\omega}\times\vec{r}_P = v$, $|\vec{\omega}|\,h\sin\alpha = v$, $|\vec{\omega}| = \dfrac{v}{h\sin\alpha}$, same result.

7.3-2 Angular Momentum in 3-D and Inertia Tensor

With the relation (7-27) between velocity and angular velocity for rigid body, we can write the general formula for angular momentum in terms of angular velocity (for rigid body), relate the two important physical quantities in rotation.

$$\vec{L} = \sum_i m_i\vec{r}_i \times \vec{v}_i = \sum_i m_i\vec{r}_i \times (\vec{\omega}\times\vec{r}_i) \qquad \text{(7-29)}$$

For rotation within 2-D, we are used to the fact that the angular momentum is parallel with the angular velocity, $L = I\omega$, where I is just a number in 2-D. We will see that such simplification generally no longer exist for 3-D, unless the rigid body possesses certain symmetry and rotational axis overlaps with the symmetry axis. One easy proof is using the relation between cross product:

$$A\times(B\times C) = B(C\cdot A) - C(A\cdot B) \qquad \text{(7-30)}$$ [70]

Then we see that:

---

[70] I just pull this rabbit out of hat as magic. Certainly you should prove it (directly from definition of cross product). Also noticed the order of vectors **A,B,C,** does (**A**x**B**)x**C**=**A**x(**B**x**C**)?

$$\vec{L}=\sum_i m_i\vec{r}_i\times(\vec{\omega}\times\vec{r}_i)=\sum_i m_i\vec{\omega}(\vec{r}_i\cdot\vec{r}_i)-\sum_i m_i\vec{r}_i(\vec{\omega}\cdot\vec{r}_i)=\sum_i m_i\,|\,\vec{r}_i\,|^2\,\vec{\omega}-\sum_i m_i\vec{r}_i(\vec{\omega}\cdot\vec{r}_i)$$

We see that the first term is similar to the $I\omega$ in for rotation around z-axis in 2-D, but the second term is not always zero in 3-D. In the 2-D case, because $\vec{\omega}\perp\vec{r}$, the last term is zero and we have the simple relation.

In the 2-D rotation, you certainly can also directly put the vector in components into (7-29): $\vec{r}_i=<x_{i,}y_i,0>,\vec{\omega}=<0,0,\omega_z>$ and get the simple relation in 2-D.

For the more general case: $\vec{r}_i=<x_{i,}y_i,z_i>,\vec{\omega}=<\omega_x,\omega_y,\omega_z>$

$$\vec{v}_i=\vec{\omega}\times\vec{r}_i=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \omega_x & \omega_y & \omega_z\\ x_i & y_i & z_i\end{vmatrix}=(\omega_y z_i-\omega_z y_i)\hat{i}+(\omega_z x_i-\omega_x z_i)\hat{j}+(\omega_x y_i-\omega_y x_i)\hat{k}$$

$$\begin{aligned}\vec{r}_i\times\vec{v}_i&=(y_i v_{iz}-z_i v_{iy})\hat{i}+(z_i v_{ix}-x_i v_{iz})\hat{j}+(x_i v_{iy}-y_i v_{ix})\hat{k}\\ &=[(y_i^2+z_i^2)\omega_x-x_i y_i\omega_y-x_i z_i\omega_z]\hat{i}+\\ &\quad[-x_i y_i\omega_x+(x_i^2+z_i^2)\omega_y-y_i z_i\omega_z]\hat{j}+\\ &\quad[-x_i z_i\omega_x-y_i z_i\omega_y+(x_i^2+y_i^2)]\hat{k}\end{aligned}$$

Now you can throw in the $\sum_i m_i$ in front of each terms.

The formula for the relation between the angular momentum and angular velocity is quite a beast. However, if we use matrix representation of vector (a column matrix represents a vector), we have a nice bookkeeping, and the above equation can be expressed in matrix form as:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad (7\text{-}31)$$

If it helps you to remember this matrix equation, you may replace the x,y,z with 1,2,3 as indices.

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2), \quad I_{yy} = \sum_i m_i (x_i^2 + z_i^2), \quad I_{zz} = \sum_i m_i (x_i^2 + y_i^2) \quad (7\text{-}32)$$

These diagonal elements are called *moments of inertia* with respect to x,y,z axis.

$$I_{xy} = I_{yx} = \sum_i -m_i x_i y_i, \quad I_{xz} = I_{zx} = \sum_i -m_i x_i z_i, I_{yz} = I_{zy} = \sum_i -m_i y_i z_i \quad (7\text{-}33)$$

These off-diagonal elements are called *products of inertia*.

The whole matrix is called inertia tensor[71], it just like the transform matrix we encountered before linking the two vectors. In short hand (7-31) can also be written as:

$$\vec{L} = \widehat{I}\vec{\omega} \qquad (7\text{-}34)$$

$\widehat{I}$ is the symbol for inertia tensor, a shorthand for the matrix and to remind you it is not a number (scalar) or a vector. For easy typing, I will use ***I*** for it in notes.

(1) The meaning of the elements in ***I***

Write out the formula (7-31):

---

[71] This is called 2^{nd} rank tensor, which can be represented as a squared matrix (scalar is also called 0^{th} rank tensor and vector is 1^{st} rank). There are higher ranks of tensors too, they will be tensors relating tensors. Those higher rank tensors cannot be expressed in terms of regular matrix. Also not all squared matrix are tensors (just like not all ordered numbers are vectors), the tensors have to satisfy transformation rules in the coordinate transformation (like vectors). However, from what is called quotient rule of tensor: A=BC, if A and C are tensors, B will be a tensor too. The *I* here is indeed a tensor bcause L,ω are vectors (1st rank tensor)

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

This tells us the I_{xx} relates to the contribution of x component of angular velocity ω_x to the x component of angular momentum L_x. This is sort of intuitive, a rotation around x axis (ω_x) will contribute the angular momentum along this direction. However, the L_x has other contributions too, from rotation around y and z axis's. I_{xy} relates the rotation around y axis to L_x. I_{xy} can be interpreted as contribution coefficient of ω_y to L_x, that is why use xy in the subscript of indices; It is also easy to remember its formula in Cartesian this way as in (7-33). I_{xz} is the contribution coefficient of ω_z to L_x.

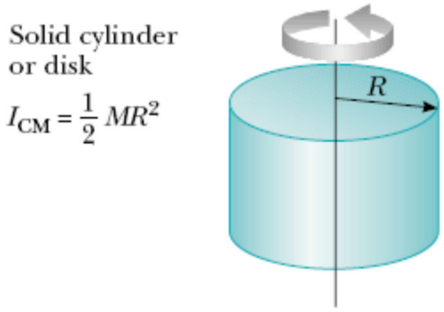
(2) ***I*** is a special matrix: Symmetric Matrix

This is clear that the matrix is symmetric with respect to diagonal, i.e. (7-33), $I_{xy} = I_{yx}$ etc. In language of linear algebra, we have:

$\boldsymbol{I}=\boldsymbol{I}^T$, $\boldsymbol{I}^T$ is the transpose of the matrix. This property is important when we talk about eigenvalues and eigenvectors for this matrix.

(3) ***I*** depends on the shape (geometry) of object and choice of coordinate system

This is clear from the definition of its elements in (7-32) and (7-33). If we choose the coordinate axis overlapping with symmetric axis of the object (if it has any), the form of inertia tensor will be simple.



Consider the cylinder I listed in section 7.1 dealing with rotation in 2-D. I treat it as 2-D object though it is indeed 3-D. The reason is if the rotation is overlapping with the symmetric axis, and this is the only rotation we have in simple cases, then I can call this rotation axis z, then $\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} = \omega_z \begin{bmatrix} I_{xz} \\ I_{yz} \\ I_{zz} \end{bmatrix}$, or $L_x = I_{xz}\omega_z$, $L_y = I_{yz}\omega_z$, $L_z = I_{zz}\omega_z$. $I_{xz} = \iiint -\rho xz dV$, but using symmetry, this integral is zero, because for every xz, you can always find on the object another point whose x'z=-xz, a perfect cancellation. Similarly I_{yz}=0 too. The only contribution to angular momentum is $L_z = I_{zz}\omega_z$, that is the formula we used to treat the rolling of cylinder and treat it as if only in 2-D.

(4) Generalized Parallel Axis Theorem

We have a relation between the elements of inertia tensor measured at the C.M. and the elements measured at some other origin (a translation of coordinate system with a shift of origin).

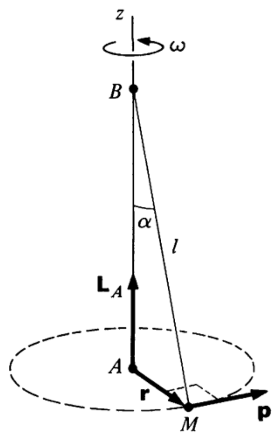
$$I_{xx} = I_{xx}^{cm} + M(Y^2 + Z^2)$$

$$I_{xy} = I_{xy}^{cm} - MXY \quad \ldots \qquad (7\text{-}35)$$

I_{xy}^{cm} is the elements with C.M. as origin, M is total mass, X,Y,Z are coordinate of C.M. in the shifted coordinate system. The proof will be trivial, similar to what I did in 2-D case, applying the property of C.M. frame. I won't repeat them here.

We'd better to work a couple examples to get used to ***I***.

Example 1. Rework KK example 6.2 with inertia tensor and compute the angular momentum and its change over time in a lab frame:



Of course for this simple problem in this example, it can be directly solved without even using the 3 methods discussed below, just directly from definition of angular momentum: **r** x **p**. The point is to using this simple example to introduce you the methods involving inertia tensor.

Method 1:Choosing B (the fixed point) as origin. It is always good to specify clearly the coordinate system we are using (the axes are translational type).

At time t=0, let's say the particle m is at <r,0,-z>, $z = l\cos\alpha = r/\tan\alpha$, angular velocity will be <0,0,ω>, so only need to evaluate those elements whose last indices is z: $I_{zz} = mr^2$, $I_{xz} = -mr(-z) = mrz$, $I_{yz} = 0$. So at time t=0: $L_z = mr^2\omega$, $L_x = mrz\omega$, $L_y = 0$. $\vec{L} =< mrz\omega, 0, mr^2\omega >$ for t=0. As time goes on, this L rotates around z.

To compute the change of angular momentum over time:

We have computed the angular momentum at t=0 in a lab frame. The change of L with time will be rotating around z axis, its magnitude won't change. Then for fixed length vector, the change over time can be calculated as:

$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} =< 0,0,\omega > \times < mrz\omega, 0, mr^2\omega >= mrz\omega^2 \hat{j}$$

(This change will be equal to the torque $mgr\hat{j}$)

Method 2: The inertia tensor computed in method 1 is in a lab fixed frame. It is at the instant t=0. At later time, as the particle rotates, ***I*** and L will change over time. In the lab frame, we have to calculate $\vec{L}(t) = \hat{I}(t)\vec{\omega}(t)$ for the angular momentum at any time.

If we choose a rotating frame, i.e. a coordinate system whose x,y axes rotate with ω around z-axis in this problem, the ***I*** will be a constant in this rotating frame, its elements will have same value as calculated at t=0 above and won't change over time. The angular momentum vector ***in the lab frame***[72] will have a simpler expression when viewed from this

[72] The reason I stress this is that what we computed will not be angular momentum measured in the rotating frame, which will be zero, because the ω will be zero in such frame. We want to calculate the L in lab frame (or other inertial frame, translational type at C.M. etc.) the reason for this is to apply the equation of motion which works for translational type frame without introducing inertial forces . If for some reason (mostly because of simpler form of inertia tensor as we shall see), the L in lab frame may be easier to be computed by expressing it in a rotating frame: $\vec{L}'=\hat{I}'\vec{\omega}'$. $\vec{L}'$, $\vec{\omega}'$ are the expressions of $\vec{L}$, $\vec{\omega}$ in the rotating frame, $\hat{I}'$ has simpler expression in the rotating frame.

rotating frame: $\vec{L}_{rot} = < mrz\omega, 0, mr^2\omega >$. The vector L when viewed from the rotating frame will be a constant vector that won't change over time, i.e. $(\frac{d\vec{L}}{dt})_{rot}$=0.

Of course the L when viewed from lab frame will be rotating, and its change over time can be computed from the expression in the rotating frame and the rotation of frame:

$$(\frac{d\vec{L}}{dt})_{lab} = (\frac{d\vec{L}}{dt})_{rot} + \vec{\omega} \times \vec{L}$$

The reasoning for the above relation is briefly discussed in footnote 66 and I will provide detailed proof in next chapter.

Since $(\frac{d\vec{L}}{dt})_{rot}$=0, $(\frac{d\vec{L}}{dt})_{lab} = (\frac{d\vec{L}}{dt})_{rot} + \vec{\omega} \times \vec{L} = \vec{\omega} \times \vec{L}$. Same as method 1.

Method 3: For this simple problem, the coordinate of the particle in lab frame is: $\vec{r}(t) = < r\cos\omega t, r\sin\omega t, -z >$, and the relevant elements of inertia tensor at any time can be computed:

$$I_{zz} = mr^2, I_{xz} = -mr\cos\omega t(-z) = mrz\cos\omega t, I_{yz} = mrz\sin\omega t$$

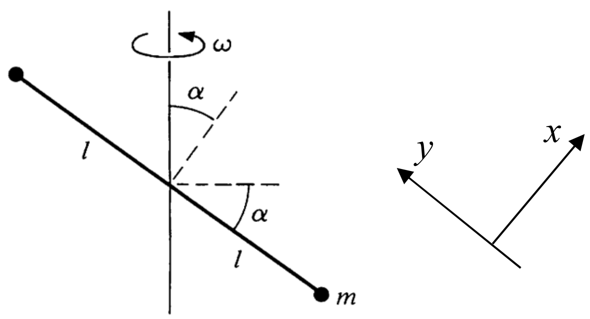
$$\vec{L}(t) = \hat{I}(t)\vec{\omega}(t) = < mrz\omega\cos\omega t, mrz\omega\sin\omega t, mr^2\omega >$$

Comments: The method 3 seems to be the most complete answer, gives the explicit expression of L in lab frame at any time. But this is least used method unless the problem only involves a couple particles in simple motion. Generally for rigid body doing some arbitrary rotation, the inertia tensor may be too complicated to evaluate at any time.

Method 1 is the preferred one before, it is has advantage to be in lab frame where the equation of motion can be applied, and physical interpretation is clearer. It has disadvantage in treating the general rotational motion, because of the same difficulty as method 3, evaluation of inertia tensor. We can compute the inertia tensor at one moment (instantaneous). At other time, due to the motion of the body, the inertia tensor has to be evaluated again and again which is not a pleasant job. However, this method is still useful when we deal with some simple problems (most problems in this chapter fit in this category), such as due to symmetry, the inertia tensor do not change over time (KK example 7.13); or some kinematic problems which can be solved from instantaneous equation of motion (as the later example shows). So this method will still be preferred one when dealing with these problems in this chapter, but you should aware its limitation. Basically the strategy of this method is: At a particular time (one instant), we set up a coordinates and compute the inertia tensor and angular momentum at this instant. At later time, if the problem is simple enough (or special enough) for us to predict what the change of angular momentum to be (like in the example above, the A.M. has fixed magnitude only rotating with certain angular velocity), we can then equate the change of angular momentum to the torque.

Method 2 will be useful when dealing with general rotation. By choosing a rotating frame that rotates with the body, it has advantage of simpler form of inertia tensor and it does not change over time in this frame, and thus the computation of angular momentum expressed in this frame. This is the basis for the Euler equation to deal with general rotation. The disadvantage is of course the non-inertial rotating frame, and the relation between the rotating frame with the lab frame can be complicated as well as the physical interpretations are not as straightforward as in method 1. The detailed handling will be the job in analytical mechanics, but I feel obligated to mention it here because it is the general method for general rotation motion.

Example 2: Simplified KK's 7.4 and 7.14.



At time zero, with the lab fixed coordinate chosen as shown (again this statement of choice of frame should be what you ought to do first), please compute the L and torque.

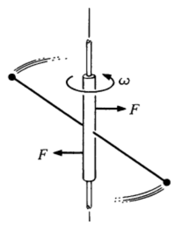
The two masses coordinate are: <0,l,0> and <0,-l,0>. $\vec{\omega} = <\omega\cos\alpha, \omega\sin\alpha, 0>$. $\overleftrightarrow{I} = \begin{bmatrix} 2ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2ml^2 \end{bmatrix}$. (the ***I*** in this coordinate has the simplest form, and that is why I choose such frame)

$\vec{L} = \widehat{I}\vec{\omega} = < 2ml^2\omega\cos\alpha, 0, 0 >$

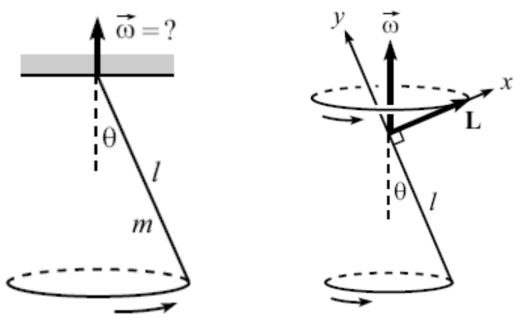
The L will rotate around the vertical axis as time changes, and its change over time (fixed vector length):

$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega\cos\alpha & \omega\sin\alpha & 0 \\ 2ml^2\omega\cos\alpha & 0 & 0 \end{vmatrix} = -2ml^2\omega^2\sin\alpha\cos\alpha\hat{k}$$

This torque is pointing inward to the paper (-k direction), and cannot be supplied by the gravity of the two masses (they cancel each other). The torque has to be supplied by the force on the 'sleeve' on the pivot axis:



Example 3.



The problem is a uniform rod is spinning as shown in the right figure. Find the angular velocity ω, assuming the angle with vertical θ is known.

The coordinate is chosen as shown in the left figure, a lab fixed one with origin at pivot so that we do not need to worry about the torque by the force at pivot. The x,y direction is chosen to ease the computation of inertia tensor. At the instant shown, the inertia tensor and $\vec{\omega}$ are:

(x and z are zero for any points on the rod), all products of inertia are zeroes. $I_{xx} = \frac{1}{3}ml^2, I_{yy} = 0, I_{zz} = \frac{1}{3}ml^2$

$$\hat{I} = \begin{bmatrix} \frac{1}{3}ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}ml^2 \end{bmatrix}, \quad \vec{\omega} = < \omega\sin\theta, \omega\cos\theta, 0 >$$

Then, the angular momentum at this moment is:

$$\vec{L} = \hat{I}\vec{\omega} = < \frac{1}{3}ml^2\omega\sin\theta, 0, 0 >$$

To find the value of ω, we shall apply the equation of motion:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau} = mg\frac{l}{2}\sin\theta(-\hat{k})$$

The angular momentum will rotate without change in length, so:

$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} = \frac{1}{3}ml^2\omega^2\sin\theta\cos\theta(-\hat{k})$$

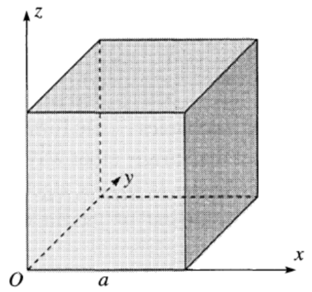
$$\frac{1}{3}ml^2\omega^2\sin\theta\cos\theta = mg\frac{l}{2}\sin\theta$$

$$\omega^2 = \frac{g}{(2/3)l\cos\theta}$$

(It is like a mass point hanging at 2/3 of l)

This example is also good to test different choices of coordinates. Instead of choosing the fixed point on the rod as I did above, you may choose CM translational type frame since we know the equation of motion works well in such frame. Please try this yourself, the inertia tensor will be somewhat different, the angular velocity would be same as above, and the torque is due to the force at the fixed point now, and the force there can be evaluated using the motion of C.M. The procedure would be similar to the detailed calculation above just a little more complicated. You may even try the lower end as origin, but then the inertial force (a force pointing away from the rotation axis here) and its torque has to be considered. Of course the last choice is most inconvenient in this example, I only state it to show you the variety. You may choose a lab fixed frame with the center of circle as origin too, but then you need to decompose the motion into CM motion and motion around CM, and the motion around the CM is just same as choosing the CM as origin.

Example 4: This is addressed to a technical issue: evaluate inertia tensor at one instant, since we have seen we need this to get angular momentum. It is here because I will use it again in next section when we talk about principle axis.



For a squared cube as shown in the figure, at t=0, the coordinates are setup as shown in the figure, the pivot is at one corner of the cube. What is the inertia tensor at this time? The cube is uniform with total mass M, side length=a. What happened if we choose the pivot at the center of the cube?

The computation of elements of the matrix is just integration based on definition in (7-32) and (7-33):

$$I_{zz} = \iiint\limits_{Volume} \rho(x^2 + y^2)dV = \rho\int_0^a dz\int_0^a dy\int_0^a x^2 dx + ...(y^2 \text{ term})$$

$\rho\int_0^a dz\int_0^a dy\int_0^a x^2 dx = \rho\frac{1}{3}a^3 aa = \frac{Ma^2}{3}$, and y^2 term will give same value, so:

$$I_{zz} = \frac{2}{3}Ma^2$$

From symmetry, it is easy to see that I_{xx}, I_{yy} will have same value.

$$I_{xy} = \iiint\limits_{V} -\rho xy dV = -\rho\int_0^a dz\int_0^a dy\int_0^a xy dx = -\frac{\rho a^5}{4} = -\frac{Ma^2}{4}$$

The rest of products of inertia will have same value.

So the matrix is:

$$\hat{I} = \begin{bmatrix} \frac{2}{3}Ma^2 & -\frac{Ma^2}{4} & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & \frac{2}{3}Ma^2 & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & -\frac{Ma^2}{4} & \frac{2}{3}Ma^2 \end{bmatrix} = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

Now provided with the angular velocity (say rotation around z axis or an arbitrary direction), the angular momentum can be computed at this instant.

The form of inertia tensor will be much simpler if we choose the pivot at center. In this case:

$$I_{xx} = I_{yy} = I_{zz} = \frac{Ma^2}{6}$$

The products of inertia will all be zero due to the cancellation from symmetry. The matrix will have a simple diagonal form:

$\hat{I} = \frac{Ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If the $\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$. The angular momentum is this case will also take a simpler form: $\vec{L} = < I_{xx}\omega_x, I_{yy}\omega_y, I_{zz}\omega_z >$.

7.3-3 Principal Axes

In the last example, we have seen that the matrix representing the inertia tensor can be complicated as the pivot-at-corner case; it can also be simple in diagonal form as the pivot-at-center case. We like the simple diagonal form, because in that situation, the relation for angular momentum and velocity is simple:

$\vec{L} = < I_{xx}\omega_x, I_{yy}\omega_y, I_{zz}\omega_z >$ for diagonal ***I*** (7-36)

It turns out for any pivot on the rigid body, we can always choose the proper directions of coordinate axes to make the ***I*** diagonal[73]. Let's suppose we have chosen the axes properly so that (7-36) satisfies, let the 3 directions of axes be $\hat{e}_1, \hat{e}_2, \hat{e}_3$. If we have the angular velocity only along direction 1, i.e. $\vec{\omega}_1 = < \omega_1, 0, 0 >$, expressed in components along $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$\vec{L} = \hat{I}\vec{\omega}_1 = I_{11}\vec{\omega}_1$ (7-37)

I used I_{11} for the old I_{xx} here. Similar relations (replacing the I_{jj} and $\vec{\omega}_j$) for angular velocity only along direction 2 or 3. In short, for special angular velocities along certain direction, we have:

$\hat{I}\vec{\omega}_i = \lambda_i \vec{\omega}_i$ (7-38)

On the left hand side, it is a matrix acting on a vector, it generally will change the input vector, give us an output vector. The output vector can be in any direction for the general case. However, for some special input vectors, the right hand side tells us the output would be a vector parallel with the input, the matrix will not change the direction of the vector, only

---

[73] This is because the matrix of ***I*** is symmetric, and linear algebra tell us for symmetric matrix, the eigenvalues are real numbers and the eigenvectors are orthogonal. The orthogonality of the eigenvectors are good, meaning they are independent, can form a new complete base on which the matrix form of ***I*** will be diagonal. The details won't be covered here or in supplementary (we already have a long math supplementary). Please pay attention in linear algebra course on this. Or read the linear algebra textbook on symmetry matrix and eigenvalue. (for example: "Introduction to Linear Algebra" by Gilbert Strang)

its magnitude by some number λ_i. These particular vectors satisfying (7-38) are called ***eigenvectors*** of the matrix, and the λ_i is called ***eigenvalue***.

The principal axes for a pivot on rigid body, is defined as the directions of the eigenvectors of the inertia tensor. So finding the principal axes given a pivot is solving the eigenvalue and eigenvector problem in Linear Algebra. The general procedure is to choose a coordinate system with the pivot as origin. This coordinate (base vectors) is sort of arbitrary, may not be the eigenvectors, and the matrix of inertia tensor in this basis may not be diagonal. After we have the matrix form in the chosen basis, we can find the eigenvalue and corresponding eigenvectors (also expressed in components of the chosen basis). The point is if we chose the eigenvectors (normalized, i.e. with unit length, since we only need the directions) as new base vectors, the matrix expressed in the basis formed by eigenvectors will be diagonal. It is easy to prove that in the *basis by the eigenvectors*, the matrix will take the form of:

$$I = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad (7\text{-}39)$$

the diagonal elements are just the eigenvalues which equals to the moment of inertia here. So by solving the eigenvalue and eigenvector problem, we get two birds with one stone: get both the principal axes and the moment of inertia.

The above general statement is a bit abstract and I'd better illustrate the points with a concrete example. Let's take the cube with the pivot-at-corner as in last example. I have shown that if we choose the axes at certain instant as x-y-z in the figure above, the matrix for inertia tensor is: $\hat{I}=\begin{bmatrix}\frac{2}{3}Ma^2 & -\frac{Ma^2}{4} & -\frac{Ma^2}{4}\\ -\frac{Ma^2}{4} & \frac{2}{3}Ma^2 & -\frac{Ma^2}{4}\\ -\frac{Ma^2}{4} & -\frac{Ma^2}{4} & \frac{2}{3}Ma^2\end{bmatrix}=\frac{Ma^2}{12}\begin{bmatrix}8 & -3 & -3\\ -3 & 8 & -3\\ -3 & -3 & 8\end{bmatrix}$. The x-y-z basis chosen is not an eigenvector basis, otherwise the matrix will be diagonal. In order to find the principle axes (i.e. eigenvectors), we need to solve (7-38), rewrite it as:

$\hat{I}\vec{\omega}-\lambda\vec{\omega}=0,(\hat{I}-\lambda\hat{1})\vec{\omega}=0$, $\hat{1}$ is the identity matrix[74].

$$\begin{bmatrix}I_{11}-\lambda & I_{12} & I_{13}\\ I_{21} & I_{22}-\lambda & I_{23}\\ I_{31} & I_{32} & I_{33}-\lambda\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}=0 \qquad \text{(7-40)}$$

In order to have non-trivial solution for the above equation (the trivial solution is all components of ω are 0, that obvious satisfies the equation. It is trivial because it basically states that if the input is zero, the output will be zero too), the matrix has to be non-invertible, equivalently the determinant of the matrix has to be zero. i.e.:

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[74] Here is another awkward moment where the conventional symbols for inertia tensor and identity matrix clash, so I use symbol $\hat{1}$ for identity.

$$\begin{vmatrix} I_{11}-\lambda & I_{12} & I_{13} \\ I_{21} & I_{22}-\lambda & I_{23} \\ I_{31} & I_{32} & I_{33}-\lambda \end{vmatrix} = 0 \qquad (7\text{-}41)$$

This will give us a *characteristic equation* (a cubic equation in 3-D here) with λ as variables, and it can be solved for the roots of λ. These roots are the eigenvalues we are looking for. For the cubic polynomials, we have 3 roots for λ and will give us 3 eigenvalues, $\lambda_1, \lambda_2, \lambda_3$. There will be cases that two of the roots are equal (also called double degenerate eigenvalue), i.e. $\lambda_1 = \lambda_2 \neq \lambda_3$, this is called *symmetric top* for rigid body rotation. If all three eigenvalues are equal (triple degenerate), it is called *spherical top*. After we solved the eigenvalues λ_i, for each λ_i, we can solve for the eigenvectors (usually we just need to find the normalized eigenvector).

Now back to the specific problem:

The characteristic equation is:

$$\begin{vmatrix} 8-\lambda & -3 & -3 \\ -3 & 8-\lambda & -3 \\ -3 & -3 & 8-\lambda \end{vmatrix} = 0$$

Here I neglect $\frac{Ma^2}{12} = u$, just treat it as unit for the eigenvalues, and attach it afterwards.

The expansion of the 3X3 determinant and factorize is a bit boring and lengthy, so I just present the results here:

$(\lambda - 2)(\lambda - 11)^2 = 0$

This will give us the eigenvalues:

$\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 11$, put in the unit $\frac{Ma^2}{12} = u$, the eigenvalues will be:

$\lambda_1 = 2u$, $\lambda_2 = \lambda_3 = 11u$

For $\lambda_1 = 2u$, put this back into (7-40) to solve for $\vec{\omega}_1$, the eigenvector associated with this eigenvalue:

$$u\begin{bmatrix} 8-2 & -3 & -3 \\ -3 & 8-2 & -3 \\ -3 & -3 & 8-2 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0, \text{ or} \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

I will solve the above linear equations with Gauss elimination method, after elimination:

$$\begin{bmatrix} 6 & -3 & -3 \\ 0 & 4.5 & -4.5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

This tells me that there is no unique solution for this equation (we expect this, because anyway the eigenvector would only be a direction), there is one free variable we can choose, here the free variable is ω_z. I will set the nontrivial one $\omega_z = 1$, then $\omega_y = 1$ and $\omega_x = 1$. So the eigenvector for $\lambda_1 = 2u$ is: $\vec{\omega}_1 = <1,1,1>$ and we can normalize it to unit vector: $\hat{\omega}_1 = \frac{1}{\sqrt{3}} <1,1,1>$. This will be the direction of one principle axis (along the main diagonal of the cube).

For $\lambda_2 = \lambda_3 = 11u$, put it back to (7-40):

$$u\begin{bmatrix} 8-11 & -3 & -3 \\ -3 & 8-11 & -3 \\ -3 & -3 & 8-11 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}=0 \rightarrow \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}=0$$

After Gauss elimination:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}=0$$

This is also expected, for this double degeneracy case, there are two free variables. I will chose $\omega_y = 1, \omega_z = 0$ for one nontrivial solution, and this gives me $\omega_x = -1$. So $\vec{\omega}_2' =< -1,1,0 >$.

Similarly if I choose $\omega_y = 0, \omega_z = 1$, I will get $\omega_x = -1$. So the other solution (also the choices of free variables make sure that this one will be independent of the $\vec{\omega}_2'$): $\vec{\omega}_3' =< -1,0,1 >$

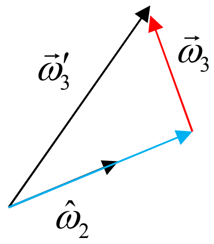
The eigenvalue and eigenvectors problem have been solved. However, I should make it better for the double degenerate case. The two eigenvectors there are associated both with eigenvalue=11u, these two vectors are independent (they are not collinear), but they are not orthogonal! I'd better choose two orthogonal vectors as eigenvectors. This can be achieved by Gram-Schmidt orthogonalization:

I will not change $\vec{\omega}_2'$, just make it normalized:

$$\hat{\omega}_2 = \frac{1}{\sqrt{2}} < -1,1,0 >$$

$\vec{\omega}_2' \cdot \vec{\omega}_3' = 1$, so they are not orthogonal. I could manufacture an orthogonal vector (with respect to $\vec{\omega}_2'$) out of $\vec{\omega}_3'$ by eliminating the projection of

$\vec{\omega}_3'$ along $\hat{\omega}_2$, as the figure below suggests:



$$\vec{\omega}_3 = \vec{\omega}_3' - (\vec{\omega}_3' \cdot \hat{\omega}_2)\hat{\omega}_2 =< -1,0,1 > -\frac{1}{2} < -1,1,0 >=< -\frac{1}{2}, -\frac{1}{2}, 1 >$$

You can check this is indeed orthogonal to $\hat{\omega}_2$ (both of them are orthogonal to $\vec{\omega}_1$, this is guaranteed by the property of symmetric matrix, that the eigenvectors associated with different eigenvalues are orthogonal. Try the dot products if you are in doubt)

Final step is just normalization:

$$\hat{\omega}_3 = \frac{1}{\sqrt{6}} < -1, -1, 2 >$$

So the principle axes for the cube inertia tensor are directions specified by $\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3$. If we choose these unit vectors as our base vectors (the direction of new x-y-z coordinate), the inertia tensor is diagonal:

$$\widehat{I}_P = \begin{bmatrix} 2u & 0 & 0 \\ 0 & 11u & 0 \\ 0 & 0 & 11u \end{bmatrix}$$, P stands for the principle axes basis.

One comment: Actually we can choose any perpendicular unit vectors in the plane spanned by $\hat{\omega}_2, \hat{\omega}_3$ to replace $\hat{\omega}_2, \hat{\omega}_3$ as basis vectors. This will not change the form of inertia tensor. This is the result of a general theorem, that if the $\hat{\omega}_2, \hat{\omega}_3$ are eigenvectors associated with degenerate eigenvalues (i.e. $\lambda_2 = \lambda_3$), then any linear combination of $\hat{\omega}_2, \hat{\omega}_3$, i.e. $a\hat{\omega}_2 + b\hat{\omega}_3$ is also an eigenvector with the same eigenvalue. Please prove it yourself from the definition of eigenvalue and eigenvectors. So the principle axes for the cube with pivot-at-corner are main diagonal of the cube ($\hat{\omega}_1$), and any two orthogonal vectors in the plane (passing the origin) perpendicular to the main diagonal.

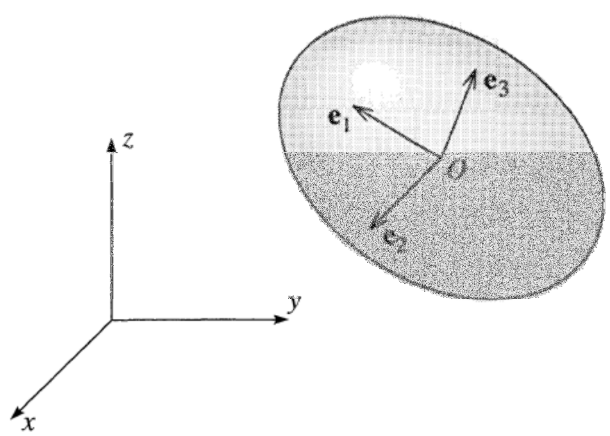
This is a pretty long example, because I tried to solve the eigenvalue-eigenvector problem in detail. You can imagine this is not a pleasant job for some odd shaped object. Fortunately for the objects considered in this course, it will have some apparent symmetry, so that the symmetry axis will be one of the principle axes, and the other two principle axes will be lying in the plane perpendicular to the symmetry axis. You will have no trouble to locate the principle axes for such object with certain symmetry.

The reason I showed you the procedure here in detail is not specifically for this course. It rather points out one important application of eigenvalue-eigenvector problem in physics. You will see more such applications in the following courses in physics, especially in quantum mechanics. You cannot survive long in quantum without linear algebra.

Suppose we find out the principal axes at certain pivot on the rigid body (stressing again, most often the pivots will be either a fixed point in inertia frame, or at the C.M. of the body), either by symmetry or detailed calculation as above or even lucky guess, we want to setup coordinate axes with principal axes. This will lead to simplest relation between angular velocity and momentum: relation like (7-36).

(1) Euler Equation[75]

We have seen that the advantage to setup a coordinate with principal axes of the rigid body. The trouble is these axes are attached to the body (you can imagine the principal axes are three orthogonal directions painted on the body). As the body rotates with some angular velocity $\vec{\omega}$ (viewed in lab frame), the principle axes rotate with the body at same $\vec{\omega}$. I will use x-y-z for the coordinates that fixed in lab (also called translational type), and 1-2-3 (represented by unit base vectors: $\hat{e}_1, \hat{e}_2, \hat{e}_3$) for the principal axes attached to the body (also called body frame), as shown in the figure below. The origin O of the body frame will be chosen always at C.M., because we know we can apply the equation of motion with this choice in the translational type coordinates.



At one instant, suppose the instantaneous 1-2-3 axes are as shown in the

[75] This part (1) is not required for this course

figure, $\vec{\omega} = <\omega_1, \omega_2, \omega_3>$ are the expression of the angular velocity in body frame; (stressing again: $\vec{\omega}$ is the angular velocity measured in lab frame, but we express it in components in the body frame, the reason to go through this treacherous detour is to get simpler expression of angular momentum in the body frame) so the angular momentum at this instant expressed in body frame is:

$$\vec{L}_{body} = <\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3> \qquad (7\text{-}36)$$

The fundamental problem is from the torque, solving the motion of the body, i.e. change of $\vec{\omega}$ over time. To apply the equation of motion:

$\vec{\tau} = (\frac{d\vec{L}}{dt})_{lab}$ (lab subscript is to stress that it is the change of L in a lab frame or translational type at C.M.). We will need the relation for the vector change viewed in lab frame and body frame, this is topic in the next chapter and also briefly discussed in footnote 66:

$$\vec{\tau} = (\frac{d\vec{L}}{dt})_{lab} = (\frac{d\vec{L}}{dt})_{body} + \vec{\omega} \times \vec{L} \qquad (7\text{-}42)$$

This is a vector equation, and in real applications we need to specify the coordinate. In order to exploit the (7-36), we need to express the equation with components along $\hat{e}_1, \hat{e}_2, \hat{e}_3$, in body frame:

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1\omega_1 & \lambda_2\omega_2 & \lambda_3\omega_3 \end{vmatrix} = \hat{e}_1(\lambda_3 - \lambda_2)\omega_2\omega_3 + \hat{e}_2(\lambda_1 - \lambda_3)\omega_1\omega_3 + \hat{e}_3(\lambda_2 - \lambda_1)\omega_1\omega_2$$

$$(\frac{d\vec{L}}{dt})_{body} = \lambda_1\dot{\omega}_1\hat{e}_1 + \lambda_2\dot{\omega}_2\hat{e}_2 + \lambda_3\dot{\omega}_3\hat{e}_3$$

Equate each component will give us so called Euler equation:

$$\lambda_1\dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_2\omega_3 = \tau_1$$

$$\lambda_2\dot{\omega}_2 + (\lambda_1 - \lambda_3)\omega_1\omega_3 = \tau_2$$

$$\lambda_3\dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1\omega_2 = \tau_3 \qquad (7\text{-}43)$$

The three coupled differential equations are Euler equation. The τ_1, τ_2, τ_3 are the projections of torque in the body frame. This makes the Euler equation awkward to work with. Because at one instant, like shown in the figure, we can project the torque along the 1-2-3 axes and solve the equation (not easy) to get the ω_i's at that instant. But as the body rotates, the projections τ_1, τ_2, τ_3 will change too. Euler solved this by introducing 3 angles, the Euler angle to relate the body frame axes to translational type axes (lab frame axes), the angular velocity and torque components in body frame will be expressed in terms of these Euler angles, solving the motion of the body becomes solving coupled differential equations to get the change of Euler angles with time. The detail on Euler angle and solving coupled equations won't be discussed in this course, it is quite messy in general case.

For some simple cases, especially the case with no external torque, Euler equation can be used to solve the rotation of rigid body with relative simplicity. Some examples are given in KK section 7.7and beyond.

(2) Rotational Kinetic Energy

We have seen that rotational kinetic energy in 2-D rotation for rigid body is: $K_{rot} = \frac{1}{2} I\omega^2$, actually from what we have discussed, it is $K_{rot} = \frac{1}{2} I_{zz}\omega^2$.

For the rotation in 3-D, what is the expression for rotational kinetic energy? With the C.M. as origin, the motion of rigid body will be pure rotational (the total kinetic energy according to Konig theorem is $K_{cm} + K_{rot}$ of course), the rotational kinetic energy in C.M. frame:

$$K_{rot} = \frac{1}{2}\sum_i m_i \vec{v}_{icm}^2 = \frac{1}{2}\sum_i m_i (\vec{\omega}\times\vec{r}_{icm})\cdot(\vec{\omega}\times\vec{r}_{icm})$$

It is quite nasty if you expand above in some coordinates, you may try it yourself and you can get the same answer as below. So I will adopt the trick used in KK (p314). The trick is using the relation of triple product of vectors:

$$(\vec{A}\times\vec{B})\cdot\vec{C} = \vec{A}\cdot(\vec{B}\times\vec{C}) \qquad \text{(7-44)}$$[76]

Let $\vec{A} = \vec{\omega}, \vec{B} = \vec{r}_{icm}, \vec{C} = (\vec{\omega}\times\vec{r}_{icm})$, we have:

$$\frac{1}{2}\sum_i m_i (\vec{\omega}\times\vec{r}_{icm})\cdot(\vec{\omega}\times\vec{r}_{icm}) = \frac{1}{2}\vec{\omega}\cdot\sum_i m_i \vec{r}_{icm}\times(\vec{\omega}\times\vec{r}_{icm}) = \frac{1}{2}\vec{\omega}\cdot\vec{L}_{cm} \quad \text{(7-45)}$$

Actually in the derivation of 7-45, I do not use any specific property of C.M., so it could be applied to any point on the rigid body, and L of course has to be evaluated with respect to the chosen point. The reason to stress the C.M., is of course to use the Konig Theorem.

If we have principle axes as coordinate axes, the (7-45) will give us

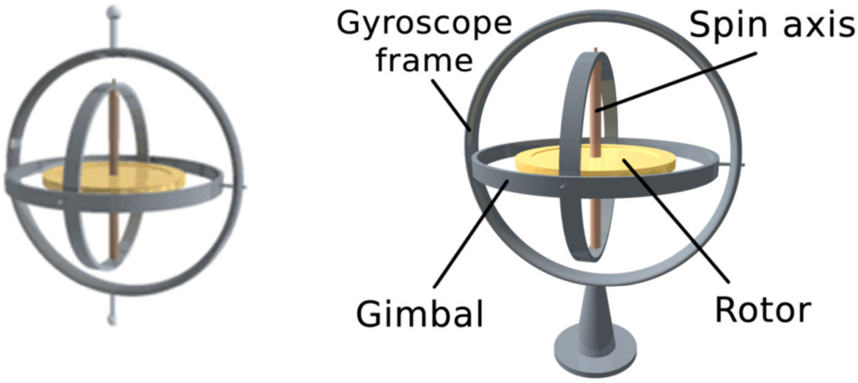
[76] This can be proved by hack way, expanding everything on both sides. There is a nicer geometric way by seeing that both sides express the volume of a parallelogram formed by these 3 vectors.

something like the 2-D case:

$$K_{rot} = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2) = \frac{L_1^2}{2\lambda_1} + \frac{L_2^2}{2\lambda_2} + \frac{L_3^2}{2\lambda_3} \qquad (7\text{-}46)$$

7.4 Gyroscope

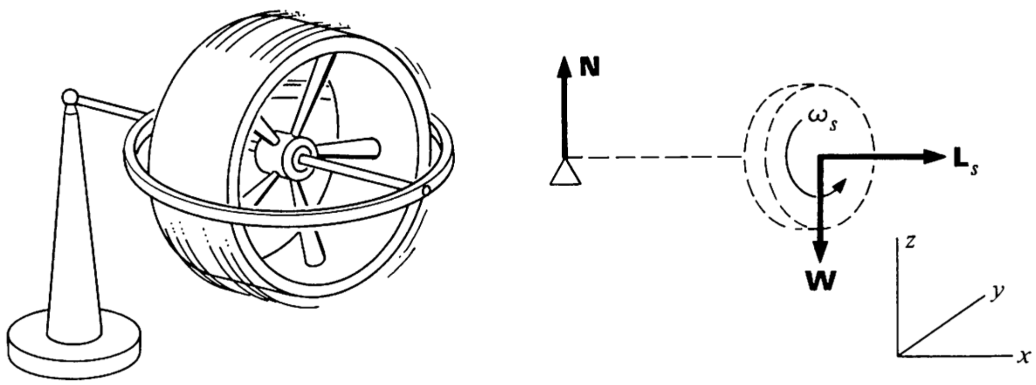
(1) Free-of-Torque Gyro



A gyroscope is a rotor, usually spins with high angular velocity only along one principle axis (usually along the axis with largest moment of inertia). The rotor is usually mounted on sets of frames called gimbal mount that each gimbal mount allows free rotation about single axis. The gimbal mount can be arranged liked the ones shown above, to make the rotor 'feels' no external torque, thus the angular momentum of the rotor will be fixed, $\vec{L}_{rotor} = I_s\vec{\omega}_s$, I_s is the moment of inertia along the spin axis(one of the principle axes), $\vec{\omega}_s$ is the angular velocity along spin axis, and if it is the total angular velocity, then from conservation of A.M. under no torque, we have constant angular velocity $\vec{\omega}_s$ too (if the angular velocity is not along one principle axis, the angular momentum is still conserved under no torque, but the angular velocity will change, it can rotate around the fixed L, the angular momentum). Such free-of-torque gyroscope is very useful to specify an inertial direction in space. The application of it includes measuring angles in a rotating frame, such as the rolling and pitching of the airplane. Put the free-of-torque gyro in the airplane, if airplane rolls (wing-wing rotation) or pitches (head-tail rotation), the gyro's spin axis will form a measurable angle with the frame of airplane, and this angle can be measured and used as feedback on controls of small fins on the wings of airplane, to adjust the fins making the airplane fly smoothly. Similar mechanism is applied to ships and yachts to make the boats more stable under waves. Now the iPhone and iPad contains a 'gyroscope' inside to detect the rotational motion of chassis. It is not a real mechanical gyroscope, but a piece of electronic design called MEMS (mirco-electromechanical system) that mimics the gyro, detecting the angular displacement of the chassis.

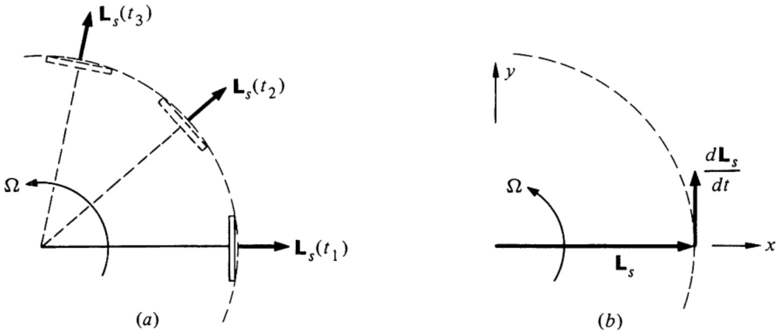
(2) Gyro under Torque

When the rotor of the gyro is subjected to an external torque, the motion of the gyroscope is a little bit unexpected. It won't rotate along the direction of the force as a stationary object would do, but rather rotates in a direction perpendicular to the force, a motion called precession (This is in analogy to the motion under gravity: a static object will fall to the ground, but a fast moving object will travel in parabola or even circular orbit provided with enough initial velocity). The reason of precession lying in the high angular speed of the rotor, the angular momentum of the gyro is almost all from its spin, L_s; under the external torque, the L_s will rotate in the direction of the torque, so that $\Delta L = \tau \Delta t$. It is along the torque not the force, and the torque by definition is always perpendicular to the force. Let's analyze it more quantitatively:



The gyro configuration for gravity pulled precession and simplified figure with coordinate setup (an instantaneous translational type coordinate, either with origin at C.M. of the rotor, or at pivot. I will choose C.M. below).

Under the torque by gravity (choose pivot as origin) or by the supporting force N at the pivot (C.M. as origin), there is actually another force at pivot pointing along the shaft to supply the centripetal force but has no torque to the C.M. As the figure below shows the gyro which spins with ω_s about the spin axis would rotate around z axis with angular velocity Ω, the precession angular velocity. We'd like to find this Ω.



Due to the symmetry of the rotor, the x-y-z axis are principle axes(instantaneous ones). Let the moment of inertia are:

$I_{xx} = I_s, I_{yy} = I_{zz} = I_\perp$.

The total angular velocity vector is:

$$\vec{\omega}_{tot} = \vec{\omega}_s + \vec{\Omega} = \langle \omega_s, 0, \Omega \rangle$$

$$\vec{L} = \hat{I}\vec{\omega}_{tot} = \langle I_s\omega_s, 0, I_\perp\Omega \rangle$$

As the gyro rotate about z axis, this angular momentum will not change magnitude but changes direction[77], i.e.:

$$\frac{d\vec{L}}{dt} = \vec{\Omega} \times \vec{L} = I_s\omega_s\Omega\hat{y}$$

The torque by the supporting force N=mg is:

$$\tau = mgl\hat{y}$$

Then: $I_s\omega_s\Omega = mgl$, so:

$$\Omega = \frac{mgl}{I_s\omega_s} \qquad (7\text{-}47)$$

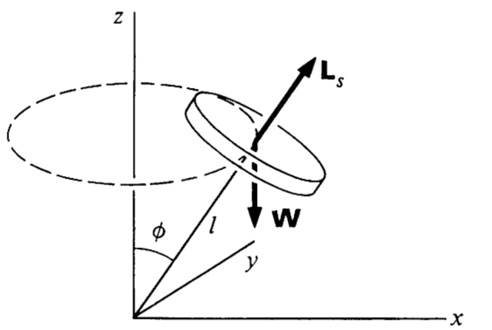
It is a little surprising that the rotor does not rotate about y direction. Actually, it did. That is called nutation. It has to be there from energy

[77] I did not prove this. The ω_s is constant because there is no torque along x direction. From Euler equation:

$\lambda_1\dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_2\omega_3 = \tau_1$, $\lambda_3 = \lambda_2 = I_\perp$, with zero torque, the ω_s will not change over time.

conservation. The rotor spins with K_{rot}, and now it also precesses around z with Ω, this increase of kinetic energy has to come from somewhere. It comes from dropping of potential energy of the rotor. The rotor will drop to some lower level and this drops can be periodic up and downs as rotation around y axis, and this motion is called nutation. The detail treatment has to use Euler equation or the method in KK note 7.2. Under the usual condition that $I_s\omega_s$ is the dominant factor, the nutation would be a small effect.

For a tilted gyro as shown in the figure:



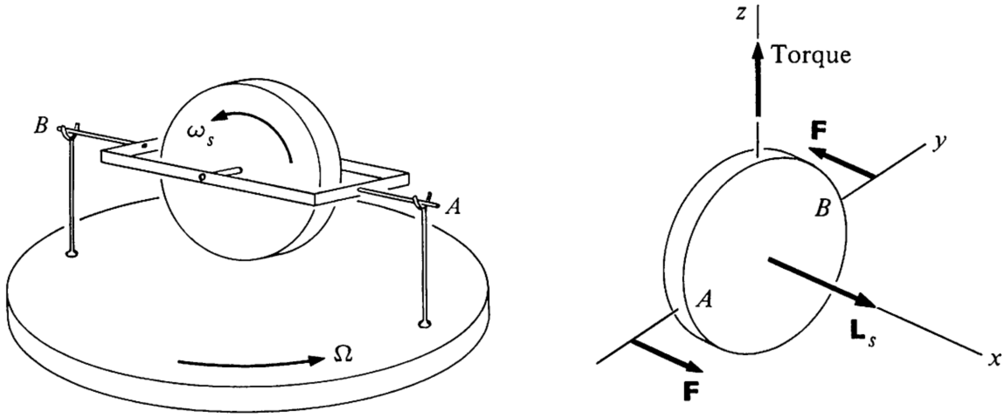
The method would be exactly similar. The origin is better chosen at the pivot (for simple torque analysis) and the x-y-z can be chosen overlapping principle axes (Not as shown in the figure, I will tilt the z axis overlap with the L_s) at this instant. Then the procedure will be similar to above, and you will need to use the *gyroscopic approximation*: $\omega_s >> \Omega$ for simple result, which will give exactly same answer of Ω as (7-47).

By using *gyroscopic approximation*, we essentially treat the total angular momentum as L_s, and since ω_s will not change (see argument in footnote 77). The change of angular momentum with time is just $\Omega \times L_s$ as in KK's example 7.7.

(3) Some Applications of Gyroscope Motion (KK 7.4)

The applications of Gyro motion are listed in the textbook as some examples (7.9-7.12). I will discuss the Gyro-compass in detail with the formal treatment, as a supplementary to the book's intuitive argument (example 7.10, 7.11). As to the precession of earth's spin and stability of rotating object (why the bullet out of rifle can hit target more accurate), please read the examples (7.9, 7.12). Noticed that for the earth precession case, this effect is the result of two facts: A) the earth is not a perfect sphere, it is actually a football shape with a bulge at equator. B) The spin axis of earth is not along the same direction as orbital axis. i.e. the spin axis has the famous 23 degree inclination to the orbit (the orbit around the sun is called ecliptic) axis. These two facts combined result a none zero torque to the C.M. of earth from the gravity of Sun (actually Moon has bigger effect, the reason is similar for the Moon case). If the earth is a perfect sphere, or no inclination of the spin axis with respect to ecliptic, then no net torque from gravity of Sun, and no precession of the earth (please figure this out yourself from symmetry).

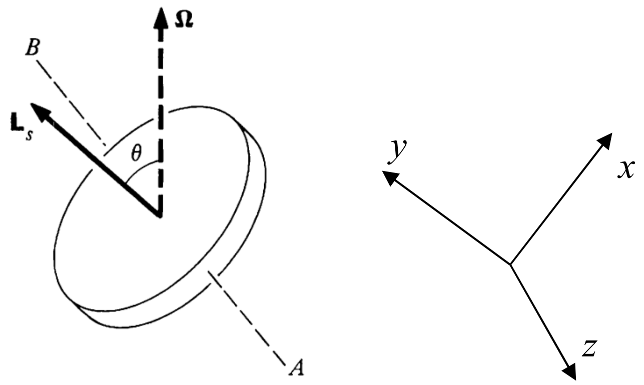
Now I will focus on the gyrocompass motion:



Put the gyro on rotating table as shown in the figure. The force on the AB shaft would create a torque as shown in the figure up right, and the L_s will flip toward that direction. So the motion of the rotor will flip towards the direction of $\vec{\Omega}$, and will oscillate around the $\vec{\Omega}$ back and forth. A detailed analysis is needed to quantitative understand this oscillation besides this intuitive argument. The KK book offers a simpler method, I intend to solve this by the formal method and introduce approximation later to get the same results. It is more systematic and rigorous than the simpler method, but also more slow.

The crucial fact is that along the AB shaft direction, there will be no torque. The external force acting at A,B, the torque with respect to the C.M. will always be perpendicular to the AB shaft. This will give us a relation in the equation of motion, and that is what we are going to exploit.

a) Choose coordinate system:



The axes are chosen overlapping with the instantaneous principle axes of the rotor, CM as origin, so that $I_{yy} = I_s$, $I_{xx} = I_{zz} = I_{\perp}$, $\tau_z = 0$

b) Angular velocity and angular momentum

Notice besides ω_s, Ω, there is an extra degree of freedom for rotation, rotation around z axis (the shaft BA): $\omega_z = \dfrac{d\theta}{dt}$

$$\vec{\Omega} = < \Omega \sin\theta, \Omega\cos\theta, 0 >$$

$$\vec{\omega}_{tot} = < \Omega \sin\theta, \omega_s + \Omega\cos\theta, \omega_z >$$

$$\vec{L} = < I_{\perp}\Omega \sin\theta, I_s(\omega_s + \Omega\cos\theta), I_{\perp}\omega_z >$$

c) Angular momentum change and equation of motion

The angular momentum change (here only need to consider the z component) along the z comes from two parts: one is the ω_z change over time, the other is the angular momentum as whole rotates with Ω in lab frame. (Combined is essentially the Euler equation along the z direction):

$$\left(\frac{d\vec{L}}{dt}\right)_z = I_{\perp}\dot{\omega}_z + (\vec{\Omega} \times \vec{L})_z = \tau_z = 0$$

$$(\vec{\Omega}\times\vec{L})_z = I_s(\omega_s+\Omega\cos\theta)\Omega\sin\theta - I_\perp\Omega^2\sin\theta\cos\theta$$
$$\approx I_s\omega_s\Omega\sin\theta$$

Here I used gyroscopic approximation $\Omega << \omega_s$

$$I_\perp\dot{\omega}_z + I_s\omega_s\Omega\sin\theta = 0$$

Use small angle approximation, $\theta \approx \sin\theta$:

$$I_\perp\frac{d^2\theta}{dt^2} + I_s\omega_s\Omega\theta = 0 \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{I_s\omega_s\Omega}{I_\perp}\theta = 0$$

This 2nd order ODE (ordinary differential equation) is in standard form with standard solution, the θ will do a periodic oscillation, with angular frequency: $\beta = \sqrt{\frac{I_s\omega_s\Omega}{I_\perp}} = \sqrt{\frac{L_s\Omega}{I_\perp}}$,

$\theta = \theta_0\sin(\beta t + \phi_0)$ where the amplitude θ_0 and initial phase ϕ_0 can be determined with initial conditions. (say initially the angle is at $\frac{\pi}{100}$ radian and zero ω_z, you work out the detail)

This gives you same result as the simpler method in the book, but I explicitly showed you what the approximations are in order to get that.

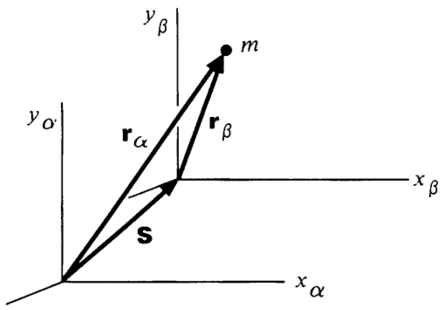
For the real setup for the gyrocompass at certain latitude on earth, use the same method above you can find the solution as that in the book, and you can see why the $\Omega_e\sin\lambda$ term of the earth spin has no effect on the motion of the gyrocompass, try it yourself.

This concludes our lengthy and detailed discussion on rotation and angular momentum. I hope you enjoyed it, if not at least you find it systematic and clear.

Chapter 8 Non-inertial Frame

All the discussions in previous chapters I stick to inertial frames on purpose. However as we had seen in last chapter we have to deal with non-inertial frames sooner or later. We actually live on earth, and we know earth orbits and spins (though the effect is small in many applications, it cannot be neglected in some cases), so strictly speaking our earth frame is non-inertial by nature. To understand some phenomena, such as tide of ocean and weather system on earth, we have to resort to non-inertial frame. So in this chapter we focus on the effect of non-inertial frame, both in translation or rotation. We shall see that *Newton's laws are still applicable in such coordinate systems, provided we add some 'extra' forces arising from the fact that the frame is non-inertial.* These forces are called **inertial force** or more appropriately the **fictitious force** (the two terms are interchangeable). It is fictitious because it originates not from real physical interaction between parties, but from our choice of coordinate system. However its effect on the party in an inertial frame is real enough to be felt by the party, you all have experience of push and pull in a accelerating / decelerating or centrifugal force in a turning automobile. Its effect on the motion observed in a non-inertial frame is just as real as a real force. We shall first look at the translational accelerating frame, and the fictitious force is simple in such frame, just like an extra gravitation field. This part is easy and straightforward. It is important for the understanding of tidal force and equivalence principle between gravitation and acceleration. Then we shall treat the rotational frame, and see how the centrifugal and Coriolis forces come into picture. This is a bit more complicated, but we had some experience from last chapter, so this one would not be as hard as the stuffs we discussed in the last chapter.

8.1 Translational Accelerating Frame



As the figure shows, there are two frames (coordinate systems), the one labeled α is an inertial frame in which the Newton's laws apply; the one labeled β is accelerating with respect to α at some acceleration A, but their axes are parallel and will not change over time, so this β is called a translational accelerating frame (with respect to the inertial one).

We know the Newton's laws apply in α and we would like to know what the equation of motion looks like in β:

$\vec{r}_\alpha = \vec{r}_\beta + \vec{S}$

$\vec{r}_\alpha$ and $\vec{r}_\beta$ are the position vectors of the same point in the two frames, $\vec{S}$ is the position vector of the origin of β in α. Take time derivatives on both sides:

$$\ddot{\vec{r}}_\alpha = \ddot{\vec{r}}_\beta + \ddot{\vec{S}}$$

$\ddot{\vec{r}}_\alpha = \ddot{\vec{r}}_\beta + \vec{A}$ times mass on both sides and applies 2nd law for the inertial frame:

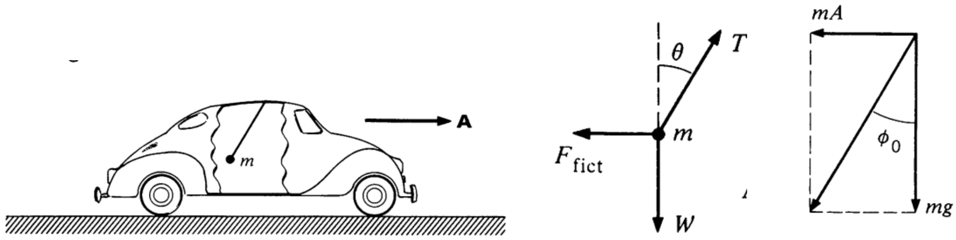
$$m\ddot{\vec{r}}_\beta = \vec{F} - m\vec{A} \qquad (8\text{-}1)$$

This is the equation of motion in the translational acceleration frame (what an observer sees in such non-inertial frame). It is still in the form of Force=ma, but now the acceleration is what observed in the non-inertial frame; and force has an extra component: besides the real force applied on the particle m, the particle also 'feels' an extra force $-m\vec{A}$. This $-m\vec{A}$ arises solely from our choice of a non-inertial frame, and is called fictitious force or inertial force (it may be a little confusing, the inertial force is 'felt' because we choose a non-inertial frame).

With the inclusion of this fictitious force, $-m\vec{A}$, the dynamics in the translational non-inertial frame could be solved exactly same as in inertial frame, as the example below illustrates:

Example: The pendulum in an accelerating car.

As the figure below shows, the pendulum is hang in a car accelerating with A, what is the equilibrium position of the pendulum and what is the period of its oscillation?



The equilibrium angle (vs. the vertical) is:

$$\tan\phi_0 = \frac{A}{g}$$

The total force of F_{fict}+mg will be along this direction and it can be treated as an effective weight, with an effective gravity:

$$\vec{W}_{eff} = \vec{W} + \vec{F}_{fict}$$
$$|\vec{W}_{eff}| = mg_{eff} = m\sqrt{g^2 + A^2}$$

So in an accelerating car, the effective vertical has angle ϕ_0, i.e. this is the direction of a plumb bob at equilibrium will tell you, and the effective gravity constant is g_{eff}, i.e. this is the pull 'felt' by the bob at equilibrium and also the weight shown by a weight-meter (the value |T |=mg_{eff}).

As to the period of the bob, the analysis will be exactly same as the pendulum under gravity with g replaced by g_{eff}:

$$\omega = \sqrt{\frac{g_{eff}}{l}}, T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g_{eff}}}$$ for small angle oscillation around ϕ_0.

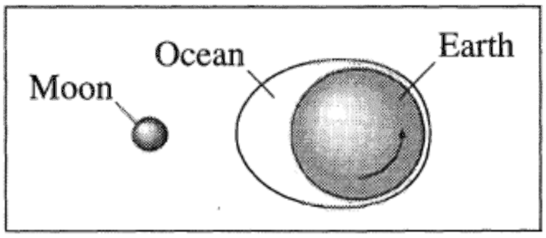
Besides simplifying analysis in some situations as shown above, the fictitious force in the translational acceleration frame finds its application in explaining the tidal force and equivalence between acceleration and gravity (also called equivalence principle in general relativity).

(1) Tidal Force

This term originates from the observation of tides in ocean on earth. We know that the ocean has high and low tides (climax and ebb) every day, and it happens *twice* daily. The question is how to explain this phenomenon. The effect comes from two facts:

a) The gravitational field of Moon or Sun is not uniform, the gravitation force is proportional to $1/\text{distance}^2$ (the potential energy is proportional to 1/distance), and the earth is big enough to sense this non-uniformity.
b) The earth is in free fall[78] towards Sun or Moon, so the earth is a non-inertial frame.

If we only consider the fact a), we will get wrong conclusion that there is only one high tide per day as shown in the figure below:



(a) Wrong

To get the correct answer, we have to consider both a) and b) and include the fictitious force due to the free fall of earth for an observer on the earth.

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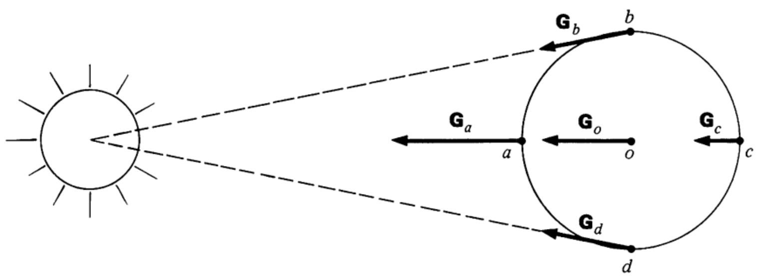
[78] The earth is orbiting around Sun, not flying straight toward it. But the effect is the same as free fall, i.e. the acceleration toward the sun is same for an orbiting earth and that of free fall.



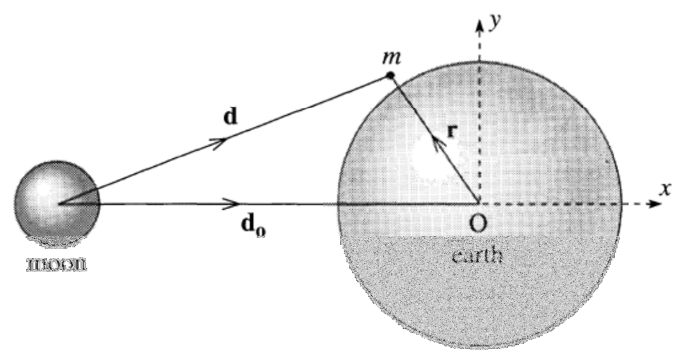
The figure shows the gravitation force on different points on the earth due to Sun or Moon, since the effect of Moon is larger, so in the calculation below I will only use Moon:

$$\vec{F} = -G\frac{M_{moon}m}{d^2}\hat{d}, \quad \vec{G} = \frac{\vec{F}}{m} = -G\frac{M_{moon}}{d^2}\hat{d}$$

G is the gravitation constant, d is the distance of point on earth to the center of moon, $\hat{d}$ is an unit vector from center of Moon towards points on earth, as the figure below shows. The earth is big enough so that different points (a, b, c, d, o in the figure) feel different forces. This is fact a) above.

Now take the fact b) into consideration, the earth is accelerating toward the Moon with acceleration:

$$\vec{A} = \frac{\vec{G}_o}{m} = -G\frac{M_{moon}}{d_o^2}\hat{d}_o$$



So the total apparent force on the points on earth for the earth-bound observer is:

$$\vec{F}'_{apparent} = \vec{F} - m\vec{A} = -G\frac{M_{moon}m}{d^2}\hat{d} + G\frac{M_{moon}m}{d_o^2}\hat{d}_o \qquad (8\text{-}2)$$

$$\vec{G}' = \frac{\vec{F}_{apparent}}{m} = -GM_{moon}(\frac{\hat{d}}{d^2} - \frac{\hat{d}_o}{d_0^2}) \qquad (8\text{-}3)$$

For the point *a* in the figure at last page: $\hat{d}_a = \hat{d}_o = \hat{x}$

$$\vec{G}'_a = -GM_{moon}(\frac{1}{(d_o - R_e)^2} - \frac{1}{d_o^2})\hat{x}$$

$$\frac{1}{(d_o - R_e)^2} = d_o^{-2}(1 - {R_e}/{d_o})^{-2} \approx \frac{1}{d_o^2}(1 + 2\frac{R_e}{d_o})$$

$$\vec{G}'_a = -GM_{moon}\frac{2R_e}{d_o^3}\hat{x} \approx -2GM_{moon}R_e\frac{\hat{x}}{d^3} = -\frac{Q}{d^3}\hat{x}$$

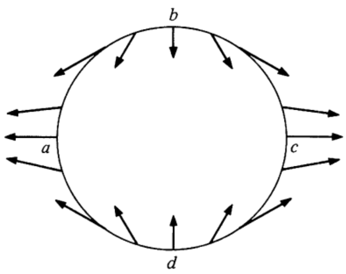
The apparent force felt by the points on earth is called the tidal force, it is smaller than the gravitation force (because it results from difference between the two gravitational force, as (8-3) indicates), and is roughly proportional to the 1/distance^3.

The calculation for $\vec{G}'_c$ is similar but will point toward $+\hat{x}$.

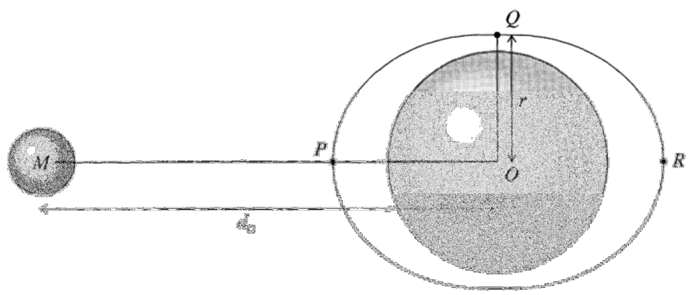
The calculation for tidal force on points b, d are give in the KK's book which will give (for b):

$$\vec{G}'_b = G\frac{M}{d_o^2}\frac{R_e}{d_o}(-\hat{y}) \approx -\frac{GMR_e}{d^3}\hat{y} = -\frac{Q}{2d^3}\hat{y}$$

The tidal force distribution over the earth surface would be something like:



The points a, c are locations for high tides and b. d for low tides, that is why every day you have twice the high / low tides (corresponding to a, c, b, d duo to spin of earth). We could also estimate the height of the tides (KK gives one account for this calculation, here is another from the argument of potential energy). Let's suppose initially the water around the earth is spherical, uniformly distributed with same height above sea-level, so that the gravity potentials due to earth are same. Due to the tidal force, the water from b, d will flow towards a, c. The sea-level at a, c will increase and at b, d will drop. This change of earth-gravitation potential energy (change of height) is compensated by the change of potential energy due to tidal force, i.e. The increase of earth-potential (increase of height) is compensated by the decrease of tidal-potential for points at a, c. At final equilibrium under earth gravity and tidal force, the total potential $U_{total} = U_e + U_t$ will be a constant for the surface of water around the earth:



At equilibrium, the PQR water surface will be an equal-potential surface for $U_{total} = U_e + U_t$. And the total force $\vec{F}_{total} = \vec{F}_{earth} + \vec{F}_{tidal} = -\nabla U_{total}$ will be perpendicular to this equal potential surface (a property of the gradient vector), so the water will not be driven by the external force on such surface (of course the water still moves due to inactions with internal factors, such as wind etc). From this equal potential, we can estimate the height of the tides:

$$U(P) = U_e(P) + U_t(P) = U_e(Q) + U_t(Q) = U(Q)$$

$$U_e(P) - U_e(Q) = mgh = U_t(Q) - U_t(P)$$

The potential of tidal force can be computed from the force formula (8-2):

The first half of the tidal force is just $-G\frac{M_{moon}m}{d^2}\hat{d}$, gravity due to moon, and the potential associated with this is (which is just the potential due to the gravitation field of the Moon):

$$U_1 = -GM_{moon}m\frac{1}{d}$$

The second part of the tidal force is a constant force which arise from the fictitious force: $G\frac{M_{moon}m}{d_o^2}\hat{d}_o$, or in the coordinate shown in the figure: $G\frac{M_{moon}m}{d_o^2}\hat{x}$, and the potential associated with this force is:

$$U_2 = -G\frac{M_{moon}m}{d_o^2}x$$

Thus, the potential associated with tidal force is:

$$U_t = U_1 + U_2 = -GM_{moon}m(\frac{1}{d} + \frac{x}{d_o^2})$$

For point Q, x=0, $d = \sqrt{d_o^2 + R_e^2} = d_o\sqrt{1+(R_e/d_o)^2}$

$$U_t(Q) = -GM_{moon}m\frac{1}{d_o}(1+(R_e/d_o)^2)^{-\frac{1}{2}} \approx -\frac{GMm}{d_o}(1-\frac{1}{2}\frac{R_e^2}{d_o^2})$$

For point P, $x = -R_e,\ d = d_o - R_e = d_o(1 - R_e/d_o)$

$$U_t(P) = -GMm(\frac{1}{d_o - R_e} - \frac{R_e}{d_o^2}) \approx \frac{-GMm}{d_o}(1+\frac{R_e}{d_o}+\frac{R_e^2}{d_o^2}-\frac{R_e}{d_o}) = \frac{-GMm}{d_o}(1+\frac{R_e^2}{d_o^2})$$

In the above derivation, I used Taylor series for $(1\text{-}x)^{-1}=1+x+x^2+\dots$ and keeps 2nd order terms so that the approximation would be similar at point P with respect to the approximation at Q, if you stop at 1+x, the approximation is a bit too crude for P.

$$mgh = U_t(Q) - U_t(P) = \frac{3}{2}\frac{GMmR_e^2}{d_o^3}$$

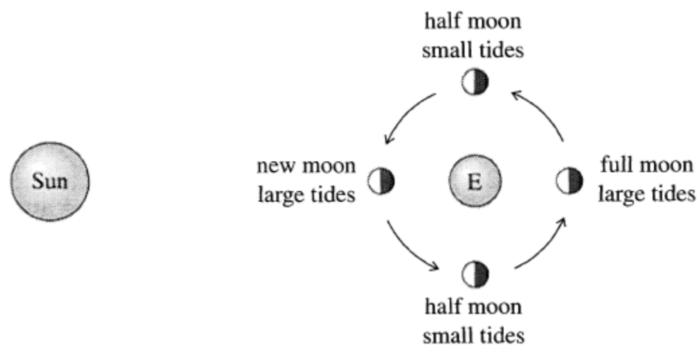
$$h = \frac{3}{2}\frac{GM_{moon}R_e^2}{gd_o^3}$$

This is exactly same as in KK's book derived from another model. You plug in the number for Moon or Sun with their mass and distance to earth, you will find that:

$$h_{moon} \approx 54cm$$
$$h_{sun} \approx 24cm$$

So the effect of Moon is twice as large as that of Sun, because of its close range and thus bigger non-uniformity of the its gravitation field over earth.

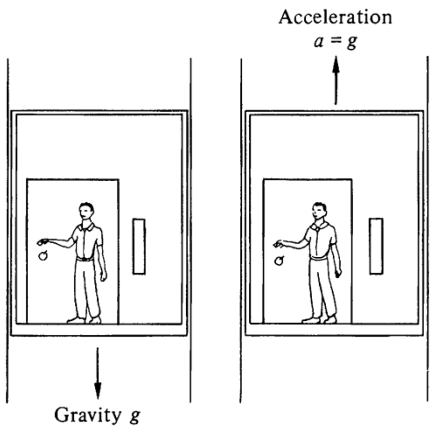
From the argument above, we see that when the Moon and Sun along the line of (PR, or ac) with respect to earth, their tidal effects enhance each other, and we will get largest high tide in the month: It happens also twice per month, when the sun and moon on the same side (new moon) or on opposite side (full moon). The height of wave will be ~78 cm. But at some geological locations, such as narrow straight or channel, due to the constraint, the tide can be much more dramatic (this is indeed the reason of the famous tide where the Qian-Tang River meets the sea). When the Sun and Moon location at P and Q direction respectively, their effect cancels and we will have smaller high tides, also twice a month.



(2) Equivalence Principle and Eotvos Experiment

We have seen that in a translational accelerating frame, the effect is equivalent to introduce a fictitious force F=-mA. This adds an additional force to every point in the system, just like a gravitation field. The effective or apparent gravity constant would be:

$$\vec{g}_{eff} = \vec{g} - \vec{A} \qquad (8\text{-}4)$$



Considering the physics in the figures above, the right figure is an non-accelerating frame with gravity g (a stationary elevator on surface of earth), and the left is an accelerating frame without gravity (an accelerating elevator in outer space). The physics observed by the observers in each individual frame would be exactly same: F=-mg for the “apple” in each elevator. The observer cannot tell the difference by doing experiment within each frame: i.e.

The physics in a non-accelerating frame with gravity g is equivalent to the physics in a frame without gravity but accelerating with A=-g.

Equivalently, suppose we are in a free fall frame, that is the fictitious force cancels the gravity, A=g. The “apple” in such free fall frame will not feel any effective gravity. If you drop the apple, it just hangs in the air; if you push it to give it some initial velocity, it just travels in straight line (suppose no other external forces of course). So the free fall system is just like a perfect inertial frame neglecting the gravity force. i.e.:

Physics in a free fall frame in a gravity field is equivalent to the physics in an inertial frame without gravity.

The above two expressions are usually called Equivalence Principle in general relativity, which is the "happiest idea" by Einstein. It shows the equivalence between acceleration and gravity, and offers a way to deal with physics under gravity. One simple application of this equivalence principle is discussed in Note 8.1 of KK, to explain the gravitational red shift of light frequency, please read that yourself though it is out of the scope of this course, we only need that fact when we explain the twins-paradox in special relativity.

There are two subtleties need to be addressed to this equivalence principle.

a) It is a local effect.

   Local means our 'elevator' has to be small enough so that the gravitational field can be treated as uniform, though the force varies with $1/\text{distance}^2$. If the gravity source is far away (be it the center of earth or other stars) compared to the size of the elevator, the local approximation is good, and the gravity can be cancelled completely with the uniform acceleration. If the size of 'elevator' is big, then the non-uniform of the gravity need to be considered, result in the tidal force we discussed in (1).

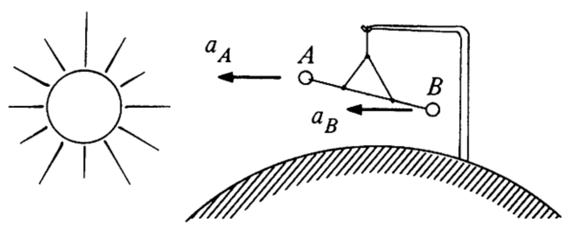
b) The equivalence between gravitation mass and inertial mass

   The equivalence principle resides on this fact. The total force felt by the object in the accelerating frame is (from (8-4)):

$\vec{F} = m_{gra}\vec{g} - m_{iner}\vec{A}$

For the equivalence principle to be always true requires that the gravitational mass (defined by the Universal Gravitational Force) and inertial mass (defined by 2^{nd} law) has to be same (or same ratio) for *all materials*. We have discussed this in chap.4 of this notes, and I asked you to think about experiments to demonstrate this. The simplest ones will be free fall objects of different compositions, such as dropping different balls from Pisa tower and confirm that they all reach the ground simultaneously (as Galileo did). Or use pendulum of same length but with bobs made of different materials and confirms that they all have same periods (As Newton did). Please do the math yourself to confirm that these experiments indeed can show the equivalence of the two masses defined.

A more accurate version to confirm such equivalence is by Hungarian nobleman and physicist von Eotvos and its modern version (by R. Dicke) shown in the figure below:



A, B are bobs that are made of different materials with same gravitational mass (say measured using a balance on earth[79]).

The total force felt by A for observer on earth:

$$\vec{F}_A = \frac{-GM_s m_{gra}^A}{r_s^2} - m_{iner}^A \vec{G}_o$$

$\vec{G}_o = \dfrac{-GM_s}{d_o^2}$, the free fall acceleration of the earth.

$$\vec{F}_B = \frac{-GM_s m_{gra}^B}{r_s^2} - m_{iner}^B \vec{G}_o$$

Since we have A,B with same gravitational mass, the first part of the force are same for A and B. The force will be same (so that the torques by A,B w.r.t. pivot will be same in magnitude but cancel each other) only if the inertial masses are same for A,B too, then there will be no torsion detected by the balance in the apparatus shown above, this is indeed the case for bobs made of different materials.

8.2 Rotational Frame

For the rotational motion, the velocity will change over time, so that you always have nonzero acceleration associated with rotation, this implies all rotational frames are non-inertial, there will be fictitious force by

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[79] As we shall discuss in the next section, due to the spin of earth, the measured balanced mass is actually a combination of gravitational mass (most part) and inertial mass (a very small part). Because of the centrifugal force of the earth spin, the effective gravity on earth is a little bit off the true gravity. Of course you can measure the mass with balance at North/South pole to null the spin effect. Actually using the effective mass measured by balance has the advantage so that we do not need to consider the effect of earth spin and concentrate on the analysis on the force due to Sun (or Moon).

choosing rotational frame. Sometimes, we have to use rotational frame. For instance, we are living on the spinning earth, so the earth-coordinates are rotational type; we had also seen in last chapter dealing with rotation of rigid body, it is has advantage to choose the coordinate axes overlapping with the principle axes of the body. Since the body rotates and such coordinate system will be a rotational frame.

The biggest difference between translational and rotational frame lies in the fact that how the base vectors change over time. In the translational type coordinate, base vectors are constant vectors over time; but in rotational type, the base vectors change over time. In order to see the formula for the fictitious force in rotational frame, we first need to study the relations in the change of vectors viewed from an inertial frame (has to be translational type) and viewed from a rotational frame, and prove the most important relation in this section, i.e.: for any vectors **A,** its time derivative $(d\mathbf{A}/dt)_{iner}$ viewed in an inertial frame is related to the derivative viewed in a rotating frame $(d\mathbf{A}/dt)_{rot}$ by:

$$(\frac{d\vec{A}}{dt})_{iner} = (\frac{d\vec{A}}{dt})_{rot} + \vec{\Omega} \times \vec{A} \qquad (8\text{-}5)$$

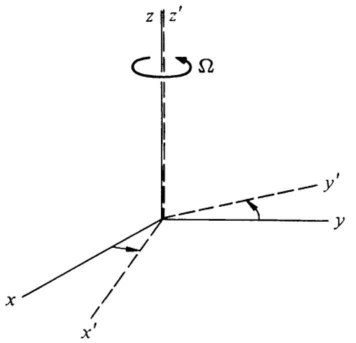
The $\vec{\Omega}$ is the angular velocity vector of the rotating frame with respect to the inertial frame. (8-5) is the relation I mentioned in the footnote 66 with some explanation but no proof, I will remedy that here.

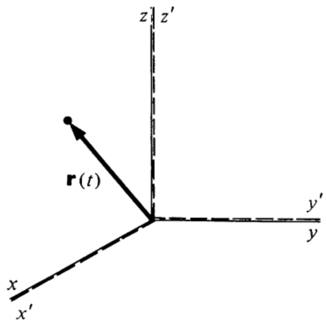
Comment: a common confusion arises from the vector **A**, so I need to address it to make it clear. The vector **A** in the formula (8-5) is the one same vector but viewed in different frames. Example: Beijing-Shanghai position vector A, viewed by ground observer is stationary and not changing with time; but for an observer in a rotating frame (say astronauts in spaceship) the same vector will appear rotating. This is easy to grasp for such vectors whose definition is independent of rotation. But there are vectors whose definition depends on the rotation, the most obvious example is that of angular velocity. The angular velocity vector viewed in one frame is $\vec{\omega}$; but in a rotating frame, the angular velocity may appear as $\vec{\omega}'$, an different vector. This is nothing new, the point I'd like to stress is the vector used in (8-5) is the same one. i.e. $\vec{\omega}$ (or $\vec{\omega}'$) on both sides of equation. You cannot put $\vec{\omega}$ on the right hand side and $\vec{\omega}'$ on the left side arguing that is the angular velocity in the rotating frame. That is not what (8-5) tells us! For the angular velocity case here, though the rotating frame observers see the angular velocity as $\vec{\omega}'$, but just imaging what happened he sees the change rate of $\vec{\omega}$. That is what the (8-5) tries to tell us, relating the change rate of ***same*** vector viewed by different observers. You may wonder why we adopt such treacherous method, this is because we need to such as in case of treating the general rotation in 3-D. On the one-hand the equation of motion applies to inertial frame, we need to express the change rate of the vector in inertial frame (lab frame); on the other hand the expression directly calculated from lab frame would be difficult (such as due to non-diagonal, time varying inertia tensor) but can be evaluated relatively easy in a rotating frame applying (8-5), that is exactly what Euler equation is about.

8.2-1 Prove $(\frac{d\vec{A}}{dt})_{iner} = (\frac{d\vec{A}}{dt})_{rot} + \vec{\Omega} \times \vec{A}$

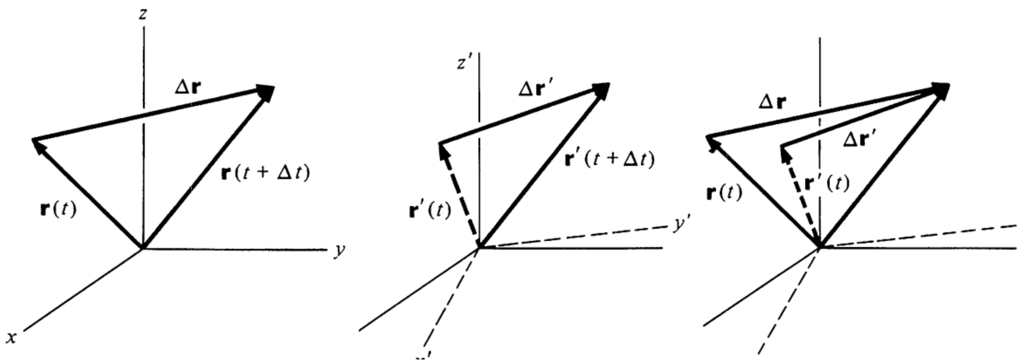
The proof can be carried out in two ways: Geometrical and Analytical. Both have its own merit so that I will show both below (KK provides the geometrical one in 8.5 and analytical one in note 8.2).

(1) Geometrical Proof





Suppose we have inertial frame x-y-z and a rotating frame x'-y'-z'. The two frames share same origin (if the rotating origin accelerating translationally, we had already discussed this situation, a fictitious force F=-mA shall be added. We neglect translational motion of the frame here to concentrate on rotational effect), x'-y'-z' rotating with angular velocity $\vec{\Omega}$ with respect to x-y-z. I shall choose $\vec{\Omega}$ to be the z and z' axes for simplicity of drawing but generally it can be any direction or magnitude.



Consider the change of vector **r** over time. In the inertial frame:

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

In the rotating frame, the initial vector will rotate with the coordinates, so for the observer in x'-y'-z', he/she will think the $\vec{r}'(t)$ as the initial vector:

$$\Delta\vec{r}' = \vec{r}'(t+\Delta t) - \vec{r}'(t)$$

In the above relation $\vec{r}'(t+\Delta t)$ is the same vector as $\vec{r}(t+\Delta t)$, but the $\vec{r}'(t)$ is not same as $\vec{r}(t)$. So there is a difference between the *change* of vectors viewed from the two systems:

$$(\Delta\vec{r})_{iner} = (\Delta\vec{r}')_{rot} + \vec{r}'(t) - \vec{r}(t)$$

Now we can derive the relation between the time derivative of vectors viewed from two systems:

The $(\Delta\vec{r})_{iner}/\Delta t$ is the time derivative of vector in an inertial frame, $(\Delta\vec{r}')_{rot}/\Delta t$ is the time derivative of the same vector in rotating frame. The question is what is $(\vec{r}'(t) - \vec{r}(t))/\Delta t$? This term is nothing but a change of vector with a fixed magnitude and rotate over time, we have known such change rate:

$$\frac{d(\vec{r}'-\vec{r})}{dt}=\vec{\Omega}\times\vec{r}$$

Of course this can also be worked out from the figure above where $\vec{r}'(t)-\vec{r}(t)=r\sin\theta\Omega\Delta t$, θ is the angle between r and Ω.

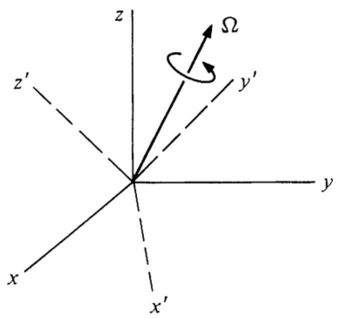
Put all above together:

$$(\frac{d\vec{r}}{dt})_{iner}=(\frac{d\vec{r}}{dt})_{rot}+\vec{\Omega}\times\vec{r}$$

I dropped $\vec{r}'$ and use $(\frac{d\vec{r}}{dt})_{rot}$, because the subscript already implies that the vector needs to be expressed in the rotating frame.

In the above derivation, nothing special about the position vector, so we can replace it with any vector **A**, and thus we get relation (8-5)

(2) Analytical Proof



The x'y'z' and xyz are shown in the figure, Ω is chosen to point in arbitrary direction (for we do not need drawing in this method). For a vector **A**, it can be expressed in both coordinates:

$\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$ in the inertial frame

$\vec{A}=A'_x\hat{x}'+A'_y\hat{y}'+A'_z\hat{z}'$ in the rotating frame

Let's take time derivative of the vector **A**:

$\frac{d\vec{A}}{dt} = \dot{A}_x\hat{x} + \dot{A}_y\hat{y} + \dot{A}_z\hat{z} = (\frac{d\vec{A}}{dt})_{iner}$ using the expression in inertial frame.

Using the expression in rotating frame, take time derivative, this time the base vectors also changes over time:

$$(\frac{d\vec{A}}{dt})_{iner} = \dot{A}'_x\hat{x}' + \dot{A}'_y\hat{y}' + \dot{A}'_z\hat{z}' + A'_x\dot{\hat{x}}' + A'_y\dot{\hat{y}}' + A'_z\dot{\hat{z}}'$$

$\dot{A}'_x\hat{x}' + \dot{A}'_y\hat{y}' + \dot{A}'_z\hat{z}' = (\frac{d\vec{A}}{dt})_{rot}$ the change of vector viewed by the observer in the rotating frame who does not notice the change of base vectors.

$\dot{\hat{x}}'$ is the change of base vectors with rotation Ω. It is the change of unit vector (length fixed) over time, and we know its formula:

$\dot{\hat{x}}' = \vec{\Omega} \times \hat{x}'$, similar formula for y',z', thus:

$$\begin{aligned}(\frac{d\vec{A}}{dt})_{iner} &= (\frac{d\vec{A}}{dt})_{rot} + A'_x\vec{\Omega} \times \hat{x}' + A'_y\vec{\Omega} \times \hat{y}' + A'_z\vec{\Omega} \times \hat{z}' \\ &= (\frac{d\vec{A}}{dt})_{rot} + \vec{\Omega} \times (A'_x\hat{x}' + A'_y\hat{y}' + A'_z\hat{z}') = (\frac{d\vec{A}}{dt})_{rot} + \vec{\Omega} \times \vec{A}\end{aligned}$$

Q.E.D.

8.2-2 Relations between Accelerations in Inertial and Rotating Frames and Fictitious Forces

Acceleration is just the second derivative of position vectors over time, also the first derivative of velocity. For the velocity vectors, we have:

$$\vec{v}_{iner} = \vec{v}_{rot} + \vec{\Omega} \times \vec{r} \qquad (8\text{-}6)$$

Now take the time derivative again for velocity, i.e. take $\vec{A} = \vec{v}_{iner}$ in (8-5):

$$(\frac{d\vec{v}}{dt})_{iner} = [\frac{d}{dt}(\vec{v}_{rot} + \vec{\Omega}\times\vec{r})]_{rot} + \vec{\Omega}\times(\vec{v}_{rot} + \vec{\Omega}\times\vec{r})$$

$$= (\frac{d\vec{v}_{rot}}{dt})_{rot} + (\frac{d\vec{\Omega}}{dt})_{rot}\times\vec{r} + \vec{\Omega}\times(\frac{d\vec{r}}{dt})_{rot} + \vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{a}_{iner.} = \vec{a}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + 2\vec{\Omega}\times\vec{v}_{rot} + \frac{d\vec{\Omega}}{dt}\times\vec{r} \qquad (8\text{-}7)$$ [80]

Multiply the mass on both sides and apply the 2nd law for the inertial acceleration:

$$m\vec{a}_{rot} = \vec{F} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) - m2\vec{\Omega}\times\vec{v}_{rot} - m\frac{d\vec{\Omega}}{dt}\times\vec{r} \qquad (8\text{-}8)$$

There are three extra terms on the right hand side of (8-8), each defines a fictitious force (I shall omit the fictitious below):

Centrifugal Force:

$$\vec{F}_{cen} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \qquad (8\text{-}9)$$

Coriolis Force:

$$\vec{F}_{cor} = -m2\vec{\Omega}\times\vec{v}_{rot} \qquad (8\text{-}10)$$

Azimuthal Force:

$$\vec{F}_{az} = -m(\frac{d\vec{\Omega}}{dt})\times\vec{r} \qquad (8\text{-}11)$$

For the most simplified cases considered in this class, the rotating frame rotates with a constant angular velocity with respect to the inertial frame,

---

[80] Comparing the (8-7) to (3-52) which is the acceleration in an inertial frame expressed in terms of polar components, you will find striking similarity. The four terms on the RHS of (8-7) are just the terms on the RHS 0f (3-52). This is no coincidence of course, because in the polar coordinate, we can choose a rotating frame so that $\Omega = \dot{\theta}$ (i.e. in such rotating frame, position vectors only have radial motion), and (8-7) will give us (3-52). So (3-52) is just a special case of (8-7). We derived (3-52) based on the fact that the base vectors $\hat{r}, \hat{\theta}$ are changing with time, same as we derived (8-7) here.

i.e. $\vec{\Omega}$ is constant, then the Azimuthal force would be zero[81]. We shall focus on the Centrifugal and Coriolis force henceforth.

With the inclusion of these fictitious force (adding them to the real forces), we can apply the 2nd law to the rotating frame: (in the absence of Azimuthal force)

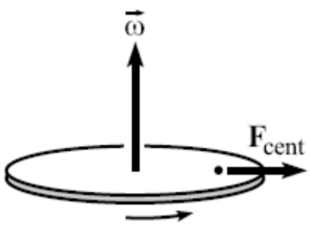
$$m\vec{a}_{rot} = \vec{F} + \vec{F}_{cen} + \vec{F}_{cor} \qquad (8\text{-}12)$$

In the following I shall give a detailed discussion on these fictitious forces.

(1) Centrifugal Force

This force is the easiest to understand since we all have experience with it sitting on a turning car, you will feel the force pushing you away from the center of turning.

Imagine you are sitting on a carousel which is rotating with $\vec{\omega}$. In the rotating frame of the carousel, you are motionless:



The force on you when observed in this rotating frame will be:

Centripetal force by friction (or tension of strings etc.) which is the real

[81] Sometimes you may see the azimuthal force is expressed in terms of $\frac{d\vec{\Omega}}{dt}$ in some textbooks, with respect to the inertial frame. This expression is same as given in the note, because for the angular velocity, $(\frac{d\vec{\Omega}}{dt})_{iner} = (\frac{d\vec{\Omega}}{dt})_{rot} + \vec{\Omega}\times\vec{\Omega}$, the last term is zero.

force:

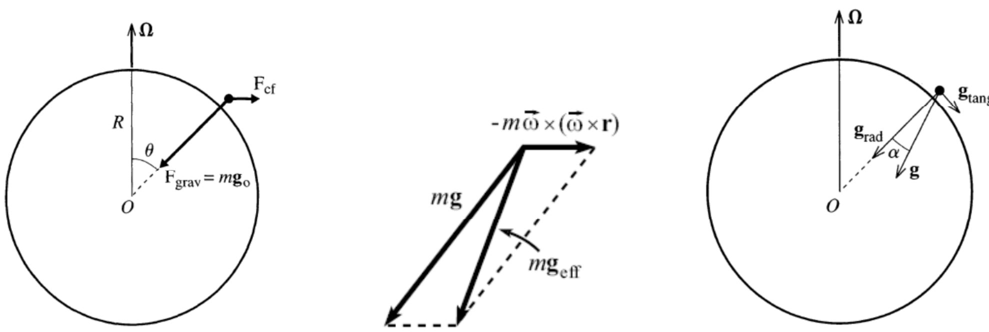
$\vec{F} = m\omega^2 r(-\hat{r})$ pointing towards center.

The centrifugal (fictitious) force in this case is:

$\vec{F}_{cen} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2 r\hat{r}$ pointing away from center (try it with right-hand rule, or use cross-product relations between $\hat{r}, \hat{\theta}, \hat{z}$)

The centrifugal cancels the centripetal in the rotating frame, as required by (8-12).

Example: Effective gravity on earth.



The real gravity on a particle at surface of earth is $\vec{F}_{grav} = m\vec{g}_0$ pointing towards center of earth[82]. For anyone lived on earth (an earthling), we choose a local coordinate (North-South as y, East-West as x, and up-down as z) which rotates with earth spin velocity $\vec{\Omega}$, which has a small value $\sim 7.3 \times 10^{-5} radian / s$. For a stationary particle (be you or me or skyscraper standing on the ground), there will be a centrifugal force with value:

$F_{cen} = m\Omega^2 R \sin\theta$ and its direction is shown in the figure.

[82] Here I assume the earth is a sphere, which is a good approximation. The actual shape of earth is a ellipsoid with slight bigger diameter in equator due to the centrifugal force.

θ is the polar angle (which is related to the latitude angle λ by $\theta = \frac{\pi}{2} - \lambda$).

Because of this centrifugal force, the effective gravity force felt by the particle is: $\vec{F}_{eff} = m\vec{g}_0 + \vec{F}_{cen}$, and the effective gravity constant is: $\vec{g}_{eff} = \vec{g}_0 + \vec{F}_{cen} / m$.

The exact value can be computed using the figure above and cosine laws of triangle (KK problem 8.10). Here I would use some approximation:

Decompose the centrifugal gravity along radial and tangential direction:

$$g_{rad} = g_0 - \Omega^2 R \sin\theta \sin\theta \approx g_0$$

$$g_{\tan} = \Omega^2 R \sin\theta \cos\theta$$

The angle α between the true vertical (pointing towards earth center) and apparent vertical (the direction shown by a plumb bob hanging from ceiling, or the vertical perceived by us) is:

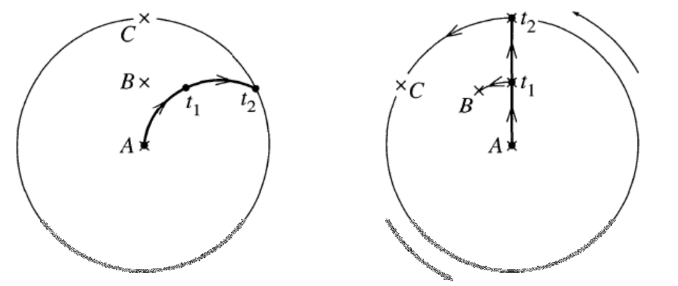
$$\alpha \approx \tan\alpha = \frac{g_{\tan}}{g_{rad}} \approx \frac{\Omega^2 R \sin\theta \cos\theta}{g_0}$$

At latitude of Beijing (approximate it at 45 degree latitude), this angle is about 0.0017rad.=0.1 degree.

(2) Coriolis Force

This fictitious force is more subtle than the centrifugal, so let me first discuss it from some intuitive point of view. Let's suppose a particle on the equator of earth, which moves along east and west direction with same speed of earth spin, which is 1000miles/hour (1 mile$\approx$1.6km). So the particle does not move along east-west direction for the earthlings at equator. Now suppose that the particle also has a North velocity so that it moves towards higher latitude in North-Hemisphere. From an inertial frame if there is no external force (the gravity is balanced off by supporting force), the particle's velocity will not change. But viewed by an earthling at higher latitude where the spin speed is smaller, say 700 miles at 30 degree latitude, the particle will appear to have an east-ward velocity of 300 mi/hour relative to the earthling, so he concludes that there must be a force pushing the particle towards east (or right if viewed along the direction that the particle travels), this force is the Coriolis force arising from the rotating frame chosen by the earthling.

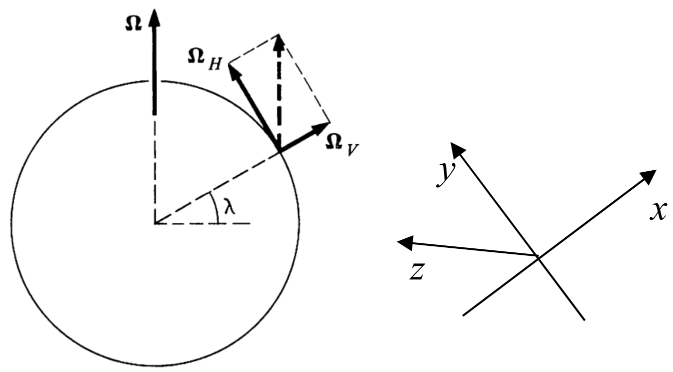
Another intuitive example is as figure shows below:



(a) As seen in rotating frame (b) As seen in non-rotating frame

A frictionless disk is rotating with angular velocity. A particle initially starts at point A and travels upward in the inertial frame. For the lab fixed observer (in non-rotating frame), the particle will simply travel in a straight line. But for an observer in the rotating frame whose up direction is defined by ABC, the particle's trajectory would be that shown in (a), bending towards right, the Coriolis force is the reason for this 'bending' for the observer in rotating frame: $-\vec{\Omega}\times\vec{v}_{rot}$ with $\vec{\Omega}$ upward for c.c.w. rotation. If the disk (rotating frame) is rotating clockwise, the Coriolis force will bend the trajectory towards left.

This same effect plays important role in the metrology system of the earth on the atmosphere motion (KK example 8.9,8.10)

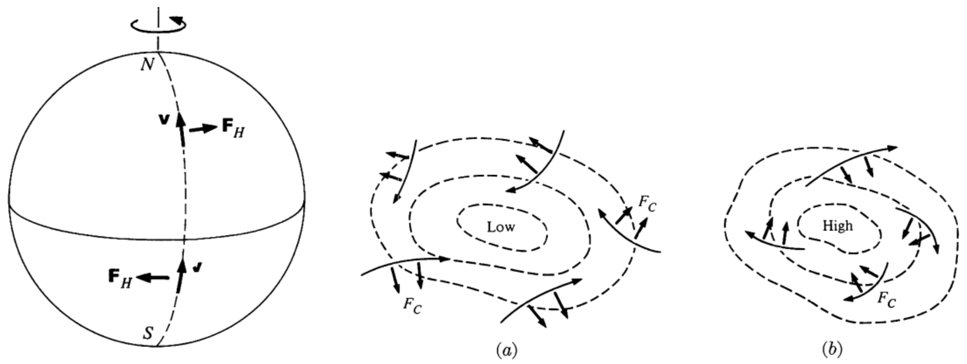


The local coordinate x-y-z on the surface of earth is shown, *with +x corresponds to east, +y to north and +z to up*. This x-y-z rotates as the earth spins. Let's consider the surface motion of some particles along the earth surface, i.e. $\vec{v}$ in the x-y plane. At latitude λ, the $\vec{\Omega}$ can be expressed as:

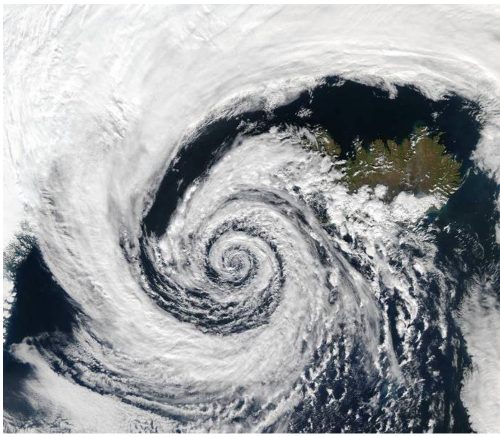
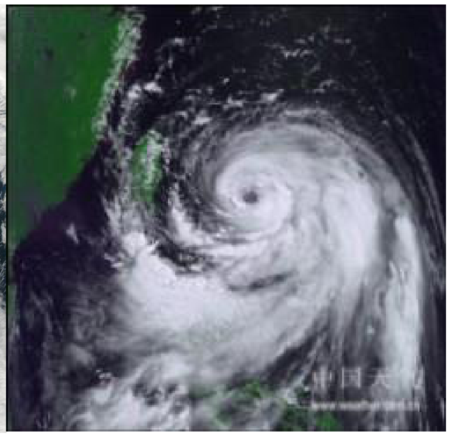
$\vec{\Omega} = <0, \Omega\cos\lambda, \Omega\sin\lambda>$, $\vec{v} = <v_x, v_y, 0>$ for surface motion

The y component (or the Horizontal H component in the figure) of $\vec{\Omega}$, will produce a Coriolis pointing along the z direction, its order of magnitude is about $2\Omega v\cos\lambda\sin\theta < 2\Omega v$. Since Ω is a small number, so as long as v is not too big, $2\Omega v$<<g (the gravity constant), and this Coriolis force due to Ω_y can be neglected for this kind of problems.

The z component (Vertical V) of the $\vec{\Omega}$, will produce a Coriolis force in the x-y plane (called F_H in KK). It always pushes the motion to the right side (viewed along the v and in the North Hemisphere):



This effect is important in the weather system on the north hemisphere, it explains why the hurricanes, cyclones or typhoons(with low pressure center) rotates counterclockwise as shown in the figure (a) above and the real photos below:

(This part is just for fun) There was once a ‘myth’ about the rotation of the toilet water, i.e. if you flush the toilet, which direction the toilet water rotates? (or you fill the sink with water and unplug the stopper, water will form vortex and what is the direction of the vortex?) The ‘myth’ attributes this to the Coriolis effect, especially publicized with one episode in the famous cartoon series “The Simpsons” (“Bart vs. Australia”, season 6 episode 16). In the cartoon, Lisa (the younger sister) claims that the vortex will rotate *counterclockwise* in the North Hemisphere, and clockwise in South Hemisphere due to Coriolis force. This intrigued Bart and he made long hour phone calls to Australia to confirm it and the fun began. The 'myth' is busted. If the Coriolis force is responsible to the formation of vortex, it will indeed make the vortex rotating *counterclockwise* as Lisa claimed. However if you observe yourself, you will find the vortex can rotate either clockwise or counterclockwise in North and South Hemisphere (The toilet in my bathroom forms clockwise vortex, but the other one (in another bathroom of course) forms counterclockwise vortex). This is because the Coriolis is a small effect compared to other forces influencing the formation of vortex in a drain (such as the design of the sink and water jets etc). I hope Myth Busters in Discovery Chanel will make one episode on this 'myth'☺.

Now back to the serious stuff. The first example is free fall of object on the surface of earth:

Example 1: Free fall of object on the surface of earth at latitude λ (KK's 8.8 is a special case on the equator).

The graph and the coordinate is same as the figure above. Initially and object is at position:

$\vec{r}(t=0) = <0,0,h>$, and initial velocity is zero. The centrifugal force will be included in the effective gravity constant as discussed in the last example, so we do not need to worry it here (i.e. our vertical +z is the direction of effective gravity and g is also the effective gravity constant). As the object fall, it will pick up the velocity downwards (-z direction) v_z, Coriolis force will push it eastwards: i.e.

$-\Omega_H(along\ \hat{y}) \times v_z(along\ -\hat{z}) \rightarrow along\ \hat{x}$. ($\Omega_V$ has no effect in this step)

Then the particle will gain velocity along $\hat{x}$, v_x. This v_x component will also generate further Coriolis effect: $-\vec{\Omega}_H \times v_x \rightarrow along\ \hat{z}(up)$, this will modify the gravity constant; $-\Omega_V \times v_x \rightarrow along\ -\hat{y}(south)$, this will create a velocity along y direction. The detailed motion will be complicated as you have seen from the above analysis.

Approximations are introduced to make the life easier: The velocity will only has significant part in –z direction, we shall neglect the Coriolis effect due to v_x,v_y (these are called higher order Coriolis effect, because v_x,v_y come from Coriolis effect in the first place). Also the modification on gravity constant due to the Coriolis effect is also neglected as argued above ($2\Omega v$<<g). With these approximations, the problem is easy to solve:

The motion along z is just a free fall under gravity:

$$\ddot{z} = -g, z = h - \frac{1}{2}gt^2, v_z = -gt$$

Along the x direction, due to the Coriolis force:

$$\ddot{x} = -2\Omega_H v_z = 2\Omega \cos\lambda gt$$

$\dot{x} = \Omega \cos\lambda g t^2 + C$, C is 0 from initial condition.

$x = \frac{\Omega \cos\lambda g t^3}{3} + C$, C is 0 too from initial condition.

Put in $t = \sqrt{\frac{2h}{g}}$:

$$x = \frac{1}{3}\Omega \cos\lambda g (\frac{2h}{g})^{\frac{3}{2}}$$

As in KK's example, if you are at equator ($\lambda = 0$), and h=50m, the deflection along x (east) is only 0.77cm. Along the y direction (a higher order effect due to v_x) is much smaller and is neglected.

If you really want to have rigorous treatment, I shall show you the procedure below:

$\vec{\Omega} =< 0, \Omega \cos\lambda, \Omega \sin\lambda >$, $\vec{v} =< v_x, v_y, v_z >$

$-\vec{\Omega} \times \vec{v} =< v_y \Omega \sin\lambda - v_z \Omega \cos\lambda,\ -v_x \Omega \sin\lambda,\ v_x \Omega \cos\lambda >$

$$\ddot{x} = 2(v_y \Omega \sin\lambda - v_z \Omega \cos\lambda)$$

$$\ddot{y} = -2 v_x \Omega \sin\lambda$$

$$\ddot{z} = -g + 2 v_x \Omega \cos\lambda$$

These are coupled 2nd ODEs, hard to get exact solutions. Approximation we used above is clear by comparing these exact equations to the approximation equations used above. If you want further improvement, you can substitute the approximation results into these exact equations and solve for better approximation. This is called reiteration process.

For the problems of effect of Coriolis force on the shooting projectile (such as battleship firing cannons toward North or East over a distance of 30km at certain latitude), the Coriolis force will deflect the projectile by

100 meters (order of magnitude). The computation will be left as homework.

Example 2: Foucault pendulum

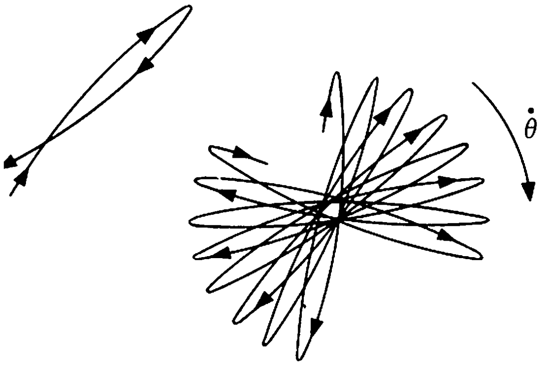
This is a device first demonstrating (by French Physicist Foucault in 1851) the Coriolis Effect and thus proved that our earth is spinning. The x-y-z coordinate for the pendulum I chose would be same as in the problem above. The detailed calculation on the motion of a pendulum on earth including the Coriolis Effect is quite complicated (similar to the above example, coupled differential equations need to be solved)[83].

Let's first see whether we can get some useful information without solving complicated equations.

a) The centrifugal force will be included in the effective gravity so no need to consider it separately here.
b) The motion of the pendulum with long length, the vertical motion (up-down) is small, and the bob is almost traveling (oscillating backward-forward) within the x-y plane. Thus the Ω_H part of the C-force will only modify the gravity constant and can be safely neglected. The period of oscillation would still be about $T = 2\pi\sqrt{\frac{l}{g}}$.
   The Ω_v component will generate horizontal 'bending' force towards the right side of travelling. So the pendulum oscillation plane will

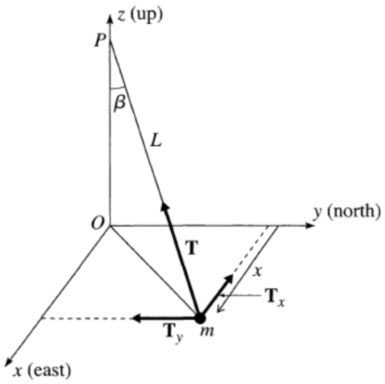
[83] A detailed account on the solution can be found at Greiner "Classical Mechanics: system of particles and Hamiltonian dynamics" (his 2nd book on mechanics), chap.3

rotate clockwise (viewing from top). The trajectory of the bob over time will be something like: (the left one is the trajectory over one period of oscillation)



(c) The precession velocity $\dot{\theta}$ can be estimated without complicated calculation. Suppose we put the pendulum at North Pole. The pendulum will oscillate in a fixed plane for an inertial observer. The earth will rotate c.c.w. with angular velocity Ω in this inertial frame. So for the earthling observer, the oscillation plane will rotate clockwise with angular speed Ω. i.e. $\vec{\dot{\theta}} = -\vec{\Omega}$ (- means clockwise). Now suppose we put the pendulum at latitude λ, the angular velocity responsible for the precession, the F_{CH} only comes from $\Omega_v = \Omega \sin\lambda$ as argued in (b). So this is just like put the pendulum on the north pole of some planet spins with $\Omega \sin\lambda$, and the same argument would lead to: $\vec{\dot{\theta}} = -\Omega \sin\lambda \hat{z}$. This is exactly the same as that given in KK's.

Now I will show you some simplified formal treatment on this (not required for this course):



The position vector of bob m is: $\vec{r} = < x, y, z > \approx < x, y, -L >$, I used the small angle approximation and going to neglect motion along z direction (this is the approximation stated in (b))

T the tension force is parallel with **r**, $\vec{T} = < T_x, T_y, T_z >$

$\vec{T} \times \vec{r} = 0$, this will give relations of the components:

$$T_z y + T_y L = 0 \to T_y = -\frac{T_z}{L} y, T_z x + T_x L = 0 \to T_x = -\frac{T_z}{L} x$$

The $T_z \approx mg$. This is in accordance to the approximation neglecting up-down motion. Then:

$$T_y = -\frac{mg}{L} y \text{ and } T_x = -\frac{mg}{L} x$$

The Coriolis Force (contribute to the F_{CH}) is same as computed before:

$-\vec{\Omega} \times \vec{v} = < v_y \Omega \sin\lambda - v_z \Omega \cos\lambda, \ -v_x \Omega \sin\lambda, \ v_x \Omega \cos\lambda >$, $v_z \approx 0$ here.

Now the equation of motion of the bob in the earth frame is:

$$\ddot{x} = 2 v_y \Omega \sin\lambda - \frac{g}{L} x \to \ddot{x} - 2\Omega \sin\lambda \dot{y} + \frac{g}{L} x = 0$$

$$\ddot{y} = -2 v_x \Omega \sin\lambda - \frac{g}{L} y \to \ddot{y} + 2\Omega \sin\lambda \dot{x} + \frac{g}{L} y = 0$$

Let's assign $\Omega \sin\lambda \equiv \Omega_z$, $\frac{g}{L} \equiv \omega_0^2$, then:

$$\ddot{x} - 2\Omega_z \dot{y} + \omega_0^2 x = 0$$

$$\ddot{y} + 2\Omega_z \dot{x} + \omega_0^2 y = 0$$

Above is a coupled 2nd ODE, and can be solved using standard method (which I shall omit here), the solution is:

$$x(t) + iy(t) = Ae^{-i\Omega_z t} \cos\omega_0 t$$

The x(t),y(t) can be solved by equating the real and complex part on both sides, the amplitude A can be determined from initial conditions. It is more illustrative by leaving the solution as above. This shows the time change of a complex number (also called 'phasor', a vector-like representation for complex numbers), the magnitude changes as $A\cos\omega_0 t$, the direction (the phase angle) rotates with $-\Omega_z = -\Omega \sin\lambda$.

Chapter 9 Two-Body Motion in Central Field

This chapter is more like an exercise to apply what we learned in the Newtonian mechanics so far. There won't be many new physical concepts in this chapter. However a useful model will be developed: We can reduce the motion of two particles in a central field by a one particle problem. The particle is somewhat fictitious with a reduce mass subject to the central field. This only works for the two-body in central field, so in this sense, it is quite special. On the other hand it is also a model finding wide applications in various physical problems such as planetary motion, hydrogen atoms and two-particle scattering process etc. This is because as I mentioned before, that fundamental forces (be gravitation, electrostatic or nuclear) are central field forces, i.e. it only depends on the relative distance between the particles, i.e.:

$$\vec{F} = f(r)\hat{r} = -\frac{dU(r)}{dr}\hat{r} \quad (9\text{-}1)$$

r is the relative distance between the particles and $\hat{r}$ is the unit vector along the line of centers, $U(r)$ is the potential of the central field.

The successful explanation of planetary motion and discovery of universal gravitation is among the earliest and most important achievements of Newtonian mechanics. We shall study this in detail in this chapter[84]. It gives the Kepler's empirical laws on planetary motion a deeper and solid physical ground.

9.1 Reduction of Two-Body Problem to One-Particle, C.M. and Reduced Mass

This is not completely new stuff. We had seen that when we deal with

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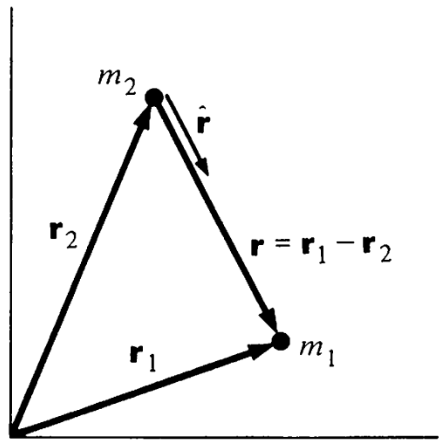
[84] The development of human knowledge from Ptolemy's Geocentric ('geo' is Latin for thing related to earth) model to Copernicus's Heliocentric ("helio" is Latin for Sun. The element Helium, second lightest, was first identified existing at Sun (He is more abundant in Sun due to nuclear fusion) before its discovery on earth) is a fabulous story. This led to Tycho's observation and Kepler's summary of his three laws regarding to the planet motion in solar system. The story is usually covered in astronomy books and won't be discussed here. Interesting students can refer to "An Introduction to Modern Astrophysics" 2nd edition by B. Carroll and D. Ostlie, Chapter 1 and 2. Or Chapter 4 in "Astronomy" 6th edition by M. Seeds and D. Backman for a lighter treatment.

multi-particles system, the common 'trick' is to decompose the motion into motion of the C.M. (a fictitious mass point containing the total mass of the system), and the motion relative to the C.M. In the rigid body case, this will be pure rotation with respect to C.M. In the general two-body which is not rigid, the decomposition of motion will become motion of C.M. plus a motion of a particle with reduced mass (another fictitious mass point). Here is the proof in math:



The two mass points are specified by vectors $\vec{r}_1, \vec{r}_2$ in an inertial frame. The same system however equivalent well represented by another pair of independent vectors: $\vec{R}, \vec{r}$, with:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2},\ \vec{r} = \vec{r}_1 - \vec{r}_2,\ r \equiv |\vec{r}\,| \qquad (9\text{-}2)$$

Reversely the $\vec{r}_1, \vec{r}_2$ can be expressed as $\vec{R}, \vec{r}$ (directly read from the figure above right):

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r}$$
$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r} \qquad (9\text{-}3)$$

In information of the system (its various physical properties, such as velocity, energy, angular momentum…and their time evolution) can be derived from the $\vec{r}_1, \vec{r}_2$, as well as from $\vec{R}, \vec{r}$. It turns out it is simpler to work out problems using $\vec{R}, \vec{r}$.

Let's take a look of equation of motion for the $\vec{R}, \vec{r}$ (noticed we have already worked out for the R before). For the central force case, we have equation of motion:

$$m_1\ddot{\vec{r}}_1 = f(r)\hat{r}$$
$$m_2\ddot{\vec{r}}_2 = -f(r)\hat{r} \qquad (9\text{-}4)$$

(9-4) 3rd law is applied here.

Adding and subtracting the above two equations will give us what we want:

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = M\frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{M} = M\ddot{\vec{R}} = 0 \qquad (9\text{-}5)$$

This is just the special case of (5-5) when we introduced C.M. in the discussion of momentum, here the total external force is zero. The C.M. will be stationary or travel with constant velocity representing the shift of the complete system. The more interesting result comes from the relative motion between particles, $\vec{r}$:

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) = \mu\ddot{\vec{r}} = f(r)\vec{r} \qquad (9\text{-}6)$$

(9-6) comes from subtracting the two equations in (9-4) (each divided by

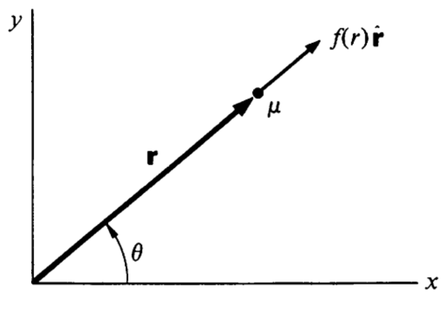
m_1 or m_2 first). μ is the reduced mass (we have seen it before in (6-63)) defined as (6-63) again:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (9\text{-}7)$$

This complete our argument that the motion of two particles in central field are decoupled into independent motions of two fictitious particles with M and μ. The interesting part is the motion of μ governed by (9-6) which is nothing but a single particle motion under a central force.

9.2 General Properties of Central Field

Here I shall concentrate on the reduced mass part, i.e. motion governed by (9-6). This is one-particle moving in a central field, and the coordinate is set up by choosing the center of the field as origin, as figure below shows:



(1) The angular momentum (associated with μ) is a constant

This is obvious from the figure, the torque by the central force is zero with respect to origin[85]. Actually we can prove this angular momentum

[85] This could also be argued from our general theory on angular momentum chapter 7. There is no external force,

(the momentum of the fictitious particle in the central field shown in the figure) is equivalent to the angular momentum with respect to the C.M. in two-particle system:

$$\begin{aligned}\vec{L}_c &= \vec{r}_{1c} \times \vec{p}_{1c} + \vec{r}_{2c} \times \vec{p}_{2c} = \vec{r}_{1c} \times \mu\vec{v} + \vec{r}_{2c} \times (-\mu\vec{v}) \\ &= (\vec{r}_{1c} - \vec{r}_{2c}) \times \mu\vec{v} = \vec{r} \times \mu\vec{v} = \vec{L}_\mu\end{aligned}$$

Here I used results from (6-62) on $\vec{p}_{1c}, \vec{p}_{2c}$, and $\vec{v} = \dfrac{d\vec{r}}{dt}$. The constant angular momentum implies that the motion of μ will be constrained to a plane. This is great, not only one particle but also a 2-dimensional problem.

(2) The mechanical energy (associated with μ) is a constant

The central force is a conservative force, as proved in example 2 in section 6.2-2 of this notes. This means the total mechanical energy is conserved, a constant of motion (This could also be equivalently argued from the two-particle system, similar to the angular momentum case). This energy is just the mechanical energy of the two-particle system with respect to the C.M.:

$$\begin{aligned}E_c &= \frac{1}{2}m_1(\vec{v}_{1c} \cdot \vec{v}_{1c}) + \frac{1}{2}m_2(\vec{v}_{2c} \cdot \vec{v}_{2c}) + U(r) = \frac{1}{2}(\vec{p}_{1c} \cdot \vec{v}_{1c} + \vec{p}_{2c} \cdot \vec{v}_{2c}) + U(r) \\ &= \frac{1}{2}(\mu\vec{v} \cdot \vec{v}_{1c} - u\vec{v} \cdot \vec{v}_{2c}) + U(r) = \frac{1}{2}\mu\vec{v} \cdot \vec{v} + U(r) = E_\mu\end{aligned}$$

---

so the total angular momentum has to be constant. The C.M. is at stationary or under uniform velocity, so its angular momentum will be constant for any origin of choice. That leads to the angular momentum associated with μ has to be constant too, this angular momentum is nothing but the angular momentum with respect to the C.M. for the two-particle system.

(3) Equation of motion of μ

The (9-6) is the vector equation and in real applications need to be expressed in a coordinate system. In the 2-D central field, the natural choice is the polar coordinate we discussed in section 3.8. Expressed (9-6) in polar coordinate is:

$\mu\ddot{r} - \mu r\dot{\theta}^2 = f(r)$ (9-8a)

$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$ (9-8b)

Above is just direct using 2nd law in polar coordinates. It is worth to take a detailed look at these two equations. As I discussed at the end of section 3.8, the (9-8b) will give us (after a small 'trick'):

$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \dfrac{1}{r}\dfrac{d}{dt}(\mu r^2\dot{\theta}) = 0$, what is $\mu r^2\dot{\theta}$? It is the angular momentum of particle μ, i.e.:

$\vec{L} = \mu r^2\omega\hat{z} = \mu r^2\dot{\theta}\hat{z},\ l = |\vec{L}| = \mu r^2\dot{\theta}$ (9-9)

So (9-8b) is a restatement of the conservation of momentum.

Let me rewrite (9-8a) to shine more light on it:

$\mu\ddot{r} = f(r) + \mu r\dot{\theta}^2$ (9-10)

This appears like a one-dimensional motion, subjected to forces $f(r) + \mu r\dot{\theta}^2$. You can easily recognize $\mu r\dot{\theta}^2$ is the centrifugal force. Suppose we are in a rotational frame that rotates with the particle $\Omega = \dot{\theta}$. In such frame, the particle μ would appear only have radial motion with a fictitious centrifugal force $\mu r\dot{\theta}^2$, and its equation of motion along r is that given by (9-10), and is equivalent to (9-8a). How about Coriolis and

Azimuthal force associated with rotating frame? They are perpendicular to r (in this 2-D case) and they are exactly the two terms in (9-8b). Since no motion perpendicular to the r in rotating frame, the acceleration perpendicular to r is zero in rotating frame. Then the fictitious force would add up to zero (real force is also zero along this direction in central field), that leads to (9-8b).

Sometimes it is also useful to express $\mu r\dot{\theta}^2$ in terms of angular momentum $l = \mu r^2\dot{\theta}$, because l is a constant of motion:

$$\mu\ddot{r} = f(r) + \mu r\dot{\theta}^2 = \frac{l^2}{\mu r^3} + f(r) \qquad (9\text{-}11)$$

It is also instructive to look at (9-8a) or (9-11) from the energy point of view. We shall see that it is equivalent to motion of a particle in a conservative potential $U_{eff}(r)$.

The total mechanical energy of the particle is:

$$E = K + U = \frac{1}{2}\mu v^2 + U(r) \qquad (9\text{-}12)$$

The mechanical energy can further be written in radial motion and angular motion of the particle:

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) \qquad (9\text{-}13)$$

The second term of the RHS is just the angular (rotational) kinetic energy (recall $K_{rot} = \frac{l^2}{2I}$). Because l is a constant of motion, this term only depends on r. It can also be treated as a central field in addition of U(r). It is also easy to see that the force associated with this potential $\frac{l^2}{2\mu r^2}$ is just the centrifugal force in (9-11): $\frac{l^2}{\mu r^3} = -\frac{d}{dr}(\frac{l^2}{2\mu r^2})$. This justifies define an effective potential for the radial motion:

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r) \qquad (9\text{-}14)$$

The $\frac{l^2}{2\mu r^2}$ part is called centrifugal potential.

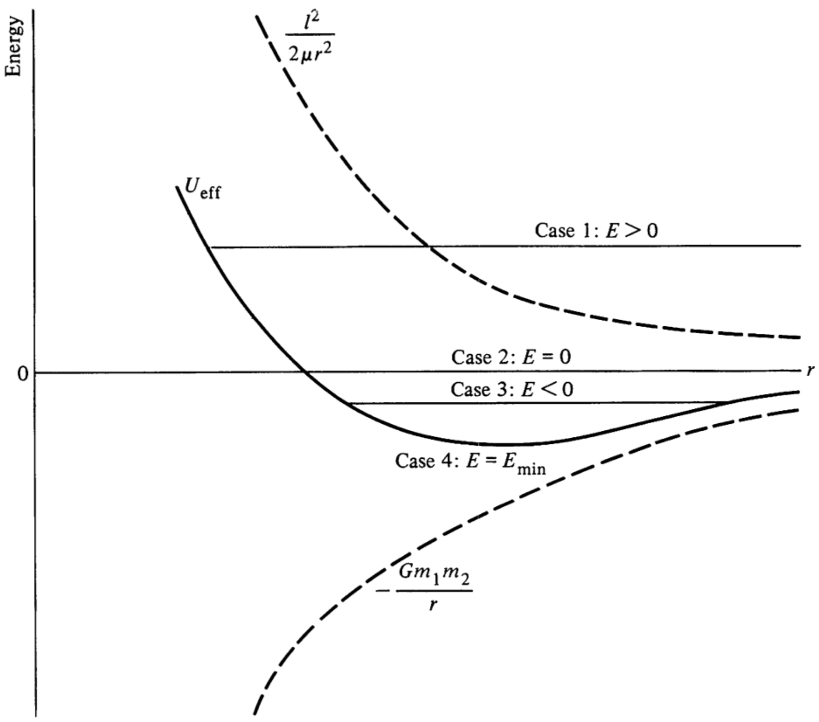
(4) Effective potential and energy diagram for radial motion

We can learn quite a few qualitative results from the energy diagram of the particle in a potential. The effective potential is provided by (9-14), it certainly depends on the detailed form of U(r). For the centrifugal potential it increase as r decrease, i.e. it is repulsive and prevents (a high potential means barrier for the particle to penetrate) particles getting too close because the need of conservation of angular momentum. Let's take the U(r) to be one important form of attractive potential for gravitation:

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{C}{r} \qquad (9\text{-}15)$$

$$C \equiv Gm_1m_2$$

The plot of potential curve is given in the figure below. The qualitative radial motion of the particle (note this is only part of the total picture of the particle motion which should also contains angular motion) can be understood by its total energy E, and energy E alone determines the type of orbit (this fact may worth remembering; of the course the detail shape of the orbit requires angular momentum, we shall see these two energy and angular momentum determine the orbit in central field problem):



1. E>0. The particle motion is 'unbound'. It will reach a certain minimum radial distance $r_{\min}$ (the intersection of the E line and the effective potential curve) and then fly away. (we shall see later the trajectory of the particle is a hyperbolic curve here)
2. E=0. Similar to case one except later we see that the trajectory is parabolic.
3. E<0. The motion is bounded between to radial limits, $r_{\min}, r_{\max}$. Corresponding to the intersections of E line and potential curve.
4. $E=E_{\min}$. The particle will have a circular motion around the center with fixed radial distance r_e.

It is interesting to look the case 4 in gravitation field. The minimum point of the potential is:

$$\frac{d}{dr}[\frac{l^2}{2\mu r^2}+U(r)]|_{r_e}=0\rightarrow f_{centrifugal}(r_e)+f(r_e)=0\rightarrow \mu r_e\dot{\theta}^2=\frac{C}{r_e^2}$$

This is the familiar form of circular motion under gravity in high school (you can also express the left hand side of the last relation as $\mu v^2/r_e$).

It is also interesting to know that if the $U(r)=-\frac{A}{r^n}$ (A positive, n is any real number), you can prove that in order to have a stable equilibrium (means a minimum in the effective potential[86]), 0<n<2, which the gravitation field satisfies (n=1).

KK example 9.2 and 9.3 further illustrate the qualitative description of motion from energy diagram, do read them yourself. (ex.9.2 can also be worked out by using what you learned in the scattering of two particles; 9.3 needs the harmonic approximation close to equilibrium point. i.e. Taylor expansion around equilibrium)

9.3 Solving the Equation of Motion

We discussed general properties of motion in the central field above, and showed a qualitative picture on its radial motion. For a complete picture of the motion, of course we need to solve the equation of motion of

[86] In such case, if the particle is away from the equilibrium (the point subject to no force, a local extreme, could be maximum or minimum in most cases), the force will pull the particle back towards the minimum. In case of unstable equilibrium happens at local maximum, if the particle is a little away from the equilibrium, the force will push the particle further away.

particle μ.

This is usually carried out by two approaches. The most direct way is solving the coupled differential equation (9-8a), (9-8b) to get the $r(t), \theta(t)$. This is unnecessarily complicated. It is much easier to start from (9-11), applying the fact the l is a constant in central field. Solve that 2nd order ODE and get r(t), then angular momentum (9-9) to get $\theta(t)$. This approach involves solving ODE, but mathematically quite straightforward thus is preferred by many authors[87].

Another method is explicitly using energy and momentum conservation, and solve r(t) from energy relation (9-13), then solve $\theta(t)$ from (9-9). This approach has the advantage of expression the integration constant in energy and momentum explicitly. KK's book adopts this approach and it will be introduced below.

Starting from the (9-13) energy relation, taking the E, l as constants. They are indeed computed from the usual initial conditions, such as initial position and initial velocity (you should know the calculation from these to get E, l by now).

$$(\frac{dr}{dt})^2 = \frac{2}{\mu}(E - U_{eff})$$

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}[E - U_{eff}(r)]} \qquad (9\text{-}16)$$

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[87] For example, in Goldstein's Chap.3; Taylor's Chap.8, they all adopt this approach.

$$\int_{r_i}^{r_f} \frac{dr}{\sqrt{\frac{2}{\mu}[E - U_{eff}(r)]}} = \pm(t_f - t_i) \qquad (9\text{-}17)$$

This is generally a nasty integral to handle unless for some specific forms of U(r) (still you may need integral table), and fortunately most important potentials do belong to these specific forms.

The angular part can be solved then(in principle):

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2} \qquad (9\text{-}18)$$

$$\theta(t) - \theta_i = \int_{t_i}^{t} \frac{l}{\mu r^2(t)} dt \qquad (9\text{-}19)$$

Both (9-17) and (9-19) are not easy to solve, and a lot of times we are interested in the orbit function, i.e. $r(\theta)$, and this can be computed by dividing (9-16) with (9-18):

$$\frac{dr}{d\theta} = \pm \frac{\mu r^2}{l} \sqrt{\frac{2}{\mu}[E - U_{eff}(r)]} \qquad (9\text{-}20)$$

At least in principle we can do the integrals to find out the time evolution of the particle $r(t), \theta(t)$; or its orbit function $r(\theta)$. Let's work out a specific example $r(\theta)$ where the U(r) is gravitational field as in (9-15)[88].

9.4 Orbit Function for Planetary Motion

The effective potential is:

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[88] Which is also applied to all forces obeys inverse squared of distance, such as Coulomb force.

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} - \frac{C}{r} \qquad (9\text{-}21)$$

Put this into (9-20):

$$d\theta = \pm \frac{l}{\mu r^2 \sqrt{\frac{2}{\mu}(E - \frac{l^2}{2\mu r^2} + \frac{C}{r})}} dr = \pm \frac{l}{r\sqrt{2\mu E r^2 + 2\mu C r - l^2}} dr$$

To solve above, I checked integral table:

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}} \arcsin(\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}) \text{ for } c < 0, b^2 - 4ac > 0$$

The solution then is:

$$\pm(\theta - \theta_i) = \arcsin(\frac{\mu C r - l^2}{r\sqrt{\mu^2 C^2 + 2\mu E l^2}}) + const$$

The integration constant can be moved to the left and combined with θ_i as some constant angle:

$$\pm(\theta - \theta_{0\pm}) = \arcsin \frac{\mu C r - l^2}{r\sqrt{\mu^2 C^2 + 2\mu E l^2}} \rightarrow \sin[\pm(\theta - \theta_{0\pm})] = \frac{\mu C r - l^2}{r\sqrt{\mu^2 C^2 + 2\mu E l^2}}$$

Solve for r:

$$r = \frac{l^2 / \mu C}{1 \pm \sqrt{1 + (2El^2 / \mu C^2)} \sin(\theta - \theta_{0\pm})} \qquad (9\text{-}22)$$

By taking the convention (this is achieved by choosing the direction of the coordinate axes), set $\theta_{0\pm} = -\frac{\pi}{2}$ and group the combination into parameters:

$$r_0 \equiv \frac{l^2}{\mu C} \qquad (9\text{-}23)$$

$$\varepsilon \equiv \sqrt{1 + (2El^2 / \mu C^2)} \qquad (9\text{-}24)$$

The orbit will be in the form of:

$$r = \frac{r_0}{1 \pm \varepsilon \cos\theta} \qquad (9\text{-}25)$$

(9-23) and (9-24) are important in a sense that they tell us the relation between the constants of motion (E,l) and parameters for orbit function (this is what I mean by saying that the energy and AM determine the orbit) This orbit function is the polar coordinate expression (shall be proved explicitly for ellipse below) of a group functions called conic sections because its geometry can be obtained by cutting a cone[89]. It represents a group of curves that are either elliptical (including circular), hyperbolic or parabolic, depending on the value of ε. ε is called ***eccentricity***[90]. (9-25) is the expression of these curves with the center at one of the foci of the curves (circle and parabola only has one focus point). $\pm$ in front of the ε depends on which foci we choose as origin and how the axes related to the curve. We shall see this in detail below.

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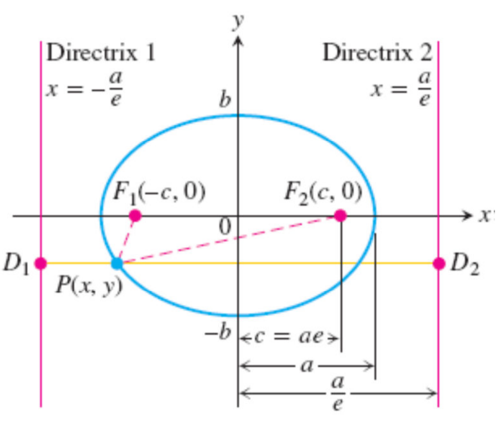
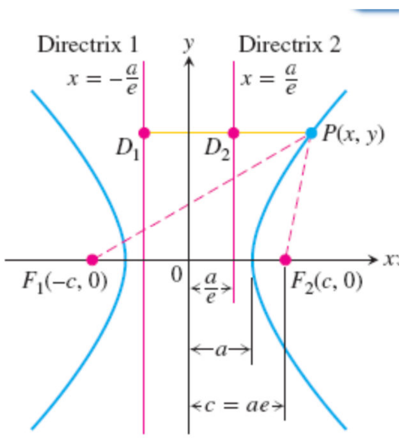
[89] Refer to Thomas "Calculus", chapter 10 for details on conic sections.

[90] This eccentricity is closely related to the one kind of geometrical definition of the conic curves, i.e. It is the ratio between the distances. The distances are that from a point on the curve to focal point and that from the point on the curve to s straight line (called directrix): $\varepsilon = PF_1 / PD_1 = PF_2 / PD_2$





(1) $\varepsilon = 0$, then r=r0. This is obviously a circle with radius r0

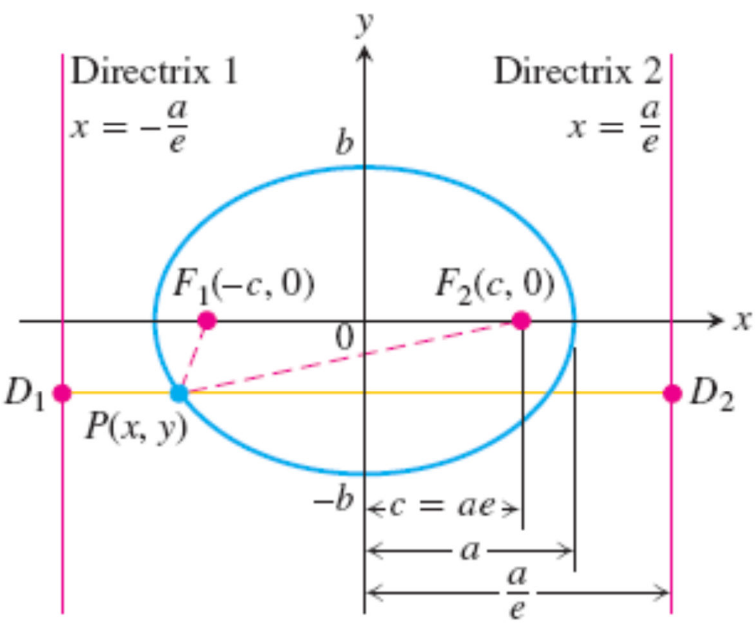
$r_0 \equiv \dfrac{l^2}{\mu C}$ here just gives us $ur_0\dot{\theta}^2 = \dfrac{C}{r_0^2} = \dfrac{Gm_1m_2}{r_0^2} = -f(r_0)$ which is the familiar equation of motion under circular orbit; and the energy is at the minimum of the effective potential curve.

$E = \dfrac{l^2}{2\mu r_0^2} - \dfrac{C}{r_0} = -\dfrac{1}{2}\dfrac{C}{r_0} = -\dfrac{\mu C^2}{2l^2}$, this indeed gives $\varepsilon = 0$.

(2) $0 < \varepsilon < 1$. The orbit function is elliptical.

Because closed orbit is among the most important in the application of central field, such as satellites orbiting around the earth and planets around star. I shall devote more time on this than the other cases.

$0 < \varepsilon < 1$ means E<0 (of course it needs to be larger than the minimum of the potential which is $-\dfrac{\mu C^2}{2l^2}$). Before I show you that the (9-25) here indeed representing an ellipse, it is maybe a good idea to summarize some of the facts about the ellipse.



The above figure summarizes the important relations of an ellipse, with the coordinate chosen above, this ellipse in Cartesian is the familiar form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The focal distance is: $c = \sqrt{a^2 - b^2}$

Eccentricity is: $e = \frac{c}{a}$ (It could be also defined as in footnote above)

$F_1P + PF_2 = 2a$ is the most common geometric definition of ellipse.

$PF_1 / PD_1 = e$ is second geometric definition of ellipse.

$r = \frac{r_0}{1 \pm \varepsilon \cos\theta}$ looks quite a different because it is written with polar coordinate and the origin is not at the geometric center but at one of the foci (F_1 or F_2). Now let's show it explicitly that it is indeed a ellipse when $0 < \varepsilon < 1$.

$x = r\cos\theta, y = r\sin\theta, r = \sqrt{x^2 + y^2}$, take $r = \frac{r_0}{1 - \varepsilon\cos\theta}$ for example (replace r, θ by x,y):

$\sqrt{x^2 + y^2} = r_0 + \varepsilon x \to x^2 + y^2 = r_0^2 + 2r_0\varepsilon x + \varepsilon^2 x^2$ rearrange it:

$(1-\varepsilon^2)x^2 + y^2 - 2r_0\varepsilon x - r_0^2 = 0$

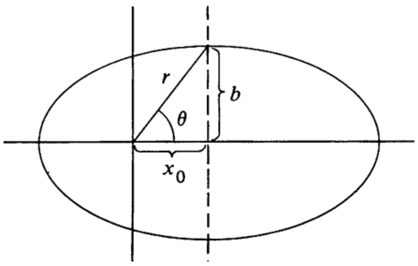
This is a quadratic function of x,y, and it is an ellipse[91]:

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

a^2, b^2, x_0 can be computed by brutal expansion and equate the coefficients:

$$a = \frac{r_0}{1-\varepsilon^2}, b = \frac{r_0}{\sqrt{1-\varepsilon^2}}, x_0 = \frac{r_0\varepsilon}{1-\varepsilon^2} \qquad (9\text{-}26)$$

This is an ellipse shifted towards right by x_0, or equivalently the origin is shifted towards left by x_0:



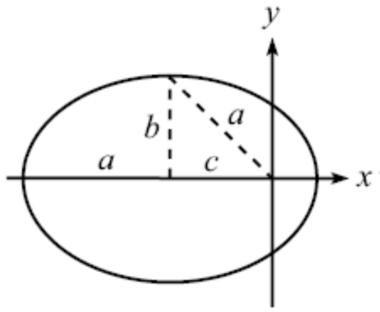
$a^2 - b^2 = x_0^2$, so $x_0 = c$ and the origin is just at the left foci of the ellipse.

$x_0 / a = \varepsilon$ so the $\varepsilon = e$, and this justifies that we call it eccentricity.

This concludes the proof that $r = \dfrac{r_0}{1-\varepsilon\cos\theta}$ is the polar representation of eclipse with origin at left foci. Similarly

[91] This can be tested using the general theory that for: $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, if $B^2-4AC<0$, it is an eclipse.

$r = \dfrac{r_0}{1+\varepsilon\cos\theta}$ is also the polar representation but with origin at the right foci:



They both represent the ellipse with different choices of origins, KK chose the left one, and others may prefer the right one. In real application, this choice seldom matters. For example the earth orbits around the sun, the sun will be chosen as origin. The earth will orbit around it with sun at one of the foci. Which foci then, the left or right? That really depends on observer's choice of coordinate. If observer A chooses the sun to be the left foci; observer B could just rotate the paper 180 degree and see the sun at right foci. What I want to state is which expression (+ or -) to use depend on how you setup your problem. So for the rest of the discussion, I will follow the convention in KK, using $r = \dfrac{r_0}{1-\varepsilon\cos\theta}$ with the choice of the coordinates stated above.

We have seen that if we know initial conditions, we can compute the E, *l*. This will allow us to determine the orbit parameters r_0, ε. Quite often, we also specify the orbits by the minimum and maximum distance $r_{\min}, r_{\max}$ (these are called perigee and apogee for satellite around the earth; or perihelion and aphelion for earth around the sun):

$$r_{\min} = \frac{r_0}{1+\varepsilon};\ r_{\max} = \frac{r_0}{1-\varepsilon} \qquad (9\text{-}27)$$

There is another property at these points: the radial velocities are 0 at $r_{\min}, r_{\max}$. This is obvious from the energy diagram on pg 305; as well as from the orbit, that at these points the instantaneous velocities are perpendicular to r, only has angular components.

So knowing any pair of these (E, *l*; r_0, ε; $r_{\min}, r_{\max}$; or *a*, b) parameters, the orbit can be determined. The relations of *a*, b with r_0, ε is given in (9-26); their relation with E, *l:*

$$a = -\frac{C}{2E};\ b = \frac{l}{\sqrt{-2\mu E}} \qquad (9\text{-}28)$$

Example: Harley Comet

The famous comet approaches the sun. You could only see it when it is close to the perihelion. Suppose you did observation and measured the perihelion distance $r_{\min} = 0.6AU$ [92]. This is not sufficient to get orbit of the comet around the Sun. Then you recall that the comet has a period of about 76 years. From these, the orbit of the comet can be determined.

One important relation need to be used is Kepler's third law: The

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[92] AU: Astronomical Unit. It is distance from the earth to sun, 1AU= about 500 light second or $1.5\times10^{8}\,km$. Measuring celestial distance is not an easy task;the simplest one would use trigonometric method.

period of the planetary motion around star is related to the major axis of the orbit, i.e.:

$T^2 = KA^3$ (9-29)

K is a constant, $A = 2a = r_{\min} + r_{\max}$

Let me first prove this famous relation (It is used often by astronomers):

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2} \to dt = \frac{\mu r^2}{l} d\theta = \frac{\mu r_0^2}{l} \frac{d\theta}{(1-\varepsilon\cos\theta)^2}$$

$$\int_0^T dt = \frac{\mu r_0^2}{l} \int_0^{2\pi} \frac{d\theta}{(1-\varepsilon\cos\theta)^2}$$

Another nasty integral to evaluate on the RHS. I just checked table:

$$\int_0^{2\pi} \frac{d\theta}{(1-\varepsilon\cos\theta)^2} = \frac{2\pi}{(1-\varepsilon^2)^{3/2}}$$

Throw in (9-23) and (9-24),express r_0, ε in E, *l:*

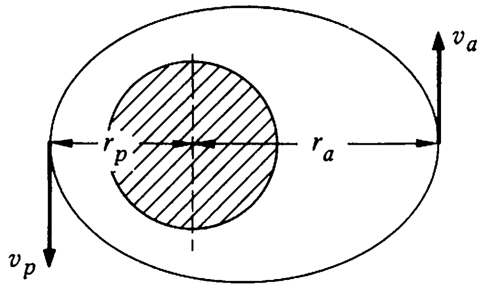
$$T = \frac{l^3}{\mu C^2} \frac{2\pi}{(-2El^2 / \mu C^2)^{3/2}} = \frac{2\pi\mu^{1/2} C}{(-2E)^{3/2}}$$

$$T^2 = \frac{\pi^2 \mu C^2}{-2E^3} = \frac{\pi^2 \mu}{2C} (\frac{C}{-E})^3 = \frac{\pi^2 \mu}{2C} A^3 \doteq \frac{\pi^2}{2GM_{sun}} A^3 \qquad (9\text{-}30)$$

Hence, we prove the Kepler's 3rd law (KK's offered another 2 methods of this proof in their section 9.7) Equipped with this relation, we can calculate $A = 2a = r_{\min} + r_{\max}$ from T=76 years, and get $r_{\max}$. From the $r_{\min}, r_{\max}$, the r_0, ε can be computed from (9-27) and the orbit will be known.

Now imagining that a small meteoroid making an inelastic collision with the comet at perihelion, how the orbit will change?

The change of course depends on the collision. We know the velocity of the comet at perihelion (it only has angular part):



The initial v_p can be computed either from energy E or angular momentum *l*, which are known from above data. First consider the simpler case, where the meteoroid also travels along v_p with mass m and velocity v_0. After the inelastic collision, the whole thing has a new velocity that is also along v_p, i.e. perpendicular to the radial direction, but with a new velocity v_0'. This v_0' can be calculated from conservation of momentum during the collision. Thus the new angular momentum and energy *l*' and E' can be calculated. Then r_0', ε' of the new orbit can be calculated too. The perihelion of the old orbit will still be a perihelion[93], this means the new orbit will still be in the form of $r = \dfrac{r_0'}{1-\varepsilon'\cos\theta}$, this solves the problem.

What happened if the collision has an angle with v_p. The final velocity can still be computed straightforwardly, but this time v_0' is not along

[93] If the meteoroid has enough momentum that is against the comet, the comet after collision can be slowed down so much that the perihelion of the old orbit may becomes the aphelion of the new orbit.

v_p anymore. It has both radial and angular components. This means the old perihelion point is NOT the perihelion (or aphelion) point of the new orbit. The new orbit will be tilted compared to the old one(there will be an angle between the major axes of the two orbits):

$$r = \frac{r_0'}{1-\varepsilon'\cos(\theta-\delta)} \quad (9\text{-}31)$$

This will be the new orbit function. r_0', ε' can still be determined similarly from the new angular momentum and energy. How to compute the tilt angle δ?

There is actually another condition, the collision point is on both the old and new orbit: $\dfrac{r_0}{1-\varepsilon\cos(\theta_i)} = \dfrac{r_0'}{1-\varepsilon'\cos(\theta_i-\delta)}$ and in this question ,since the collision is at perihelion of one orbit, $\theta_i = \pi$, and δ can be determined.

This example, though the detailed values are not computed, outlines the method you may use to compute the orbit and orbit change in the central field problems.

(3) $\varepsilon > 1$, the orbit is hyperbola

This is when E>0, the particle has non-zero kinetic energy at very large distance.

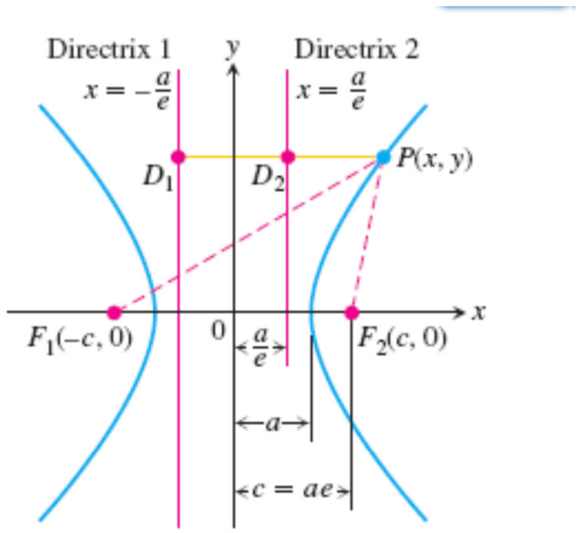
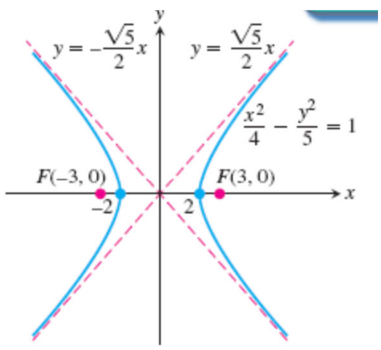


FIGURE 10.20 The foci and directrices of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. No matter where P lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.



A standard hyperbola with the origin at geometric center is shown above with the form of: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Focal distance: $c^2 = a^2 + b^2$

Eccentricity: $e = c / a$

Asymptotes (dashed lines in the small figure): $y = \pm \frac{b}{a} x$

$PF_1 - PF_2 = 2a$ is the common geometric definition (for right branch)

$PF_1 / PD_1 = e$ is the second geometric definition.

For $r = \frac{r_0}{1 - \varepsilon \cos\theta}$, if we use the convention $r > 0$, this will limit the range on the angle:

$\cos\theta < \frac{1}{\varepsilon}$, $\cos\theta = \frac{1}{\varepsilon}$ corresponds to $r = \infty$ and is the angle of the asymptotes. We will see this corresponds to the right branch of the hyperbola.

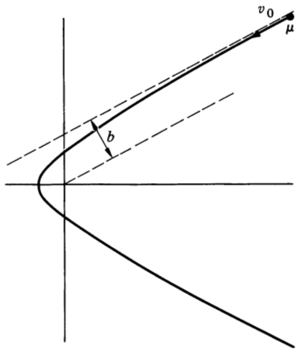
Similar procedure from polar to Cartesian will lead us:

$$(\varepsilon^2-1)x^2-y^2+2r_0\varepsilon x+r_0^2=0$$

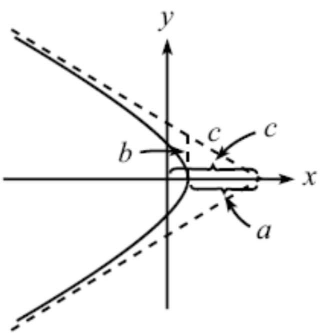
$$\frac{(x+x_0)^2}{a^2}-\frac{y^2}{b^2}=1$$

$$a=\frac{r_0}{\varepsilon^2-1}, b=\frac{r_0}{\sqrt{\varepsilon^2-1}}, x_0=\frac{r_0\varepsilon}{\varepsilon^2-1} \qquad (9\text{-}32)$$

This is the right branch of a hyperbola shifted to the left, or the origin is shifted to the right by x_0. x_0 is the focal length and the origin overlaps with the right foci.



for $r=\dfrac{r_0}{1-\varepsilon\cos\theta}$



for $r=\dfrac{r_0}{1+\varepsilon\cos\theta}$

The $r=\dfrac{r_0}{1+\varepsilon\cos\theta}$ case will be similar but the origin is shifted to the left foci and the hyperbola is the left branch. Still the choice of + or – sign depends on how you set the problem and we shall adopt the KK's convention, this corresponds to the particle flies from infinity on the right side towards center or from center flies towards right.

The parameter b is called impact parameter, it is related to the angular momentum and energy as:

Suppose the speed of the particle at infinity is v_0, then:

$$E = \frac{1}{2}\mu v_0^2; \, l = \mu v_0 b \qquad (9\text{-}33)$$

So the motion can be described by a pair of parameters such as E, l; r_0, ε; v_0, b, or other combinations (such as a, b or asymptote angle etc).

Example: Suppose a particle with mass m with speed v_0 and impact parameter b flies towards Sun, what will be the closest distance to the sun?

We could work out this two ways.

First without using orbit function, the closet distance will be a perihelion point where the particle will have only angular speed, no radial speed, then: $E = \frac{1}{2}mv_0^2 = \frac{l^2}{2mr_{\min}^2} - \frac{C}{r_{\min}} = \frac{mb^2v_0^2}{2r_{\min}^2} - \frac{C}{r_{\min}}$

$$v_0^2 r_{\min}^2 + 2qr_{\min} - b^2 v_0^2 = 0, \; q = C / m$$

Take the positive root for r:

$$r_{\min} = \frac{-q + \sqrt{q^2 + b^2 v_0^4}}{v_0^2} = \frac{b^2 v_0^2}{q + \sqrt{q^2 + b^2 v_0^4}} = \frac{b^2 v_0^2 / q}{1 + \sqrt{1 + b^2 v_0^4 / q^2}}$$

The reason I am writing this is to compare with the second method.

The second method is using the orbit function:

$$r_{\min} = \frac{r_0}{1+\varepsilon}, \quad r_0 = \frac{l^2}{mC} = \frac{mb^2v_0^2}{C} = \frac{b^2v_0^2}{q}$$

$$\varepsilon \equiv \sqrt{1+(2El^2 / mC^2)} = \sqrt{1+mv_0^2m^2v_0^2b^2 / mC^2} = \sqrt{1+b^2v_0^4 / q^2}$$

The two methods gives the same results as expected. The famous Rutherford scattering experiment also apply the hyperbolic orbit in data analysis, there the interaction is Coulomb interaction (similar to gravitation).

(4) $\varepsilon = 1$, the orbit is a parabola

This is when E is exactly 0, not a usual situation in real life.

$r = \dfrac{r_0}{1-\cos\theta}$ corresponds to the parabola opens towards right, with the origin at the focal point of the parabola. $r = \dfrac{r_0}{1+\cos\theta}$ corresponds to the parabola opens towards right.

(5) There is actually another possibility, that is $l = 0$, no angular momentum with respect to the origin. This corresponds to a particle dropped with initial zero velocity (or velocity along the center line). The particle clearly will travel in a straight line and in the collision course. $r = \dfrac{r_0}{1-\cos\theta}$ also represents this situation. Here, $r_0 = 0, \varepsilon = 1$, r will take none zero value only at $\theta = 0$.

In all above discussion, strictly speaking, I solved for the fictitious particle with reduced mass. To get the real picture of the two-particle motion, I will need to combine this with the motion of center of mass and use relation (9-3) to get the motion of each particle. In many applications, such as planets orbiting around star, satellite around earth, or electron in the field of nuclei. One of the particles (the sun, earth, nuclei) will have the dominant mass, and the C.M. can be treated overlaps with this heavy particle, and the motion of the reduced particle we focused above will be that of the light particle.

9.5 Kepler's Laws of Planetary Motion

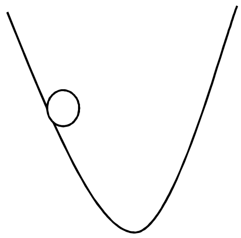
Actually we have already proved three Kepler's laws.

The first law which states the motion of planet around the sun has elliptical orbit with sun at one focus. This is just the case of motion with total energy <0 in central field.

The second law states the planet moves with constant area velocity, this is the consequence of constant angular momentum, and we proved this constant area velocity in Chapter 7.

The third law states that the square of the period of the motion is proportional to the cube of the major axis of the orbit. This is proved in the previous example leads to (9-30)

Chapter 10 Vibration (Oscillation) and Waves



As the figure shows, if you drop a ball in a bowl (frictionless), the ball just rolls up and down periodically. This kind of periodic motion is vibration. It happens in many natural systems (both in macroscopic such as the rocking motion of floating object in water, wind etc; and in microscopic, such as molecular vibration) and such vibrations are the source of the wave (the sound wave coming from vibration of musical instrument; light coming from vibration of electric dipole). It is this kind of motion we shall study in this chapter.

The simplest model will be that of harmonic oscillator[94]. A mass is under a linear restoring force, i.e.:

$$m\ddot{x} = -kx \qquad (10\text{-}1)$$

The mass-spring system, the simple pendulum or the physical pendulum we discussed in chapter 7 all fit in this category, and many more (such as a block floating in water due to balance between buoyancy and weight,

[94] There is little difference between vibration and oscillation, so I shall interchange them freely. Harmonic comes from the original study of Greek on the sound coming from musical instrument, such as Harp. They found that when the string satisfies certain lengths for a harp, it produce sound pleasant to ear.

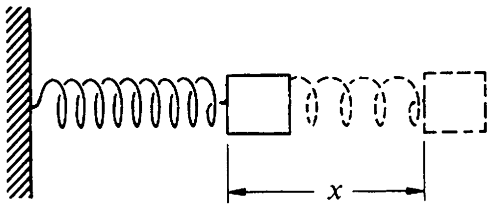
push it down and release, the block will display oscillation). The wide application of this simple model comes from the fact as the above figure shows and as we already discussed in section 6-3: For any potential energy curve (such as that of a bowl, it is more hyperbolic than parabolic in the figure above) close to the minimum point, it can be approximated by:

$$U(x) = U(x_0) + \frac{dU}{dx}|_{x_0} (x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2}|_{x_0} (x - x_0)^2 + ...$$

The first term is taken (set) as zero for the minimum (or any number, since it's the potential difference that matters, the absolute value has little importance here); the second term is zero due to the fact of local minimum; and the third term will be the dominant factor of potential change due to displacement from equilibrium (higher orders are neglected at small $x - x_0 = \Delta x$). Usually we shall use x to symbolize this displacement from equilibrium (effectively as taken equilibrium point $x_0 = 0$), then the potential will be in forms of $U = ax^2$ and force will be $F = -\frac{dU}{dx} = -kx$, and the equation of motion will be that of (10-1) close to the equilibrium point for all kinds of potentials. The simple periodic motion with single frequency is the result of this equation of motion as we shall see below.

The math we need in the derivation below involves solving ordinary differential equation (ODE) and is discussed in some detail at supplementary III, please make sure referring to it for the math details there if you are new to this math.

10.1 Free Oscillator without Damping



We shall use the mass-spring model for the rest of discussion.[95] A free oscillator is the mass is under no other external driving force besides the restoring force –kx of the spring; no damping means no dissipation force such as friction. This is of course the simplest the one can get. (10-1) will be rewritten in a standard form:

$$\ddot{x} + \omega_0^2 x = 0 \quad \omega_0 = \sqrt{k/m} \qquad (10\text{-}2)$$

(1) Displacement Motion

Solving it with the standard method of 2nd ODE:

The guess of the solution will take forms of $e^{\lambda t}$, and throw the guess into equation:

$$\lambda^2 + \omega_0^2 = 0 \rightarrow \lambda_1 = i\omega_0; \lambda_2 = -i\omega_0 \qquad (i^2 = -1)$$

[95] Other models will be reduced to (10-1) too, with a different meaning of m, k and x. For example, in pendulum case, the m will be related to moment of inertia, and –k will be related to torque coefficient and x will be angular displacement. Discussions on these other systems besides mass-spring can be found in French's 'Vibration and Waves' chap.3.

From the discussions in supplementary III (under the 2nd order linear ODE), the general solution of (10-2) is:

$x(t) = \tilde{c}_1 e^{i\omega_0 t} + \tilde{c}_2 e^{-i\omega_0 t}$

The displacement function is obviously a real function, so:

$x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t = A\cos(\omega_0 t + \phi)$ (10-3)

The constant A, ϕ depends on the initial conditions: $x(0), \dot{x}(0)$.

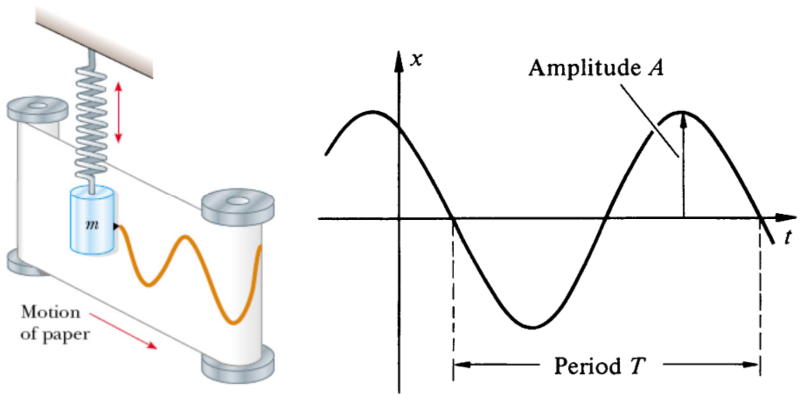
So the motion is really periodic with **period** T of:

$$\omega_0 \Delta t = 2\pi \rightarrow T = \frac{2\pi}{\omega_0} \qquad (10\text{-}4)$$

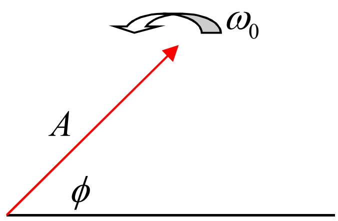
The reciprocal of period is called **frequency** and ω_0 (previously in rotation we call it angular velocity) is called **angular frequency**:

$$\upsilon = \frac{1}{T} = \frac{\omega_0}{2\pi} \qquad (10\text{-}5)$$

A is called **amplitude** of the oscillation and ϕ is called **initial phase**. If ϕ>0, the cosine curve will be shifted towards left (origin is shifted towards right); ϕ<0. The curve is shifted towards right (origin is shifted towards left). The convention is to choose the interval for ϕ to be either $[0, 2\pi]$ or $[-\pi, \pi]$. The meaning of period and amplitude and initial phase is shown in figure right below, it is a cosine shifted towards left, corresponds to some positive ϕ (the figure on the left shows a simple device to plot the cosine motion of oscillation, electronic device such as an oscilloscope uses similar principle):



One useful geometric representation of (10-3) is rotating 'vector' (also termed 'phasor') method as the figure below shows. The arrow has length A (the amplitude) and forms a initial angle ϕ with the axis of choice, it rotates with angular velocity ω_0 (c.c.w. for positive ω_0). The projection of such arrow along the horizontal direction is exactly that in (10-3):



We won't use this method much in this course. However, this 'phasor' representation will be very useful in the discussion of superposition of oscillations and waves, which will be among the most important principles when we talk about wave theory and light in later courses.

You may notice that this rotating 'vector' representation is very much similar to the 'vector' representation of complex numbers (as shown in supplementary III), both are called 'phasor'. This is no coincidence, since complex numbers are used mathematically to describe oscillation and waves; the origin is the Euler formula.

(2) Velocity and Energy

Take time derivative of (10-3), we get velocity:

$$v = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi) = A\omega_0 \cos(\omega_0 t + \phi + \frac{\pi}{2}) \qquad (10\text{-}6)$$

The velocity has a phase difference of $\frac{\pi}{2}$ (it is called lead[96] in phase by $\frac{\pi}{2}$) comparing to the displacement. This difference in phase makes perfect sense from energy conservation. As the mass reaches largest displacement (highest potential) when $\omega_0 t + \phi = 0, 2\pi ...)$, the velocity and kinetic energy are zero, and vice versa. The potential and kinetic energy are:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t + \phi)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi)$$

$$E = U + K = \frac{1}{2}m\omega_0^2 A^2 = \frac{1}{2}kA^2 \qquad (10\text{-}7)$$

The total energy is conserved (no change over time) and it is proportional to the square of the frequency and amplitude.

The angular frequency is determined by the parameters of the system in (10-2); it is called natural frequency of the system and won't vary once

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[96] This can be understood by picking a reference point, say the maximum, and the velocity will reach maximum at earlier time ($\omega_0 t+\phi+\frac{\pi}{2}$ =0) than the displacement.

the system is fixed. The amplitude depends on the initial conditions and does vary in many cases. So we sometimes claim that the energy of the oscillator depends on the amplitude. This is a general property of oscillators and waves generated by such oscillators. We shall see in optics that intensity of light is also proportional to the square of the amplitude.

Though the total energy is conserved, the kinetic and potential energy do vary with time. It is interesting to calculate their time average, which is defined as:

$$<A>=\frac{1}{T}\int_0^T A dt \qquad (10\text{-}8)$$

T is the time period of average process.

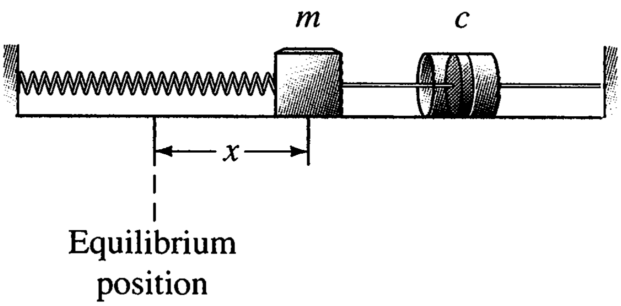
$$<U>=\frac{1}{T}\frac{1}{2}m\omega_0^2A^2\int_0^T \cos^2(\omega_0 t+\phi)dt=\frac{1}{T}\frac{1}{2}m\omega_0^2A^2\int_0^T \frac{1}{2}[1+\cos 2(\omega_0 t+\phi)]dt$$

$$=\frac{1}{T}\frac{1}{2}m\omega_0^2A^2\int_0^T \frac{1}{2}dt+\frac{1}{T}\frac{1}{2}m\omega_0^2A^2\int_0^T \frac{1}{2}\cos 2(\omega_0 t+\phi)dt=\frac{1}{4}m\omega_0^2A^2$$

The $\cos 2(\omega_0 t+\phi)$ term drops out because if we average over a long time T (need to be larger than the period of oscillation), this term is negligible because up and downs cancels with each other for the cosine function in the integration (or if the time of average is just the period of oscillation, then the integral of sinusoidal functions over its period will always be zero; isn't this obvious?). Similar result is for the kinetic energy (a time average of $\sin^2(\omega_0 t+\phi)$:

$$<K>=\frac{1}{4}m\omega_0^2A^2=<U>=\frac{1}{2}E \qquad (10\text{-}9)$$

10.2 Damped Free Oscillator



We shall study in this section a more realistic case where the oscillator is subjected to a friction force (called damping force), such as the suspension system shown in the figure above. The damping force would be always against the motion of the oscillator, and is assumed to be proportional to the velocity (which is equivalent of Taylor expand the friction force vs. v and keeps the lowest non-zero order) i.e.:

$$f_d = -bv$$

The equation of motion will be:

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x}$$

And this is usually expressed in another form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \qquad (10\text{-}10)$$

$$\gamma = b/m; \omega_0 = \sqrt{k/m}$$

Characteristic equation for (10-10) is:

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0 \qquad (10\text{-}11)$$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

Case A： $\frac{\gamma^2}{4}-\omega_0^2<0$ weak damping (most important for us):

Introduce:

$$\omega_1=\sqrt{\omega_0^2-\frac{\gamma^2}{4}} \qquad (10\text{-}12)$$

$$\lambda_{1,2}=-\frac{\gamma}{2}\pm i\omega_1$$

General solution is:

$$x(t)=e^{-\frac{\gamma}{2}t}(c_1\cos\omega_1 t+c_2\sin\omega_1 t)=e^{-\frac{\gamma}{2}t}A\cos(\omega_1 t+\phi) \qquad (10\text{-}13)$$

When the damp is very weak, $\omega_0 >> \gamma$, the solution is like a oscillator with a slow decaying amplitude $e^{-\frac{\gamma}{2}t}A$ and a shifted frequency (comparing to its natural one), as the figure below (on the left) shows. The right figure shows a comparison between the 3 cases.

Case B: $\frac{\gamma^2}{4}-\omega_0^2=0$, critical damping, $\lambda_{1,2}=-\frac{\gamma}{2}$

The general solution would be:

$$x(t)=(c_1+c_2 t)e^{-\frac{\gamma}{2}t}$$

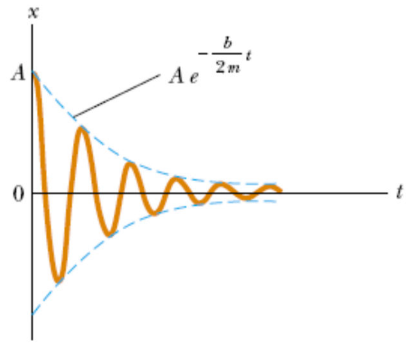
It approaches to zero at longer t. It actually approaches zero faster than case C. This is called critical damping, $\frac{\gamma^2}{4}-\omega_0^2=0$ is the critical damping condition. This has wide applications in situations where oscillation is not wanted, such as suspension system in automobile.

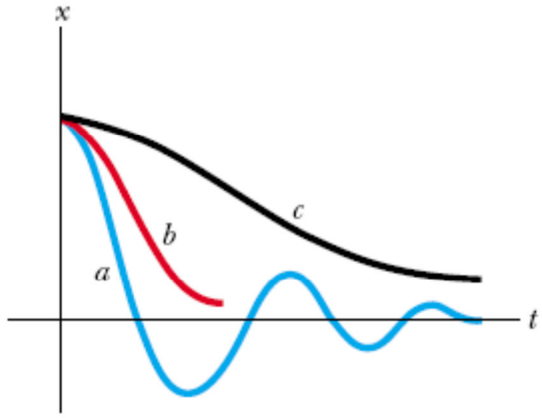
Case C: $\frac{\gamma^2}{4}-\omega_0^2>0$, strong damping

λ_1,λ_2 are real numbers, general solution:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Note λ_1, λ_2 are negative numbers, so the x(t) will decay to almost 0 at long run.





Energy and Quality Factor Q for weak-damping case:

Let's consider the situation $\gamma << \omega_0$[97], then the calculated total energy can be approximated very good by (details in KK) the formula similar to (10-7), except the constant amplitude in no-damping oscillator needs to be replace by the decaying amplitude in (10-13):

$$E = \frac{1}{2} k A^2 e^{-\gamma t} \qquad \text{(10-14)}$$

The lifetime can be defined as the time for the energy drop to e^{-1} from the initial value:

$$\tau = 1/\gamma \qquad \text{(10-15)}$$

The quality factor Q is defined as:

$$Q = 2\pi \frac{initial\ energy}{energy\ loss\ per\ cycle} = \frac{initial\ energy}{energy\ loss\ per\ radian} \qquad \text{(10-16)}$$

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97 Such case is quite common in microscopic system. For example, the electrons inside atoms oscillates on the order if 10^{14} Hz, its decay time $\tau = 1/\gamma$ is on the order of 10^{-9} s.

This tells you how good the oscillation is, the higher Q, the oscillator would be more like a harmonic oscillator. From (10-14) the energy loss per radian (or cycle) can be computed:

$$\Delta E / cycle = \left|\frac{dE}{dt}\right| \Delta t = \gamma E T = \gamma E \frac{2\pi}{\omega_1}$$

$$Q = 2\pi \frac{initial\ energy}{energy\ loss\ per\ cycle} = \frac{\omega_1}{\gamma} \approx \frac{\omega_0}{\gamma} \qquad (10\text{-}17)$$

Another interpretation of Q is also from energy point of view:

The time required for the energy decay is characterized by $\tau = 1/\gamma$, this lifetime corresponds to how many cycles of oscillation?

$$number\ of\ cycles = \frac{\tau}{T} = \frac{\omega_1}{2\pi\gamma} = \frac{Q}{2\pi} \qquad (10\text{-}18)$$

So Q could also be interpreted relating to the number of cycles in a lifetime, i.e. time for energy drops by factor e^{-1}.

10.3 Oscillator under Driving Force and Resonance

Up to now we only considered free oscillator, here we shall investigate the oscillators under a driving force. We only consider a very special type of the driving force[98]: that is the driving force itself is also a harmonic function, with angular frequency ω. An interesting phenomenon called resonance will arise when the driving frequency is close to the natural

[98] For the treatment of a general force other than harmonic or exponential type, we will need Fourier Transform (or Laplace transforms; but for frequency analysis of the system response, Fourier Transform is more common). We shall study Fourier Transform in detail in optics, so I shall not dig into it here.

frequency of the system. The system will response violently (energetically) in resonance.

$F_{drive} = F_0 \cos \omega t$

(1) Equation of motion and solution

The standard form of equation of motion will be:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \qquad (10\text{-}19)$$

The solution for this 2nd order ODE is quite different from that (10-10). The details are in supplementary III under 2nd order ODE. (10-19) is an inhomogeneous 2nd order ODE, and its solution contains two parts:

$x(t) = x_p + x_c$

The x_p is one solution that satisfies (10-19), it is called particular solution of the equation; x_c is called complementary solution which is the general solution to the homogeneous equation (10-10). I have already shown you the solution for x_c in (10-13), so only x_p is needed here.

The derivation of x_p using method of complex number and Euler formula is presented in the supplementary, so only the result is shown here:

$$x_p = \frac{F_0}{m} \frac{\cos(\omega t + \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} = A\cos(\omega t + \phi)$$

$$A = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \qquad (10\text{-}20)$$

$$\phi = -\phi' = \tan^{-1}\left(\frac{\gamma\omega}{\omega^2 - \omega_0^2}\right)$$

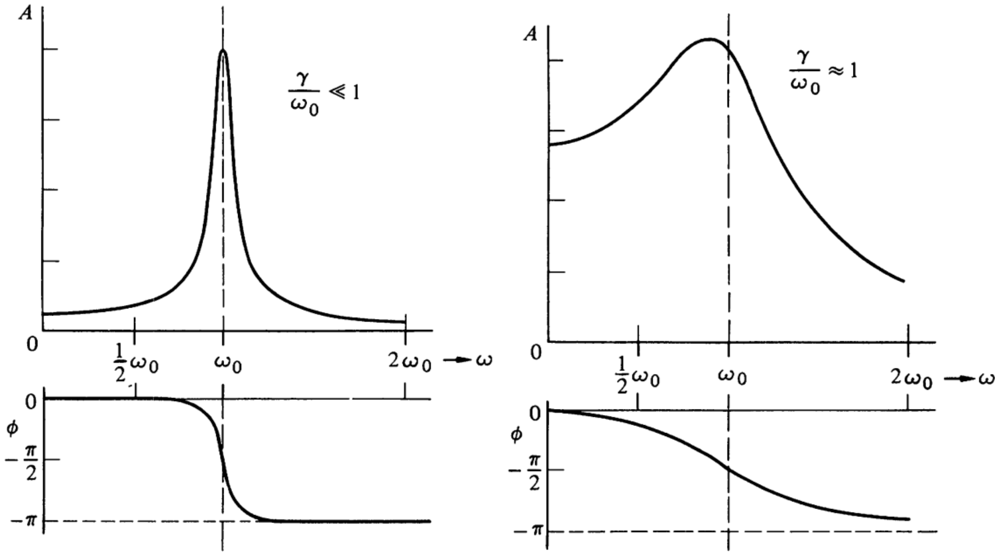
Take a time and look at the result. The particular solution is also an oscillation that has same frequency as the driving force, and it also has a phase delay or lead depends on the sign of ϕ. You may wonder where is the property of the oscillator itself, how about its natural frequency ω_0, damping γ? Well it is also reflected in the amplitude A, we shall see this will give rise to resonance. What is more, (10-20) is not full story for the solution yet, but it will be the dominant one at longer time. The complete solution of equation (10-19) is:

$$x(t) = x_p + x_c = A\cos(\omega t + \phi) + A' e^{-\frac{\gamma}{2}t} \cos(\omega_1 t + \phi_1) \quad (10\text{-}21)$$

A' is the amplitude for the complementary solution. At longer time, due to the damping, the complementary will decay into obliteration. The dominant part will be the x_p given by (10-20), and we shall only focus on this part in the later discussion. This does make sense that the oscillator will finally yield to the driving force if the force persists, so it will oscillate at the same frequency. However, the characteristics of the oscillator itself are not lost. They are reflected in the response through the amplitude and phase delay.

(2) Resonance

Here we discuss how the characteristics of the oscillator affect its response to the external driving force. The A and ϕ of the response is given by (10-20) and are plotted in the figure below:



We can see that when the driving frequency is close to the natural frequency, the amplitude has its maximum. Actually we can calculate the driving frequency that gives maximum amplitude by usual differentiation method: $dA/d\omega = 0$, this will gives:

$$\omega_{A-\max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}}$$

At this driving frequency, the amplitude of the response of the oscillator is the largest, and this is called resonance. Under the weak damping condition, where $\gamma << \omega_0$, then:

$$\omega_{A-\max} \approx \omega_0, \quad \text{for } \gamma << \omega_0 \qquad (10\text{-}22)$$

The oscillator will have largest amplitude and its phase will be $\frac{\pi}{2}$ behind the driving force (i.e. $\phi = -\frac{\pi}{2}$)[99]. For other frequency, the

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[99] Actually when $\omega = \omega_0$, from (10-20), we see that $\tan\phi = \infty$. This implies that the phase difference could either

amplitude and phase are also shown in the figure.

The energy curve of the response could also be computed from (10-20):

$$\dot{x} = -A\omega\sin(\omega t + \phi)$$

$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \to < K >= \frac{1}{4}m\omega^2 A^2$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega t + \phi) \to < U >= \frac{1}{4}m\omega_0^2 A^2$$

$$< E >=< K > + < U >= \frac{1}{4}mA^2(\omega^2 + \omega_0^2) = C\frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

Under the weak damping, we have seen that the amplitude is very small when the ω is far away from ω_0, this implies the energy of the oscillator will also be small in such cases; so we shall focus on the case when ω is close to the ω_0. The above equation on the average energy will be simplified to:

$$< E >= C\frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \approx C\frac{2\omega_0^2}{(\omega_0 + \omega)^2(\omega - \omega_0)^2 + \gamma^2\omega_0^2} \approx C\frac{2\omega_0^2}{4\omega_0^2(\omega - \omega_0)^2 + \gamma^2\omega_0^2}$$

$$= \frac{C}{2}\frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2} \qquad (10\text{-}23)$$

The energy above is in a function form called Lorentian:

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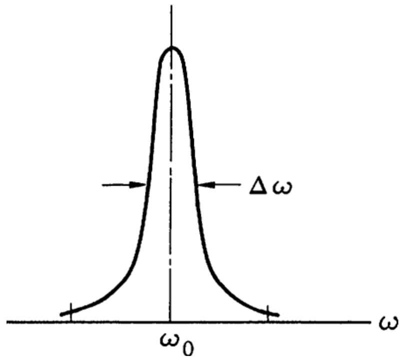
be $\pm\frac{\pi}{2}$. The reason we only take $-\frac{\pi}{2}$ is a physical one: the driven oscillator’s response would be delayed in time from the driving force, so that its phase would be behind that of the driving force.



For $\gamma << \omega_0$, its maximum at ω_0 (as expected from the maximum of A), and has a width of $\Delta\omega$ at the waist, where the energy dropped to half of its maximum value at the waist. This width $\Delta\omega$ defined this way is called Full Width at Half Maxima (FWHM). $\Delta\omega$ can be computed as:

$$(\omega - \omega_0)^2 = (\frac{\gamma}{2})^2 \rightarrow \omega_{\pm} = \omega_0 \pm \frac{\gamma}{2} \rightarrow \Delta\omega \equiv \omega_+ - \omega_- = \gamma \qquad (10\text{-}24)$$

If you recall the definition of Q and expression of it, we see that Q can be expressed as (under weak damping):

$$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\Delta\omega} \qquad (10\text{-}25)$$

From the (10-24) and (10-25), we can see that the smaller γ or larger Q, the sharper the peak of resonance. The reason of close relation between the line width $\Delta\omega$ and damping constant γ (10-24) carries an important physical model: The uncertainty relation between frequency distribution and temporal distribution. Let me rewrite the (10-24), and express the damping constant with the lifetime of the oscillator using (10-15):

$$\Delta\omega = \gamma = \frac{1}{\tau} \text{ or } \tau\Delta\omega = 1 \qquad (10\text{-}26)$$

We will see (in later courses) that (10-26) is a result can be proved with

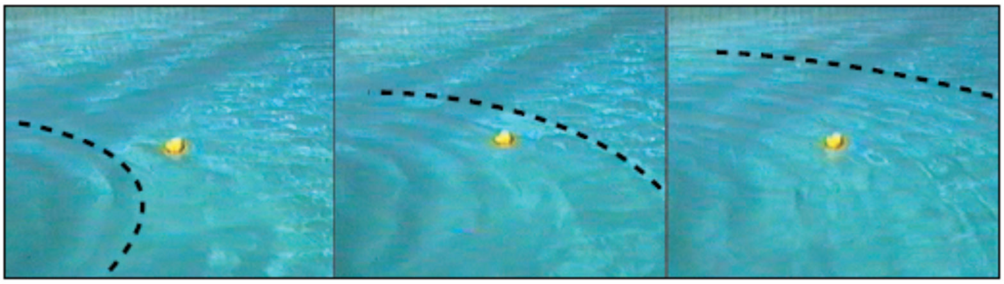
Fourier Transform in the study of Optics and is a general property for oscillation and waves; it will also give uncertainty relations in quantum mechanics.

The phenomena of resonance have good and bad effects. The good ones involves receiving electro-magnetic signal by antenna (fundamental in communication, and the hearing mechanism of human ear is also based on the resonance) and probe the atom/molecule with light (the frequency of light (the driving force) has to be tuned close to the natural frequency of the atom/molecule (or electrons inside it) to be effectively absorbed). The dramatic example of destructive resonance is the collapse of Tacoma Bridge in 1940: whether the famous story that a marching Napoleon's army caused a collapse of a bridge be true or not is a mystery, the collapse of a suspension bridge, Tacoma Narrow Bridge in state of Washington, is kept in film. The wind (not too wild) drove the bridge into rocking motion and that frequency matched that of the bridge, and the bridge shook violently under resonance and finally collapsed as the figure below shows.

(a) (b)

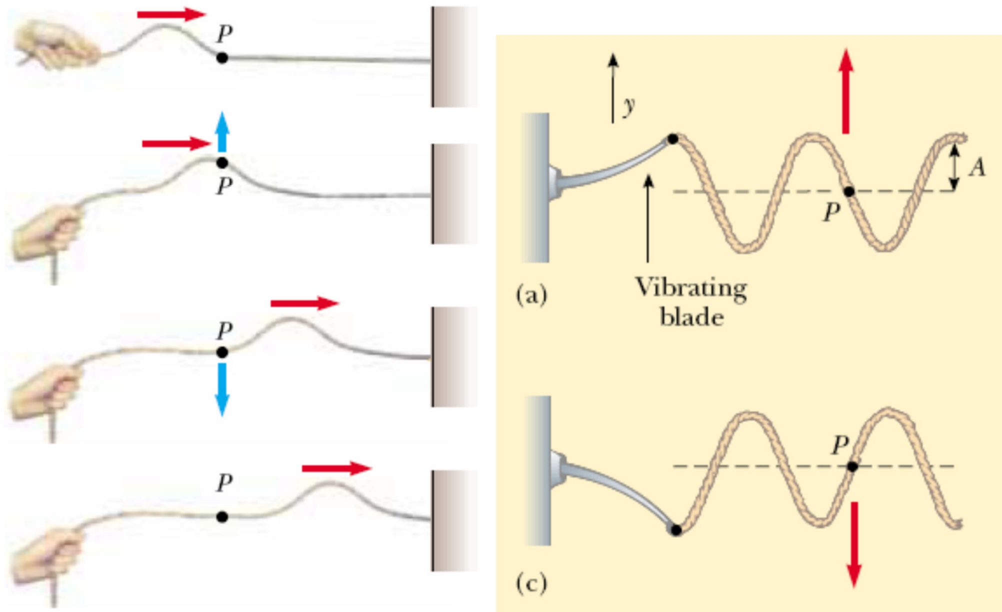
10.4 Waves (Mechanical Wave)

Let’s first consider the picture where the term wave comes from, a water wave. You perturb the water locally with some device (drop a stone at some point, set off a bomb under water surface, or just keep tapping one point on the surface of water etc.), you will observe water wave propagate from the source (the center of perturbation): a peak-trough water front (the ripples) moves from the center and across the surface of water. However, locally the water molecules do not travel far away from its original position as the wave propagates. This is best demonstrated by putting a rubber duck Dave on the surface of water. You tapped water and created a wave. When the water wave passes through the position where Dave is, the duck will start moving upward-downward, or left-right, a kind of oscillation up-down or left-right. It does not drift away as fast as the water wave.



f / As the wave pattern passes the rubber duck, the duck stays put. The water isn't moving forward with the wave.

Similarly if you attach a rope with a vibrating end, you will observe a wave motion across the rope:



As shown in the left, you can create a lump of displacement and it will travel along the rope (this lump is called wave packet), you probably see this in the show of artistic gymnastics with ropes. The one on the right is with a vibrating end that keeps oscillating, and this will create a sinusoidal wave along the rope. These are not strange phenomena, the point I want to show is like that in water wave: The local perturbation creates wave (a jerk at the end on the left; an oscillation on the right here);

the wave propagates along the media (the rope here); the local point (P point above) only does some oscillation around its equilibrium position (up-down oscillation above), while the wave propagates far away. We see it clear that it is not the particle P moving forward along the rope; it acts as a media so that its oscillation will relay the perturbation originated from the source to the point next to it, and such perturbation (carries energy) travels along the media. What we observe of the wave is the propagation of this energy, generated at the source of perturbation and relayed through the media. This wave is a ***collective*** motion of many particles in the media (many water molecules, particles in the rope, and for sound wave many air molecules in the relay of passing the sound), individual particles only do some local oscillation while through interaction with others, the energy is passed on further away.

The examples above are called mechanical waves. From the discussion above, we see that it has following properties (as I already stressed above, below is a summary or reiteration): A) The wave is created by a source of perturbation. B) It propagates through a media, and its energy is transferred forward through the physical interaction between particles of the media like in a relay race, i.e. the energy propagates forward[100] though the media particles only do some local oscillations. So the wave motion is the result of collective motion of many particles in the media.

[100] From the study of relativity, we shall see the equivalence of energy and mass. So physicists assign 'particle-like' stuffs to the wave propagation: phonons for sound wave and photons for electro-magnetic wave.

An analogy of this is motion of a parade or demonstration where tens-of-thousands people crowded in the street. Though each individual person only moves as particle-alike, observed from far away (say on a helicopter above), the motion of the mob as whole (a collective motion) will be like a wave.

Besides mechanical wave, we are going to study another important wave later, the electro-magnetic wave (E-M wave). It has similarity and difference from the mechanical waves we discussed here. It still requires source of perturbation (an accelerated charge for instance), but this wave can propagate through vacuum (no need for media molecules) due to the elector-magnetic interactions are self-induced. The electrons that created the field may only drift at a very small speed (order of cm/sec. in a conducting wire), and the field (and the energy of E-M interaction) will propagate at the speed close to that of light in the wire.

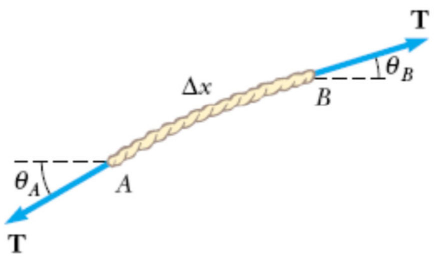
The study of wave can be quite different from our study of motions of single particle where this course has treated in detail already. It will be a major part when we study Optics and Quantum Mechanics. Here I shall only briefly discuss the mechanical wave and leaves the detailed treatment on wave to later courses.

(1) Wave Equation

We are going to work out the equation that gives the motion of the mechanical wave here. But first question is how we represent this wave? i.e. what is the function form describing the wave? I shall use the rope in the figure above as an example. Under the disturbance, each point on the rope is doing some kind of oscillation, i.e. taking point P as example: its displacement from equilibrium is changing over time:

$y_P(t) = y(x_P, t)$. y is the vertical displacement in the rope case, x_P is to specify its position on the rope (say we cut the rope in small segments and consider the pth segment) and this displacement is changing over time. For the collective motion of all points on the rope, the description would be a function with variables of both x and t, i.e. $y(x,t)$, and it is called wave function.

Let's take a very small segment of rope centered around some point x with small length Δx:



The tension along the rope is same everywhere (we shall prove this below with assumption that the rope does not accelerate along the x direction, and the piece is small enough to neglect weight). Now let's bend this small piece a little from its equilibrium position (y=0). The tension of the rope will be always along the tangential direction (this is true for the soft rope that there will be no shear force in it, i.e. if you make a cut of the rope with a cross section, the tensional force will be normal to the cross section and the shear force will be parallel with the cross section. Here we assume only tensional force exist). The force at the two end forms angles θ_A, θ_B respectively w.r.t. x axis. The rope has density ρ.

With the conditions given above, we can analyze the dynamics of the rope. First the force:

$$F_x = T\cos\theta_B - T\cos\theta_A$$
$$F_y = T\sin\theta_B - T\sin\theta_A$$

For a small displacement of y, the angles will be small ones; and the difference between the angles θ_A, θ_B would be also very small, i.e.: $\cos\theta_B \approx \cos\theta_A \approx 1$; $\sin\theta_B - \sin\theta_A \approx \theta_B - \theta_A = \Delta\theta$:

$$F_x \approx 0$$
$$F_y \approx T\Delta\theta$$

There will be no motion along the x direction for this little piece as expected.

Along the vertical displacement:

$$T\Delta\theta = \rho\Delta x(\frac{d^2 y}{dt^2})_x = \rho\Delta x\frac{\partial^2 y}{\partial t^2}$$

Also from geometrical constraint, at certain time t:

$$\tan\theta = (\frac{dy}{dx})_t = \frac{\partial y}{\partial x}$$

$$\Delta(\tan\theta)_{\theta_A\to\theta_B} = \frac{\partial y}{\partial x}|_{x_B} - \frac{\partial y}{\partial x}|_{x_A} = \frac{(\frac{\partial y}{\partial x}|_{x_B} - \frac{\partial y}{\partial x}|_{x_A})}{\Delta x}\Delta x = \frac{\partial^2 y}{\partial x^2}\Delta x$$

$$\Delta(\tan\theta)_{\theta_A\to\theta_B} = \frac{d\tan\theta}{d\theta}\Delta\theta = \frac{\Delta\theta}{\cos^2\theta} \xrightarrow{small\ \theta} \Delta\theta$$

So: $\Delta\theta = \frac{\partial^2 y}{\partial x^2}\Delta x$ and

$$T\frac{\partial^2 y}{\partial x^2}\Delta x = \rho\Delta x\frac{\partial^2 y}{\partial t^2}$$

Rewrite the above into a standard form:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad (10\text{-}27)$$

$$v = \sqrt{\frac{T}{\rho}} \qquad (10\text{-}28)$$

The (10-27) is called wave equation and (10-28) is an expression of v in terms of tension and density for the rope model. We shall see that the physical meaning of this v is the velocity of wave propagation[101]. Though (10-27) was derived with our specific simplified model of rope under tension, it turns out that other waves will satisfy same equations including the E-M wave. This justifies that it is being called wave equation[102].

(2) General Solutions and Harmonic Wave

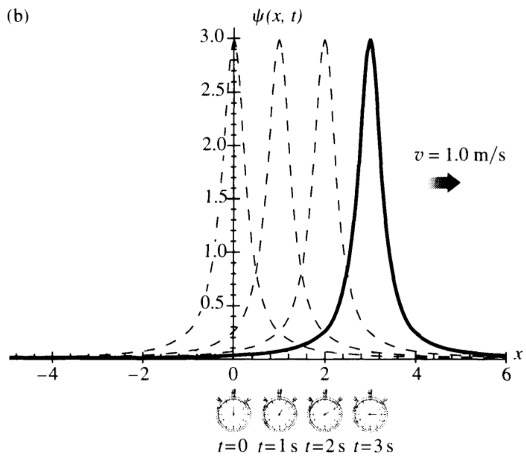
[101] Strictly speaking, this velocity is the phase velocity of the wave. There are other velocities on the propagation of wave and we shall not discuss this in depth here (We will in Optics course).

[102] Of course the expression for v will be different in different cases. Also the waves here are limited to the classical waves, which obeys Newton's equation or Maxwell equation in the classical theory. In quantum, the wave equation (the Schrödinger Equation) will be different from (10-27.)

The wave function $y(x,t)$ (or $\psi(x,t)$ more generally) need to be solved from wave equation (10-27), this requires solving partial differential equations (PDE) with boundary conditions or initial conditions. It is out of the scope of this course to solve PDE[103], so I shall just give out the general solution directly:

$y(x,t) = f(x \pm vt)$ (10-29)

The general solution to the wave equation (10-27) is in this form, f is some function (need to be defined by initial conditions) with spatial and temporal variables grouped as $x \pm vt$! You can test that (10-29) does satisfies the wave equation by plugging it into (10-27). The $f(x-vt)$ represents a wave with some initial form $f(x)$ (at t=0) traveling towards right (positive x direction) as tine progresses and it is shown in the figure below, displayed as 'snap shots', i.e. displays the wave function at some fixed times:

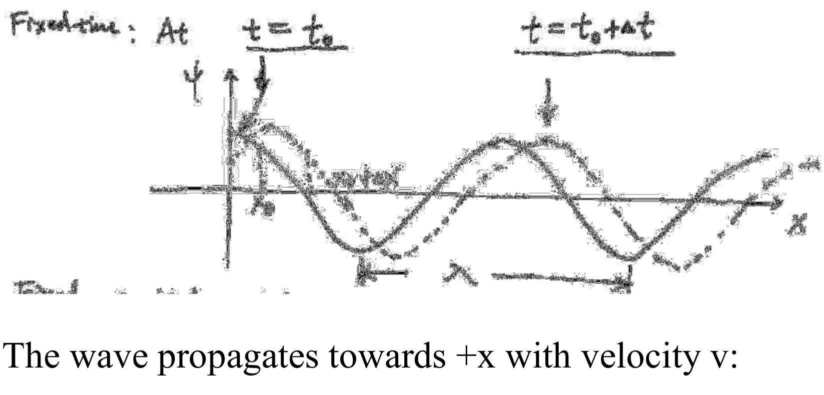


[103] The PDE in forms of (10-27) is not too hard to solve. It can be solved with separation of variables or with the Fourier Transform method. We will come back to it in the Optics course.

$f(x+vt)$ will represent a wave with initial form travels towards left. The detailed form of f depends on initial conditions and can vary from simple functions to complicated one depends on situation. The simplest form and also the most useful[104] of the wave function will be that of harmonic functions, i.e. f is in form of sinusoidal forms, and the waves represented by such functions are called harmonic waves. The convention is to choose cosine function (of course you may choose the sine function, but that only means a phase difference of $\frac{\pi}{2}$):

$$y(x,t) = A\cos[k(x - vt) + \phi] \qquad (10\text{-}30)$$

This is the general function form for a harmonic wave. A is the amplitude and ϕ is the initial phase. It represents a **traveling wave** propagating towards +x direction. The meaning of k is going to be discussed next. First the figure below shows a traveling harmonic wave at two 'snap shots':



The wave propagates towards +x with velocity v:

[104] It is useful not only because these harmonic function are generated easily by a harmonic oscillation, but also because of the method of Fourier Analysis in which other complicated function forms can be expressed as superposition of these harmonic functions. So the harmonic wave functions discussed here are going to be building blocks for complicated wave forms.

As the figure shows, the wave function in (10-30), at $t=t_0$, for a point x_0, its displacement is: $y(x_0, t_0) = A\cos[k(x_0 - vt_0) + \phi]$, as shown by the vertical dashed line. Suppose you monitor how this point on the wave travels (of course you can pick other points such as the maxima or minima or zero). At later time $t = t_0 + \Delta t$, this displacement moves to the position at $x_0 + \Delta x$, this means the phase inside the cosine function has to be same, i.e.:

$$k(x_0 - vt_0) + \phi = k[x_0 + \Delta x - v(t + \Delta t)] + \phi$$

$$v\Delta t = \Delta x \rightarrow v = \frac{\Delta x}{\Delta t}$$

Which is exactly the meaning of velocity (since this velocity is how the equal phase point travels, it is called phase velocity for the wave).

Also in these snap shots of the wave, at a fixed time t and for simplicity I just take t=0, the **wavelength** of the wave is defined as spatial interval at fixed time so that the phase difference is 2π:

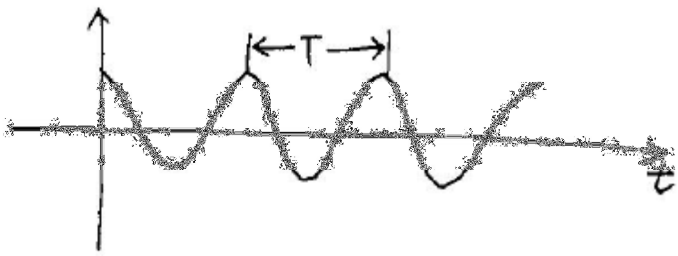
$$k(x_0 + \lambda) + \phi = kx_0 + \phi + 2\pi \qquad (10\text{-}31)$$

$$k = \frac{2\pi}{\lambda} \qquad (10\text{-}32)$$

k is called **wave number** (angular wave number to be more precise, since wave number sometimes is defined as $\bar{\upsilon} \equiv \frac{1}{\lambda}$)

Now pick a fixed point x_0, and see how its displacement $y(x_0, t)$ changes over time, it is a harmonic oscillation as the sketch below shows:

Fixed spatial point, $x = x_0$



T is the period of oscillation which is defined as time interval so that the phase difference is by 2π:

$$k[x_0 - v(t_0 + T)] + \phi = k(x_0 - vt_0) + \phi - 2\pi$$
$$kvT = 2\pi$$
$$v = \frac{2\pi}{Tk}$$

We shall define frequency υ and angular frequency ω as;

$$\upsilon \equiv \frac{1}{T}; \quad \omega \equiv 2\pi\upsilon = \frac{2\pi}{T} \qquad (10\text{-}33)$$

The phase velocity would be:

$$v = \frac{\omega}{k} = \frac{\lambda}{T} \qquad (10\text{-}34)$$

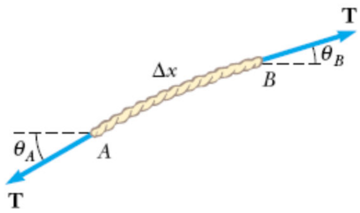
(10-32) to (10-34) are basic definition and relation between the parameters used to describe the harmonic wave, so given a pair of parameters such as λ, T or ω, k, you should be able to compute the others. The harmonic waves are usually expressed in terms of ω, k:

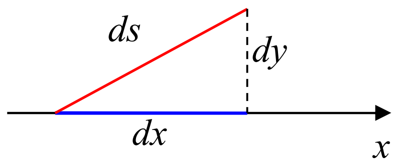
$$y(x,t) = A\cos(kx - \omega t + \phi) \qquad (10\text{-}35)$$

This represents an ideal wave with single wavelength and single frequency. It is analogous to a mass point with definite momentum and energy in the particle case which is an idealization, but a building block for complicated cases.

(3) Rate of Energy Transfer

As we stated earlier, though the individual particles does not move far away from its equilibrium position, such as those on the rope, there is energy carried by the wave that propagates along the rope. In the rope case, the vibrational energy from the oscillating source transfers along the rope in form of wave. This energy carried by wave in the rope case is stored as the kinetic and potential energy of the rope and thus can be computed.





Consider a small segment of the rope with length of Δx along the x direction (this is the length when the rope is totally relaxed). The mass of it is: $dm = \rho dx$. Its kinetic energy is:

$$K = \frac{1}{2}\rho dx v^2 = \frac{1}{2}\rho dx (\frac{\partial y}{\partial t})^2 \qquad (10\text{-}36)$$

For the general solution this is $y = f(x - vt) = f(q)$, $q \equiv x - vt$:

$$K = \frac{1}{2}\rho dx v^2 (\frac{df}{dq})^2 = \frac{1}{2}\rho dx v^2 [f'(q)]^2 \qquad (10\text{-}37)$$

It is more proper to define an energy density, i.e. the energy per unit length in this case:

$$\rho_K = \frac{K}{dx} = \frac{1}{2}\rho v^2 [f'(q)]^2 \qquad (10\text{-}38)$$

For the special case of harmonic wave, $f(q) = A\cos kq$:

$\rho_K = \frac{1}{2}\rho v^2 k^2 A^2 \sin^2(kx - \omega t) = \frac{1}{2}\rho\omega^2 A^2 \sin^2(kx - \omega t)$, its time average would be:

$$< \rho_K > = \frac{1}{4}\rho\omega^2 A^2 \qquad (10\text{-}39)$$

The above is taking the time average of the (10-38) while use the sinusoidal relation: $\sin^2 x = \frac{1-\cos 2x}{2}$, the sinusoidal (cosine here) time average is zero. (10-39) is quite similar to the (10-9), the average kinetic energy by an oscillator. There is no surprising, as we have seen that at certain fixed special point, the harmonic wave function reduced to a particle doing harmonic oscillation. Then following the result of (10-9), you probably could guess that the averaged potential energy density stored in the rope $< \rho_U >$ would be also in form of (10-39) and the total energy density would be $\frac{1}{2}\rho\omega^2 A^2$. This is an excellent guess, with the sound physical argument that it is harmonic oscillation for any fixed spatial point. However, let me be a little more rigorous and derive the potential energy density and not limited to harmonic waves.

The potential energy of the rope must due to the stretch (or compress) of the rope from its natural length dx. As the wave propagates through the rope, this dx segment would be stretched to length ds, and the potential energy due to the tension is:

$$U = T(ds - dx)$$

$$ds = \sqrt{dx^2 + dy^2}$$

The small change of dy due to the dx in this case is :

$$dy = \frac{\partial y}{\partial x} dx$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{\partial y}{\partial x})^2}\, dx$$

If the $\frac{\partial y}{\partial x}$ is a small number for a small transverse displacement:

$$ds \approx [1 + \frac{1}{2}(\frac{\partial y}{\partial x})^2]dx$$

The potential energy density would be:

$$\rho_U = \frac{U}{dx} = \frac{1}{2} T (\frac{\partial y}{\partial x})^2 \qquad (10\text{-}40)$$

For the general wave function $y = f(x - vt) = f(q)$:

$$\rho_U = \frac{1}{2} T [f'(q)]^2 \qquad (10\text{-}41)$$

This does equal to that of ρ_K in (10-38) if you recall that $v = \sqrt{\frac{T}{\rho}}$.

For the special case of harmonic wave function, the relation $\rho_U = \rho_K$ still holds and the averaged total energy density is (for harmonic waves):

$$< \rho_E > = < \rho_K > + < \rho_U > = \frac{1}{2} \rho \omega^2 A^2 \qquad (10\text{-}39)$$

The energy density for a harmonic wave stored in the rope is proportional to the square of the amplitude, which is expected from the results of harmonic oscillation.

The rate of energy transfer of the wave is defined as the energy flow through a certain point per unit time, which would be just the energy density times the velocity of the wave:

$$P_{power} = <\rho_E> v = \frac{1}{2}\rho\omega^2 A^2 v \qquad (10\text{-}40)$$

This is the formula of rate of energy transfer for a traveling harmonic wave, since we did not consider dissipation (loss of energy due to friction etc) in the derivation, it is an idealization. (10-40) certainly makes sense that if you want to create a wave with high frequency, large amplitude and travels fast, you need high power input at the source to sustain the wave.

Example: Suppose I drive a string of $\rho = 0.05kg/m$ with a tension of $T = 80N$. The amplitude of the harmonic wave I created will be 6cm, and my hand is shaking with a frequency of 20Hz. How much power must be supplied by me to sustain the wave?

The power of the energy transferred by the wave is given in (10-40), and this power has to be supplied by the driver.

Given the conditions, this is just plug the number into formulas:

$$v = \sqrt{\frac{T}{\rho}} = 40m/s$$

$$\omega = 2\pi\upsilon = 125s^{-1}$$

$$P_{power} = \frac{1}{2}\rho\omega^2 A^2 v = 56 watts$$

The wave discussed above is what is called *transverse* wave, where the vibration motion of each element is perpendicular to the direction of propagation of the wave (the direction of energy propagates). Examples of transverse wave include the wave on the rope, electro-magnetic wave (light), etc. There is another type of wave called longitudal wave, where the vibration motion is parallel with the direction of wave propagation. A typical example for this kind of wave is the sound wave, which caused by the molecular density (the pressure) variation of the air. The treatment on longitudal wave is same as the transverse case, so I will not derive the wave equation for the sound wave[105].

[105] If you are curious on the details, please refer to Serway and Jewett's "Physics for Scientists and Engineers" chap 17. Or H.J. Pain's "The Physics of Vibration and Waves" chap.6 for more serious reading.

Part II

Introduction to Special Relativity

Outlines of the Topics in Special Relativity:

Chapter 11 Birth of Special Relativity

Topics: 1) Historical background: The difficulty in E-M with Galileo Transformation, a special inertial frame 'ether' and efforts to detect such 'absolute ether' frame: Michelson-Morley experiment and efforts to save the old mechanics (ether drag model and bullet-gun model, Lorentz transform based on ether frame etc) 2) Fundamental postulates by Einstein 3) Direct results from these postulations: (definition of events first) time dilation, length contraction and most important that these are caused by the simultaneity problem, simultaneity is a relative thing, depending on the reference frame. These will pave the way for the Lorentz transformation.

Chapter 12. Lorentz Transformation

Topics: 1) Derivation of Lorentz Transformation from homogeneity of space-time (results in linear transform) and isotropy of space (only time and coordinate along the direction of motion will be coupled) + length contraction and time dilation already discussed can be used to get Lorentz transform. 2) From L-Transform to look at time dilation, length contraction and simultaneity. More examples to apply this transform as the following topic will demonstrate 3) Doppler effect. 4) Paradox: yard-barn; star war and twin paradox. 5) Minkowski diagram. 6) cause-effect requires signal speed is less than c to avoid contradictory, absolute future and past (time-like) and absolute alibi (space-like). Topics paved way for the next chapter: 7) velocity relations (transform)

Chapter 13 Relativistic Dynamics: Momentum and Energy; 4 vectors

Basically the relativistic momentum and energy will be introduced and defined by two methods. First directly from extending the conservation laws of classical mechanics and work out the relativistic momentum and energy by requiring such conservation still holds in all inertial frames. Second define the space-time 4 vectors and the most important: the invariant spatial interval under L-transform, using another form of the invariant, proper time, to define energy-momentum 4-vector.

Topics include: 1) Relativistic momentum and mass from conservation. 2) Energy from work-energy theorem and its equivalence to mass. 3) massless (rest massless) particles and speed limit on particles. 4) Space-time 4-vector and invariance spatial interval; re-derive L-transform from this invariance (in analogy to rotation but with hyperbolic sinusoidal function, this is optional). 5) Proper time and extend 4-vectors from space-time to 4-velocity, and energy-momentum and shows that this approach will give same results as in topic 1) and 2).

Chapter 14. Relativistic Dynamic

Topics: 1) Transform of Acceleration. 2) Force and its transform; 4-vector form of the force and equation of motion in relativity (in special relativity only, no geodesic approach) 3) Examples: work out some simple examples of trajectory of motion in SR.

Chapter 11 Birth of Special Relativity

Here we shall discuss the historical background leading to the birth of special relativity. The materials I chose to present may not be strictly chronological but to serve a logical argument[106]. We shall see the difficulties facing the physicists more than 100 years ago, i.e. the less harmony between classical mechanics and classical electro-magnetic theory (E-M) and the efforts tried to correct this. The two fundamental postulates proposed by Einstein solved difficulty and unveil the truth in space-time which has long and profound (even shocking and confusing to his contemporary and beginners) influence. We shall study the experimental facts (though Einstein was not aware that of Michelson and Morley's result) that led to the postulates and see what are the direct results from them, i.e. time dilation, length contraction and relativity on simultaneity.

11.1 Event and What are Changed and Unchanged in Special Relativity

The classical mechanics is incomplete (I mentioned this in chapter 1 and

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[106] For a chronological account of the development, you may read W. Pauli's "theory of relativity" (1921), part 1 or C. Moller's "theory of relativity" (1952). Both are sort of advanced treatment using tensor analysis which is not necessary for special relativity. They do include extensive references to the original papers. Especially the Pauli's book which in my point of view, is read like a long review article.

here is a restatement) in a sense that it applies to low speed situations ($v/c \ll 1$, c the speed of light in vacuum)[107]. Under high speed, the old theory needs to be modified and even reformulated and quite a few changes will be introduced. Rather starting to talk about this, I feel it may be helpful if I first summarize what are changed and more interesting what are not changed even in the light of new theory. This is what I am doing in the following paragraphs and I shall delay the birth of special relativity to the next section. This is done without rigorous proof or argument yet. An important concept "event" needs to draw our attention. We had used it in old theory and daily life almost taking it for granted, but it will prove useful if we explicitly state it in relativity.

Thing are not changed: 1) Measurement of space and time in one inertial frame. We still measure the time (with a clock) and space (distance etc. with ruler stick) in an inertial frame of our choice. So Beijing-Shanghai is 1000km apart has the same meaning as in old mechanics, except I will stress that this is measured with respect to a frame (say rest on earth). A big soccer game (the Euro-Cup game when I am writing this) is broadcasted to begin at 8pm is still meaningful, and still I shall say this is time according to the clock that is rest on earth. So basically we still describe anything with space and time as usual,

[107] Actually, the classical mechanics is also incomplete in a sense that it does not apply to the microscopic situations where $x \sim h/P$ (P momentum, h Planck constant), that requires quantum theory which will not be discussed here.

provided we refer to the same inertial frame. These things (all physical phenomena) are called **events**, an event is just something happened at some particular location (space) and at some particular time (a baby is born on 6/18/2006 in Beijing; a particle hits detector at (x,y,z) in lab and on time t; some poor guy was killed at some location and time…you name it. Even the common phrase that someone is "at wrong place and wrong time" carries this flavor). Of course the clocks need to be *synchronized* to report the correct time and measuring sticks need to be calibrated. If the broadcaster's clock and your clock are not synchronized, yours is hours behind, you will miss the soccer game that is 8pm by the broadcaster's clock.

To define an event, we attach these *space-time coordinate* to it, say (t,x,y,z). This will specify the location and time of the event. Say a plane will arrive Beijing airport at 5pm, this is (5pm, 120 longitude, 42 latitude, h=200m); a particle hits a target at (1s, 2m,3m,4m) etc. These labels are meaningful and useful as long as we agree on the measuring of time and space: we use same ruler stick and same synchronized clocks. Otherwise confusion may arise and agreement cannot be reached between observers even within the same inertial frame.

There is some ambiguity to the time of an event really happened, we have to define how we record them. For example, a fire work is launched at downtown and you are watching it from Tsinghua. You know the downtown spatial coordinate with respect to you (you measured with ruler beforehand), but the time you recorded the fire work is t_1 with your stopwatch (this is the time you see the firework with your own eyes; or the event that the firework signal reaches your eyes). You know that this time t_1 is not the exact time the firework launched (this is a different event); it happened at t_0 which is tiny-winy before t_1. So in physics (in real life most people would disregard such small difference), we assume we have observers all over the world, record events at exactly where and when it happens , and write down its time and space coordinate, so that we have a true space-time coordinate in this sense(*a network of observers attached to the inertial frame*, the poor buggers are fixed in the coordinate and you equip them each a synchronized Rolex to record time, their sole purpose is to record the space-time coordinate as an event happens). As in the example above, you could ask a friend at the firework launching site to record the exact time of the launch, and show you the data later so that you may have an accurate recording of the event. Clearly synchronization is essential here. In order to trust your friend's data, you have to make sure that your watch and his are synchronized. It is interesting on how you achieve this. You two may compare the watch locally, say at Starbucks outside and make them synchronized. But the motion from Starbucks to downtown will change the synchronization as we shall see, so this method is not very reliable in high precision measurement (unless you move the clock very slow).

The correct way (at least in principle) is to synchronize the clocks with signal. Still in the last example, you send the recording of t of your clock through electromagnetic wave (light, radio-signal and the signal may not even need to carry the t, you two may agree upon that at exactly 8pm according to your clock, you will send a light to the friend). The distance of the light travel can be determined precisely (say 100km through optical fiber), so the travelling time of light Δt can be computed (of course I used the assumption that the speed of light is a constant here), and your friend at launch site will set his clock to t+Δt. This way both of you will be sure that two clocks are synchronized. All the clocks in the world can be synchronized this way, so in principle we can measure the space-time of any events with confidence.

Still I shall stress that all these are referring to the same inertial frame, say rest on earth (on a large scale, the non-inertial effect of earth need to be corrected too, but we simplify this by treating earth as a true inertial frame here). In summary, the space-time coordinate of an event, has the same meaning here as those in old mechanics. It can be measured precisely provided: ***A) Same distance and time unit length; B) Synchronized clock.*** In the later sections I shall frequently use terms like "for an observer in a frame S, or to the observe in coordinate system S, or just in S etc or as the observer sees…", all these mean the same thing just as I stated above: a network of observers in a particular coordinate system equipped with synchronized clock to record the space-time coordinates of any events.

2) Then the velocity and acceleration can be defined and determined within the inertial frame of choice as usual (small change of space divided by small change of time. etc.).

3) The conservation laws in momentum and energy are not changed. They are more robust than Newton's laws and survived in all branches of physics. As I stated at the very beginning (Chapter 1), these conservation are related to symmetry of our space and time (Homogeneity and isotropy of space and time) and inertial frame is such a frame that the space-time are homogeneous and isotropic.

Things are changed[108]: Though for an observer Adam in one inertial frame (say S, rest on earth) can define an event with the space-time coordinate as I talked above. For another observer Bob in another inertial frame that is moving with respect to S (S', Bob is on board a moving train), he could also measure the space-time coordinate, according to his ruler stick and clock and by his hordes of observers in S'). The Bob's recording for a specific event will disagree with that of Adam's. This is not too surprising, because old mechanics also predict a difference. 1)

[108] Only an account is provided here, with detailed arguments in later sections.

The striking thing is that the difference is not the classical Galileo's transform, but will be the Lorentz transform in special relativity. *The simultaneity of events (events happened at the same time recorded by the clocks in S or S'), time rate and ruler stick length will all depend on the inertial frames.* So the conditions A), B) above do not satisfy between observers in different frames though within one frame they can be satisfied.

For the events happened simultaneously in S (say rest on earth, Adam's wife Eve gives birth to a baby in Beijing at exactly 8pm according to Adam's watch; his friend Tom's wife Lisa gives birth to a baby in Shanghai also exactly at 8pm according to Tom's watch which is synchronized with Adam's; the two events are simultaneous in earth frame), and its clock rate say 1 second and its ruler stick length say 1m, would all appear differently from Bob's point of view (in S', say a very speedy flight from Beijing to Shanghai). Bob would say those events are not simultaneous, Tom's kid was borne first; and that Adam's 1 second in time is 1.2 second according to Bob's clock and Adam's meter is 0.83 meter according to Bob's length. For Adam he would draw the same conclusion on Bob's, i.e. what simultaneous in S' are not simultaneous in S, Bob's 1 second is 1.2 second and Bob's 1 meter is 0.83 meter according to Adam's measurement. These are in stark contrast to our 'common sense' based on daily life (Einstein cynically criticized ‘common sense is a prejudice developed before 16’)[109].

We do not experience ‘time dilation’ and ‘length contraction’ in daily life. You meet your friend at 5pm at airport who is flying from Shanghai. You will meet him at 5pm on both of your watches despite the fact your friend took a flight which is certainly a moving frame with respect to you. The reason is of course such effects are only important and become noticeable at high speed (or with high precision measurement of time as in Hafele-Keating around the world flight experiment[110]), the speed is compared to that of light. Our daily speed are just too small (compare to the speed of light) for us to notice these differences. In short, the simultaneity, time interval and space interval will depend on observer’s inertial frame, and the relations between them obey Lorentz transform. This brings a chain of changes, because the measurement of space and time is so fundamental, it is the foundation of Newtonian mechanics.

As an added comment on this, I should mention that though different observers may disagree on simultaneity events happened at *different* places (such as the babies born in Beijing and Shanghai in the last example), they will *agree* on the simultaneity of events happened at the *same* place. Both observers agree that any event happens actually already

[109]What he against is the ‘common sense’ based purely on experience. There are common sense based on logic and beliefs on fundamental principles, which are good. So do not use this against the good common sense, especially those in social science, such as the famous booklet “common sense” by Thomas Paine on human right and government

[110] Hafele and Keating, *Science* 177, pg 166-168;pg 168-170. (1972)

implies this, but let me illustrate this point more to make it explicit and clear: The statement of some event happens actually can be rephrased as different events happen at same place and time. A baby is born in Beijing means the mother and the baby both are at the same place and time; a dog was run over by a car means the poor dog was at the same place and same time (you can say at the wrong place and wrong time) with the car. Say as the incident happened, observer in S recorded (x_1,t_1) for both the car and the dog, and he said the dog was killed at (x_1,t_1). For the observer on the car S', he would record the incident as both the car and the dog at (x_1', t_1'), though these numbers may be different from the recording of S, but the fact that the car and the dog were at the same place (x_1' for S', x_1 for S) and same time ($t_1'$for S' and t_1 for S) are truth for both observers and they both record the death of poor animal, or the birth of the baby(the baby and mother at same place and same time, though the exact value of the space-time coordinate may be different).

If different observers cannot agree on the simultaneity of events at the same place, real contradiction will appear: the dog was not run over by a car or the baby was not coming out of mother's womb according to one observer; while it is otherwise according to another. Then the meaning to say an event happens will be lost, there will be no truth in the world. So (a repeat of myself) different observers will agree on the simultaneity of events happening at same place, what they cannot agree are the simultaneities of events at different places.

For the students who already learned some SR, the above statement may make their head nodding; for those new to the SR, the same statement may drive them scratching the heads. Don't worry, I shall explain all above in detail in later section, so carry on!

2) The velocity and acceleration will have different transform property (i.e. relations between velocities measured by different observers in different inertial frames).

3) We shall see the correct formulas for mass, momentum and energy will appear differently from those in old mechanics (but very important that it will reduce to the old formula at low speed limit).

4) Force, the important quantity in Newtonian, will lose its dominant position. Actually Adam or Bob could still calculate the force with the old formula in their individual inertial frames and measure it through experiments (such as from change of momentum). But since the force depends on space, mass or even time (the general force, not the fundamental ones). The force will appear differently in different frames (in old days, the force depends on relative positions, relative velocities, mass and time which are invariant in Galileo transformation, so the force appear same for different observers) and to our old friend, equation of motion F=ma which is so essential in Newtonian is not true anymore

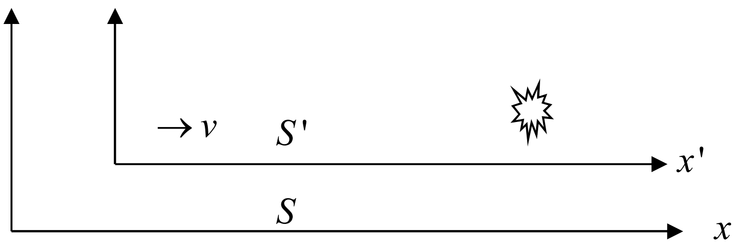
under high speed. The correct formula in relativity that is corresponding to (means reduced to F=ma at low speed) equation of motion is quite complicated and less useful. (This reduced the importance of force, which is introduced in old mechanics as a measurement of interaction, because we have a simple relation between the force and acceleration F=ma) So energy, momentum approach will be the easier and preferred ones.

Finally after this brief summarizing the changed and unchanged aspect of mechanics under relativity, I shall stress that there is something *invariant* to wet your appetite. The two observers from different frames may not agree upon a lot of things, however, there will be something unchanged and all observers will agree upon. We will see what is it in the due course and this unchanged thing (called invariance of transform) is very important in the theory of special relativity (in the 4-vector section later), and a whole theory can grow out of this.

11.2 Historical Background

11.2-1 Galileo Transform and Relativity Principle for Mechanics

We had talked about Galileo Transform in section 4.2 and Relativity Principle in chapter 1, here I shall restate it limiting to the mechanics only.



As the figure shows the two *inertial* frames are travelling relative to each other. In the stand point of view of S, S' is moving with +v towards right along x direction; while for S', S is moving with –v towards –x' direction, where v is a constant as required by the inertial frames. Galileo Transform is the relationships between space-time coordinates of any event viewed by the inertial observers in S or S', say for a fire cracker exploded, and both observers recorded this event in their respective frames as (x,y,z,t) in S and (x',y',z', t')in S', Galileo Transform tells us:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \qquad (11\text{-}1)$$

From this we will have simple relations between velocity and accelerations viewed by S and S':

$$\begin{aligned} &u'_x \equiv \frac{dx'}{dt'} = u_x - v; \quad u'_y = u_y; \quad u'_z = u_z \\ &a' \equiv \frac{du'}{dt'} = a \equiv \frac{du}{dt} \end{aligned} \qquad (11\text{-}2)$$

From Newtonian Mechanics, the mass will be independent of motion (see chapter 4 on this premise) and the force will be function of relative positions which will be same in S and S' (at a specific time, the distance between two particles are same in both frames). So this tells us $\vec{F} = m\vec{a}$ applies equally well for observers in all frames moving with constant velocity to each other. The jargon (a fancy way to state the above) is that the equation of motion (Newton's law) is *invariant* upon transformation (change from one frame to another). To keep the law invariant, its components (here force, mass and acceleration) has to transform accordingly, which is called *covariant* as space-time coordinates changes upon transformation. Of course here the components (F, m and a)are invariant too upon transformation (Galileo), but more generally if the observation only requires the law (here $\vec{F} = m\vec{a}$) invariant, its components can change as long as keeping the law same, to put it more strictly in math: say upon transformation between S and S':

$\vec{F}' = \hat{L}\vec{F}$ where $\hat{L}$ represents the transformation (it is a matrix if the transformation is linear, generally it is equations relating the forces observed in different frames), and if the other part of the equation *ma* also transform the same way: $(m\vec{a})' = \hat{L}(m\vec{a})$, then formula for equation of motion would be same in both frames: $\vec{F} = m\vec{a}$ for S and $\vec{F}' = (m\vec{a})'$ in S'.

This may appear a bit abstract, so let me illustrate it with the example by Galileo: Suppose the S is a stationary observer on the bank of a river, S' is an observer aboard a ship sailing with v relative to the bank. A stone is dropped from the mast of the ship (an event). If both observers follow the motion of the stone (a sequence of events), what do they observe? For S', the observer on board, the stone will be just free fall dropped from above (no trace of motion of the ship); if he throw the stone upward, he will catch it back later without moving. For the observer on the bank, he would see the stone following not a straight line free fall but a parabolic path. However both observers conclude that the motion of the stone obeys Newton's law, i.e. if the observers apply the Newton's law within his own frame, his prediction based on it will be exactly what he observed within his frame (the difference in observed trajectory is due to difference in initial conditions but not the physical laws).

This is the essential meaning of ***Relativity Principle for Mechanics***: *The mechanical laws are same (invariant) for all inertial observers*. Another way to say the same thing (a corollary) with a different flavor (or stress) is that the absolute motion of the inertial frame cannot be detected by mechanical experiment *within the frame* only. The man on a constant moving ship doing all kinds of mechanical experiments and he will get same answer as if on ground, so without referring to outside reference point (suppose the ship is traveling in dark in a starless night), he cannot tell that he is on a moving ship or on ground.

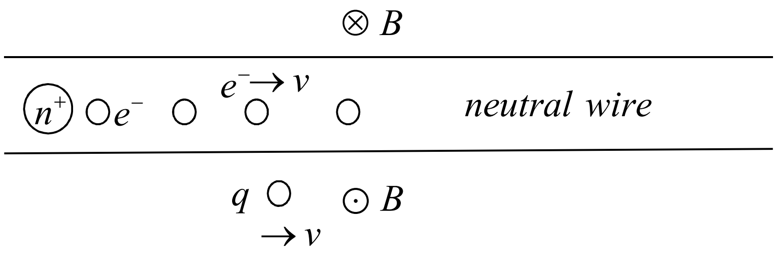
Newton's idea of an absolute space-time as the 'mother' of other inertial frames is actually redundant since all inertial frames are equivalent---principle of relativity for mechanics. However, in Galileo Transformation, *the length and time is absolute in a sense that the measuring stick and clock are all same in different inertial frames*. This seemingly apparent reasonable assumption turns out faulty at high speed. But instead of just giving out correct form of transformation, let's see first what led to people finding the flaws.

11.2-2 Trouble of Relativity Principle with E-M under Galileo Transformation

All the above, the Galileo Transformation and Relativity Principle work fine and dandy for mechanics, the trouble is with the E-M theory. The fundamental equations in the E-M theory are Maxwell equations, which are differential relations between electric and magnetic field given the charge and current distribution. They are 2nd order partial differential equations. I will not explicitly workout the Maxwell equation under Galileo Transform here, however if you are intrigued, please pick one and start partial differentiation yourself. Or you may try the direct result on the wave equation derived from the Maxwell equations:

$$\frac{\partial^2 E(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} \qquad (11\text{-}3)$$

This is the wave equation (recalled wave equation formula we derived in chapter 10) for the electro-magnetic wave (light) and c is the speed of light. Please try the Galileo Transform on this and you will see that its equation form depends on the frame[111]. Instead of taking this mathematical approach, let's consider the following physical example:



As the figure above shows, a neutral conducting wire (with equal number of positive and negative charges, only one positive charge is shown in the figure) with current flows in it. The electrons inside the wire is moving with velocity v (current then is flowing backward towards left). This current will generate a magnetic field B according to Ampere's law. Another point charge q also moves with same velocity outside the wire. This charge q will experience a force, the Lorentz force $qv \times B$. (if q positive, it is repelled from the wire; if q negative, it is attracted towards the wire) This is the observation from a lab-fixed frame. Now consider a frame that is moving with same velocity v as the charges. In this frame, the electrons and the point charge q are stationary, while the positive charge in the wire moves with –v, so there is still same current in the wire, and magnetic field from this current would also be same. However, the

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[111] I shall only give a start here:
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} \rightarrow \frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'} \rightarrow \frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial x'^2}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} \rightarrow \frac{\partial E}{\partial t} = -v\frac{\partial E}{\partial x'} + \frac{\partial E}{\partial t'} \rightarrow \frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial t'^2} - 2v\frac{\partial^2 E}{\partial x' \partial t'} + v^2\frac{\partial^2 E}{\partial x'^2}$$

charge q is motionless in this frame so there will be no Lorentz force. Then is there a force on the point charge q viewed from the moving frame? If we stick to Galileo Transformation and apply the Maxwell equation in its original form in the lab frame, we will have no force in the moving frame (which was wrong). The observation would really depend on the inertial frames then. If we stick to Galileo Transformation and in order to get the same effect (the charge q is attracted or repelled from the wire), the Maxwell equations will need to be changed (into some nasty forms) in order to get non-zero force on charge q in the moving frame.[112] Either way, the Relativity Principle seems do not apply for the E-M theory: **Maxwell equation is not transform invariant with Galileo Transformation between frames.** There appears lacking of unity here in the physical laws where the Mechanics follows the Relativity Principle (under Galileo Transform) while the E-M does not.

This difficulty suggests either of the 3 below or a combination maybe the remedy: 1) The relativity principle is not a universal rule and not applicable to E-M theory, the **Maxwell equation is only true under one special frame**. 2) The E-M theory needs to be modified or corrected. 3) The E-M theory is correct. The Galileo transform of space-time is

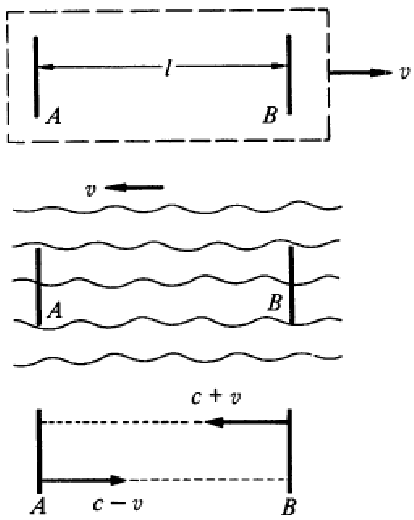
[112] Of course on retro respect, this difficulty arises from the Galileo Transform of space-time is wrong. Under Lorentz Transform, the observation would be same in both frames and the origin of the forces can be satisfyingly explained. One is due to the Lorentz force; the other (in moving frame), the force arises from length contraction, so that the local total charge density is not zero anymore, the positive charge will have higher density than the negative in the wire locally due to length contraction we shall talk about later, and the force on charge q in this moving frame is caused by the Coulomb force.

incorrect, but ***this really means we have to abandon our intuitive view of space and time and modify Newtonian mechanics*** that built on it. Naturally (put yourself in shoes of those physicists 100 years ago) most people will choose 1) in which the representative was Maxwell or 2) in which the representative was Lorentz. Only Einstein chose the path 3 and gave birth to special relativity and revolutionized our view of space-time. What Einstein believed is that the E-M theory is correct and the Relativity Principle would hold for ***all*** physical laws in all inertial frames. This suggests it applies to Electro-magnetic theory too. But the Galileo Transform between constant velocity moving inertial frames is not suitable for E-M theory. This means the correct transform would be otherwise; and since Newtonian Mechanics obeys Galileo Transform, so under the correct transform relation between space-time, Newtonian formula would be probably be frame-dependent, violating the relativity principle and need to be reformulated. Of course there is another possibility that both the classical E-M and Newtonian (which are the complete physical theories known at that time) are not transform invariant which means neither satisfies the relativity principle, and both need to be reformulated. It turned out that the E-M theory satisfies the correct transform (this does not say that E-M is the final correct form, it needs to be modified in quantum domain, that is Quantum Electrodynamics or QED) while the Newtonian does not. So we shall focus on the reformulation of Newtonian Mechanics by special relativity here. Though we know the path 3 is the right one now, it is helpful to understand other paths too.

If you, like Maxwell, treat the E-M equations only work for a special frame, there is then certain absoluteness in this frame, i.e. The E-M theory has a preferred frame. Such frame is called ether frame (old spelling is aether). Ether was believed by Maxwell and his contemporaries as some substance all over the space (the wording of ether actually has much longer history, coming from Aristotle of Greek time) , permeating the whole universe. It is also acting as media that transmits electro-magnetic wave, the light (in analogy to water transmitting ocean waves). Of course this is a very natural choice if you believe path 1), but it is only a belief or hypothesis. What is important in science as we stated in chapter 1, is such hypothesis should be subject to rigorous tests (experiments). Same would be true for other hypothesis. And indeed many experiments were conducted to test the existence of this mysterious ether, the most famous (and most accurate and convincing at that time) one is Michelson-Morley experiment.

11.2-3 Michelson-Morley Experiment---Null Result of Detection of Motion Relative to Ether

As Maxwell believed that his equations only applies in one special frame which was called ether frame. The wave equation (11-3) for E-M wave (light) which can be derived directly from the Maxwell equation tells us that the speed of light is c in this ether frame. The ether was thought like water to water wave and air to sound wave that carries the light. And the speed of light propagates in this mysterious ether is c. This suggests a method to detect the motion relative to the ether, assuming the Galileo Transformation was correct (pretending we know nothing about Einstein's theory at present, only our old friend Newtonian Mechanics).

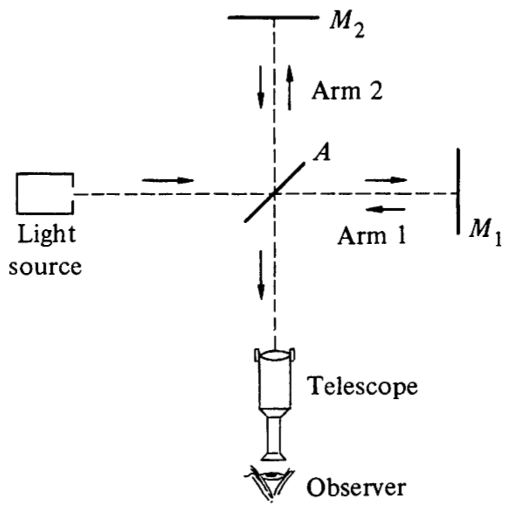


As the figure shows the model of light propagating in ether (watery lines), if the light source (fixed relative to mirror A or B, e.g. you can imagine a light bulb attached to the mirror A) is moving with respect to ether, then the speed of light along the direction of motion of the source (upstream against ether flow) will be c-v and the speed of light against the motion (downstream) will be c+v (velocity is –c-v) with respect to the lab (the AB) frame. So we could measure the speed v of motion of our lab frame in which the mirrors and light source are fixed by measuring the speed of light in the lab frame directly which implies we can detect the absolute motion of one frame relative to ether. But the accurate determination of speed of light at that time is not available. It requires precise measurement of distance and time intervals and precise synchronization of clocks at A and B. Fizeau used round trip method (a total time travel from A to B and back) to avoid synchronization, but the time interval of this round trip is:

$$T = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = 2\frac{l}{c}\left(\frac{1}{1 - v^2/c^2}\right) \approx 2\frac{l}{c}\left(1 + \frac{v^2}{c^2}\right)$$

The term that depends on v is a second order v/c, taken the possible v the motion relative to ether to be the speed of earth traveling in space known at that period: the orbiting speed of earth around the sun which is about 30km/s, this suggests $(v/c)^2$ is only about 10^{-8}, very difficult to detect for a long time. Please note the significance of such experiment that if we can accurate determines T then it seems that we could carry out one experiment in one inertial frame (the mirror one) and determine the traveling speed of our inertial frame without referring to any outside world. This would contradict the relativity principle. We now know the trouble is caused by our assumption that light travels at constant speed in the frame of ether, so this assumption picks out a preferred reference frame for light. It also caused by the Galileo transformation where the time is absolute in all moving frames, this leads to simple addition or subtraction of speed.

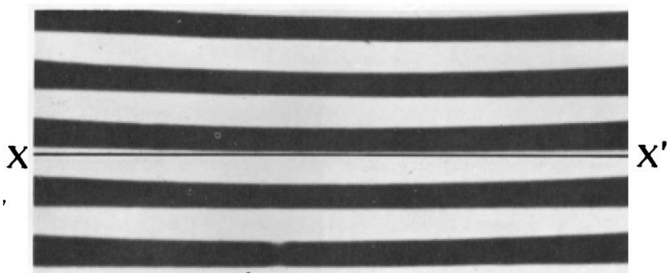
Michelson overcame this difficulty by measuring the interference pattern of light (1881) with the interferometer invented by him, and he was a pioneer of high precision measurement in modern physics. The sketch of the original setup is shown in the figure below:



The light source and mirrors are fixed in lab frame. The incoming light was split by the beam splitter A into two beams traveling a round trip along two arms. The reflected light from the two mirrors are recombined by the A and interference[113] between the two light beams will happen. Basically due to the different optical path length (or different time) the light travels along the two arms, there will be a phase difference between the waves when they meet again on the observing screen. If the phase difference is 2π (this happens when the optical path length difference is λ, the wavelength of light; or equivalently the time difference is T the

[113] This will be thoroughly discussed in the Optics course

period of light), the two waves from the two arms will add up together enhancing each other---constructive interference, and a bright pattern will be seen; if the phase difference is π, the two waves will cancel each other---destructive interference, and a dark pattern will be seen. We can arrange the mirrors to form different interference patterns (equal thickness or equal inclination patterns), the typical equal thickness pattern is shown below (a pattern when length 1 and 2 are fixed):

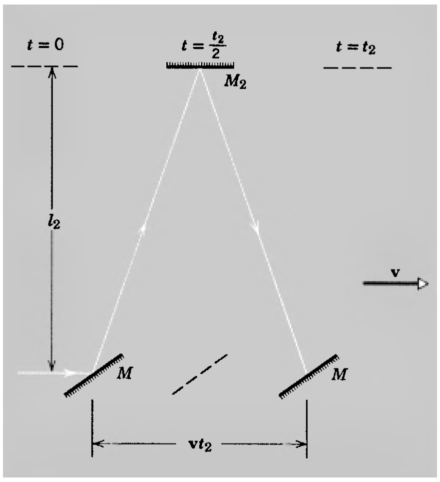


If the optical path length difference is changed, say arm one is shortened etc. (the one we shall see corresponds to the arm one will be shortened while the arm 2 will be lengthened), for every total optical path length difference change by λ, the interference pattern will shift by one fringe, i.e. if the round trip along arm one is shortened by λ, the bright stripe below (or above, depending on the arrangement) the cross wire XX' will move up to the position of the XX' (in the figure above, this means the lowest bright stripe will move up to the XX'). Such shift can be observed and since the shift is caused by a length difference on the order of wavelength of light (590 nm for sodium light; or a time difference on the order of period of light 10^{-14}s), it is essentially a high precision measurement in length or time *difference*.

Suppose the lab frame is traveling with respect to ether with velocity v along the direction of arm 1, then the time interval for the round trip along arm 1 (between AM_1) is just what I already calculated above:

$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = 2\frac{l_1}{c}(\frac{1}{1-v^2/c^2}) \approx 2\frac{l_1}{c}(1+\frac{v^2}{c^2})$$

For the light traveling along arm 2 (between AM_2) which is perpendicular to motion v, say the time for round trip is t_2, the figure below shows a view from ether point of view (the lab frame is moving with v)



During the flight of light towards M_2, the total distance from Pythagoras theorem is[114]:

$$l_2' = \sqrt{l_2^2 + (\frac{vt_2}{2})^2}$$

Since here we take the ether point of view, the light travels with speed of c, then:

[114] In the original work, Michelson just used l_2 by mistake. This causes some numerical error but won't change the results. .

$$t_2 = \frac{2l_2'}{c} = 2\sqrt{l_2^2 + (\frac{vt_2}{2})^2} / c$$

$$c^2 t_2^2 = 4l_2^2 + v^2 t_2^2 \rightarrow t_2^2 = 4l_2^2 / (c^2 - v^2)$$

$$t_2 = \frac{2l_2}{c}\frac{1}{\sqrt{1 - v^2 / c^2}} \approx \frac{2l_2}{c}(1 + \frac{1}{2}\frac{v^2}{c^2})$$

Of course you can calculate the above from lab point of view, in this case the distance will be $2l_2$, and the speed of light will be $\sqrt{c^2 - v^2}$ (just like a swimmer in a flowing river, and the speed relative to river is c), this will give the same transit time.

So in this arrangement, the total time difference between path 1 and 2 are:

$$\Delta t = t_1 - t_2 = 2\frac{l_1}{c}(1 + \frac{v^2}{c^2}) - \frac{2l_2}{c}(1 + \frac{1}{2}\frac{v^2}{c^2}) = \frac{2(l_1 - l_2)}{c} + \frac{v^2}{c^2}(\frac{2l_1}{c} - \frac{l_2}{c})$$

Such time difference will create an interference pattern, what Michelson did is observing the interference pattern with this arrangement, then he *rotated* the whole apparatus by 90 degree, so that now it is path 2 that is along the v. Similar calculation will show that in this case:

$$\Delta t' = t_1' - t_2' = \frac{2l_1}{c}(1 + \frac{1}{2}\frac{v^2}{c^2}) - 2\frac{l_2}{c}(1 + \frac{v^2}{c^2}) = \frac{2(l_1 - l_2)}{c} + \frac{v^2}{c^2}(\frac{l_1}{c} - \frac{2l_2}{c})$$

The difference between the rotated and the original setup is then:

$$\Delta T = \Delta t - \Delta t' = \frac{(l_1 + l_2)}{c}\frac{v^2}{c^2}$$

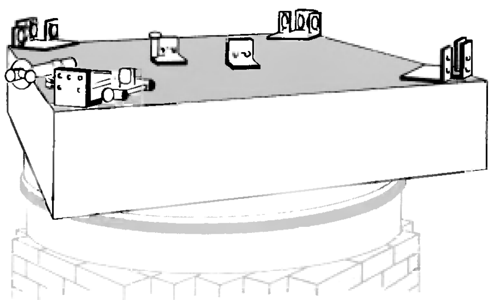
The total phase change between these two setups will be then:

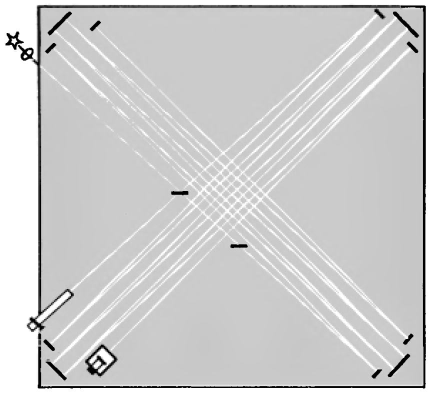
$$\Delta\phi = \omega\Delta T = \frac{2\pi}{T_{period}}\Delta T = \frac{2\pi c}{\lambda}\Delta T$$

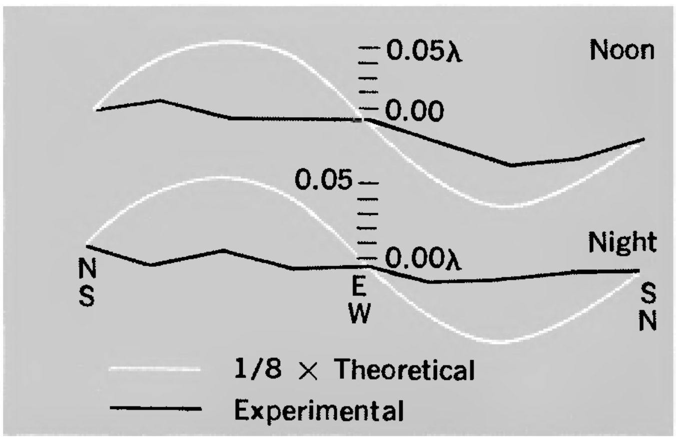
Since every 2π change in phase causes one fringe shift, then the total shift of fringe is:

$$N = \frac{\Delta\phi}{2\pi} = \frac{c}{\lambda}\Delta T = \frac{(l_1 + l_2)}{\lambda}\frac{v^2}{c^2}$$

In the original setup, the length is about 1.2 meters, and the motion in the ether is expected to be that of earth orbiting the Sun, i.e. v=30km/s, or $v^2/c^2 \sim 10^{-8}$. $\lambda \sim 600nm = 6\times10^{-7}m$ (it is actually 589nm), put all these into the formula, will give us $N \sim 0.04$. This is indeed very small shift, and the Michelson's resolution limit is about 0.01. So as the first experiment result came out, it was questionable to many contemporaries. In 1887, in collaboration with Morley, Michelson made an improvement by essentially extending the length of each arm by a one order of magnitude[115]:







[115] Taken from the A.A. Michelson and E.W. Morley, Am. J. Sci. **134** (volume) , 333 (page), (1887)

With length increases to 10m, the $N \sim 0.4$. However the result is a disappointment to Michelson and Morley. There is no observed fringe shift as expected. If you stick to the ether hypothesis, then this means that the v=0. Our lab (or earth) is stationary relative to ether. This is of course possible for the experiment in a particular day that earth happens to be stationary w.r.t. ether. However as the earth is orbiting around the Sun, it cannot always be stationary all year, but the experiment carried out at different times of the year still showed null result, no fringe shift was detected. This experiment really put the ether hypothesis (along with it the path 1 we talked about in 11.2-2) in a big question mark. Naturally there were further efforts to save the ether hypothesis and we shall take a brief look below to see why these efforts failed.

11.2-4 Efforts to Save the Ether Hypothesis[116]

These are the efforts to explain why there is a null result in Michelson-Morley（MM）experiment, while still holding the ether hypothesis.

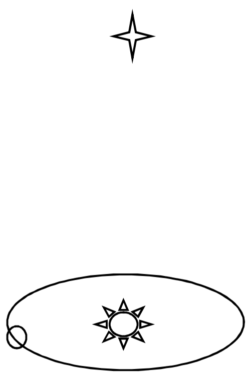
(1) Ether Drag Model

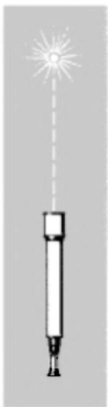
In order to explain the null result of MM experiment from the ether hypothesis, one of the most straightforward models is the ether drag. That is to say the earth will drag the ether along its motion in space,

[116] Optional material, I include it here for a complete background. You may skip the reading for the first time.

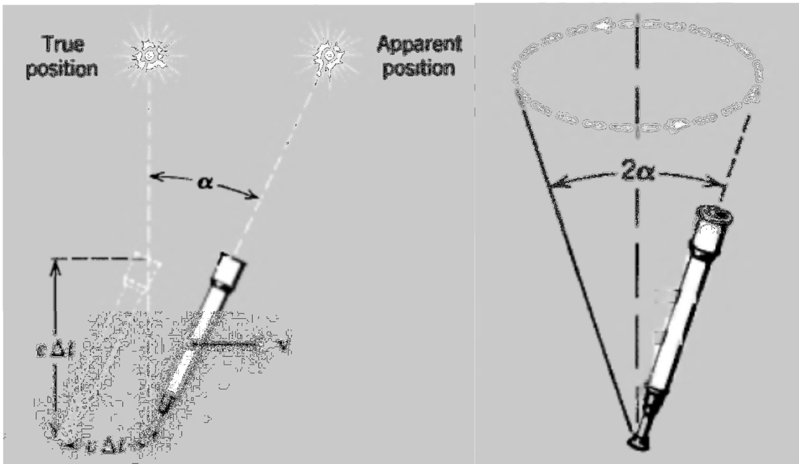
just as the air inside the train is dragged when the train is moving, even when the train is traveling with supersonic speed (as if such train exists), people can still hear each other since the air inside train is dragged along and is stationary relative to the train. In this model, the ether surrounding the earth will be always stationary relative to the earth; it is dragged by the earth and move along with it. The speed of light in MM experiment would be c for paths along both arms and the v (the motion of earth relative to ether) is always 0. Puzzle solved by the ether drag model in MM experiment.

The problem of this model is that it contradicted other experimental facts that were observed long before the MM experiment: the most important one is the stellar aberration in the observation for stars:





As the figure shows, imagining a star hanging over the earth-sun orbit, and the star is really far away, so that its position is almost always directly above the earth even though earth's position changes along the orbit. So if the earth is not moving, then you just steer your telescope straight up like the figure right shows and you will observe the star. Things will be different considering the motion of earth, and the figure below illustrate the situations (figure taken from Resnick's):



Due to the motion of the earth, the light emitting out from the star will appear titled by an angle α, and this α can be computed simply from addition of velocity vectors (viewed by earthling, the star appears moving with v in the reversed direction) :

$$\tan\alpha = \frac{v}{c}$$

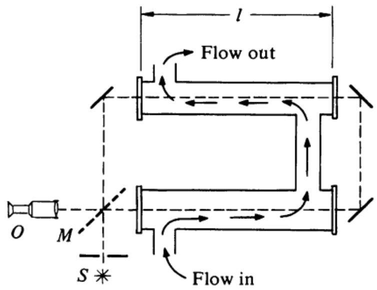
This is very like the rain drop model: suppose the rain is falling down straight towards ground, if you are running, you will feel (observe) that the rain is not only falling down but rushing against your face as well. Situation here in observing the light is similar (though the computation from wave theory, such as that of Huygens principle will be a little more complicated, but will give same result).

So to observe the star, the telescope needs to be tilted by this angle α.

If the earth is traveling in a straight line, then it will appear that the star's apparent position as indicated in the figure and you will not notice any abnormality. However since the earth is orbiting around the sun, so after half year, the star will appear as traveling in the reverse direction and the telescope need to be tilted by angle 2α comparing with half year ago. In fact the telescope will trace a cone whole year around as shown in the figure above right. This angle change for a far away star is not by the different position of the earth (the change of angle due to this is small enough to be neglected) along the orbit but by the different velocity. This angle change of apparent star position is called stellar aberration and was discovered in 1727 by Bradley and he used this to measure the speed of light, since the 2α can be measured and the speed of earth orbiting the sun can be estimated at that time.

The relevant question is that the ether drag model will contradict this stellar aberration observed. If the earth really drag the ether along with it, just imaging a heavy runner is running while drag the air around it also moving with same speed as he run, then the rain will appear to the runner as if just falling straight down. Similar argument would predict that if the earth is really dragging the ether, the light from the far away star will appear just straight downward and no tilt of the telescope is needed and so no aberration. This contradicts the observation.

There is another experiment to a less extent contradicts the ether drag model, the Fizeau experiment (1851) measuring the speed of light in a flowing water tubes:



It is essentially another interferometer, one path (the ccw one) will travel along the flowing water, while another one (the cw one) will travel against the flow of water. The light travels inside a stationary water is: c/n, where n is the index of refraction of the water (~1.33). Now with the water is flowing with speed of v as shown in the figure, and suppose that this flow of water will drag the either along with it with same speed v. In the water frame, the speed of light is still c/n; but in the lab frame this speed will be: c/n+v for the ccw path and c/n-v for the cw path. This will cause a phase difference between the two paths (comparing to the stationary water case) and a fringe shift will be seen when the water is flowing from zero speed to v. A little computation (left as an exercise for you) shows that the fringe shift would be:

$$N = \frac{4n^2 l}{\lambda c} v$$

This was NOT observed in the real experiment. The actual result of Fizeau experiment is:

$$N = \frac{4n^2 l}{\lambda c} f v \quad f = (1 - \frac{1}{n^2})$$

We shall see the satisfactory explanation for this result from velocity addition in relativity. Zeeman repeated this kind of experiments in 1914-1922 as confirmation for the special relativity.

Historically the $f = (1 - \frac{1}{n^2})$ is called Fresnel drag coefficient in the ether theory developed by Fresnel in 1818, in which he assumed that the moving medium (such as water here) only drags ether *inside* the medium by a fraction which is the relation given above[117].

(2) Bullet-Gun Model

This is officially called emission model, but I think the bullet-gun is more vivid. This was a model adopted by Ritz trying to modify the Maxwell's theory and ether hypothesis by requiring that the light is not traveling with speed c w.r.t. ether, but to the source instead. Just like firing a gun in a moving car, the relative speed of the bullet to the gun is always c, but to the observers on the ground this speed will change. This model would explain the MM's null result easily, since the light source is fixed in the lab frame as the mirrors did, the speed would be c to both paths and no fringe shift could be detected. This also saved the E-M theory to the relativity principle, i.e. you cannot detect the motion of frame by carrying out experiment inside the frame, such as by measuring the speed of light inside the frame.

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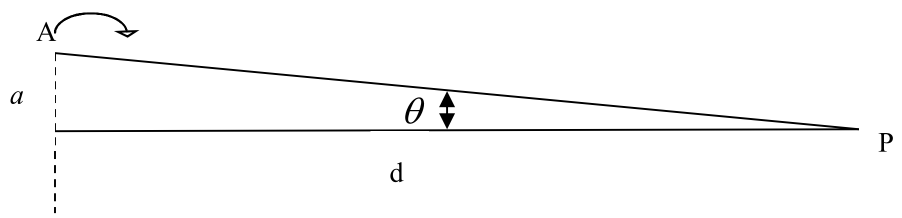
[117] For a brief discussion of the Fresnel drag model, please refer to C. Moller's 'theory of relativity' section 7, or 刘佑昌 '狭义相对论及其佯谬' 附录 1.

But this assumption leads to other contradictories with observations. One of the observations would be the recording light coming from a double star system. The double star system is two stars rotating around a common center, as shown in figure below:



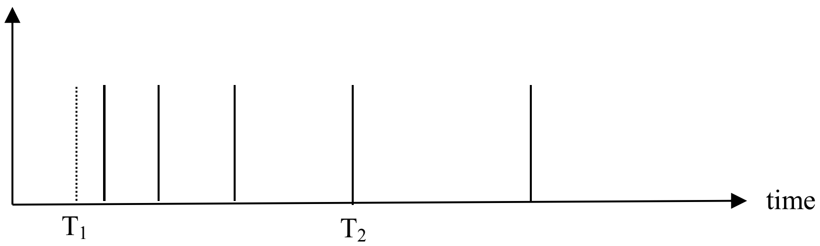
A is one of the double star rotating clockwise, *a* is radius of its orbit and d is the distance to the observer P on earth, d>>*a*, so the θ is very small. Suppose linear velocity of the star A is v, at the location 1 as shown (A is perpendicular to d), the light from A would travel at velocity ~c+v towards earth according to bullet-gun model, and the distance is approximately d (small angle makes the approximation good). The time for light to reach the observer P would be:

$T_{1A} = \frac{d}{c+v}$ Now suppose some time later, the A will reach position 2, which is on the line of d. Here the velocity of light towards P would be c only, and distance is d-a, so the time interval for light to reach earth would be:

$T_{2A} = \frac{d-a}{c}$ Take their difference:

$$T_{1A} - T_{2A} = \frac{d}{c+v} - \frac{d-a}{c} = \frac{d}{c}(\frac{1}{1+v/c}) - \frac{d-a}{c} \approx \frac{d}{c}(1-\frac{v}{c}) - \frac{d-a}{c} = \frac{a}{c} - \frac{d}{c}\frac{v}{c}$$

Here I used approximation of $1/(1+x)\sim 1-x$ for small x.[118] Though v is usually much smaller than c, but d can be very large in astronomical case. So it is not unusual that $\frac{d}{c}\frac{v}{c}$ can be larger than $\frac{a}{c}$, and the light from position 1 of A will reach P with shorter time, this will cause the light intensity recorded by P oscillating with time. As the figure below show:



The vertical axis is recorded light intensity, the horizontal is time. $T_1=T_{1A}$, and $T_2=T_{2A}+\Gamma/4$, where Γ is the orbital period of star A. Between T_1 and T_2, also shows the light from positions of A moving from 1 to 2. The characteristic is the time interval would increase due to the longer travel time of light. Of course the star emits light continuously with almost even intensity (that is why each line will be about same height), and if you record the light continuously, you will add light intensities within certain time interval determined by instrument, the graph will be total light intensity within certain time interval, and that intensity will have high and lows periodically similar to a sine curve. But that is not what was observed. The recorded light is

118 Taylor expansion.

almost a flat line. So this contradicts the model of the bullet-gun for light propagation. The puzzle of all these will be solved in special relativity, which you probably know the answer, that the light always travel at speed c, in all inertial reference frames.

There are more experimental facts contradict this model (though at much later years when the special relativity had long been established): A supernova explosion was observed in 1980's, which is 10^4 light years away from us. The light from the major explosion was collected from this supernova explosion on earth which only lasted for 10's of seconds (the afterglow lasted much longer). If you follow the bullet-gun model, the light from the explosion may come from all pieces that are flying at high speed in all directions. For the sake of estimation, let me assume that the flying speed of the pieces is the same order as earth orbiting the sun, then $v/c \sim 10^{-4}$. This implies that the light from the explosion would last for 1 year, quite different from what was really observed!

Another experimental fact directly demonstrates that for the light emitted by a particle moving with high velocity, the speed of light viewed from the lab frame is still c, has nothing to do with the speed of the particle[119]. The experiment demonstrated that for a pion (a neutral meson, an unstable particle which quickly decays and emitting

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[119] Alväger et al. *Phys. Letters.* **12**, 260 (1964)

light), which was created by bombarding nucleons with high energy (20GeV) protons. The pion created will travel almost to the speed of light (0.99975c), but γ rays (light with high energy or short wavelength) produced by the pions were detected in the lab frame traveling with same speed of light c.

(3) Lorentz Theory[120]

Since we are going to derive the famous transformation from fundamental postulate of special relativity, so that I will not talk too much on Lorentz theory here, interesting student may refer to Bohm's book (chap.6-10) for some detailed account.

Basically in order to reconcile the classical E-M theory with the MM-experiment result. Lorentz propose an ad hoc method (ad hoc here means making some hypothesis in order to explain a particular phenomena)[121]. What Lorentz did was proposing that for things moving w.r.t. ether, its length will be contracted. For example, the length of path 1 in the MM experiment is l_1, if it is rest in ether.

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[120] It is also called Lorentz-Fitzgerald theory, because Fitzgerald was first to propose length contraction in an ad hoc way and Lorentz put it in a theory. In the 1904 review paper, Lorentz showed the transform that will make the Maxwell equations hold its form in all inertial frames. He did not call the transform after his name. It is Poincare gave the name in 1905 before Einstein's publish. It is interesting to learn that Einstein in developing the special relativity was ignorant of Lorentz theory.

[121] This ad hoc method is very common in the development of science. Of course the hypothesis may explain certain experimental fact but may fail in general cases. Examples here that Lorentz theory explain the MM experiment by using length contraction; but it will not explain the Kennedy-Thorndike experiment (1932) with Michelson interferometer with different arms length and fixed in lab only rotating with earth over the year. Another example of ad hoc approach is the Bohr-Somerfield theory on atomic structure in the development quantum mechanics. It explains the structure of hydrogen-like atoms or ions nicely but failed in more general cases.

However if it is moving with speed v w.r.t. ether, the length will be contracted to $l_1 / \sqrt{1 - v^2 / c^2}$, smaller by a factor of $1 / \sqrt{1 - v^2 / c^2}$. If you replace this contracted length in the computation of MM result in the previous section, you will find that there will be no fringe shift.

Of course Lorentz gave a model on the physical reason why you have this contraction based on the atomic-electron structure available then. In fact he developed a theory that also has time dilation and transform between inertial frames for the E-M equations. One result of this theory is that it pointed out the futility in detecting the motion relative to ether, because the theory implies that for all observers in inertial frames (moving with constant velocity), the detection of light speed due to length contraction and time dilation will always give out the value c. Poincare claimed that this mysterious ether always escaped our detection.

Lorentz theory though a big step forward, is still rooted in the ether hypothesis and in fact a modification of E-M theory with ad hoc hypothesis. Since this ether and motion relative to it was never detected, then should physicists still hold its existence? Shouldn't the science only consider what had been and can be observed or tested instead of mysterious god or absoluteness (which may be the topics for philosophers, especially those ancient ones in our history)? We will see in the next section that it was Einstein took this attitude and with two simple postulates unifying the mechanics and E-M theory in a sense that they both follow the relativity principle.

11.3 Einstein's Postulates

Let me quote his own words on the issue. "The relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties using a few very convincing assumptions."[122] It is these assumptions that we shall learn in this section. We have seen that "the theory of relativity has grown out of the electrodynamics and optics. In these fields it has not appreciably altered the predictions of theory, but it has considerably simplified the theoretical structure…" "Classical mechanics required to be modified before it could come into line with demands of the special theory of relativity. For the main part, however, this modification affects the laws for rapid motions, in which the velocities of matter v are not very small as compared with the velocity of light."[123] We shall focus on this in the rest of the course.

Here is a reminder on the postulates once more: the postulates cannot be derived from other fundamental laws, they are the fundamentals. They

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[122] Einstein and Infeld *The Evolution of Physics* (1938)

[123] Both the quotes from Einstein *Relativity* , section 15.

should be tested on the basis of the conclusions drawn from them.

Postulate 1 (Relativity Principle): All physical laws are same for all inertial observers.

Comments: This is an extension of the Relativity Principle we discussed above (11.2-1), there it only applies to mechanics. Now Einstein believes that it applies to all physical laws including the E-M theory. It turns out this is true for other physics. It is a test for the correctness of physical laws. For example during the early development of quantum mechanics, one of the fundamental equations is the Schrödinger's equation describing the time changes of wave functions. The equation does not have the same form in different inertial frames, suggesting it only applies to low speed limit. The one satisfying the relativity principle was given by Dirac, which is called Dirac equation. The relativity principle holds in quantum too and it is believed serving as a heuristic test for the correctness of any theory, i.e. the physical laws has to be invariant (its components have to be covariant) under transformation from one inertial frame to another.

In special relativity (SR), we shall only deal with inertial frames (coordinate system), this implies this cannot be the whole story. To include the accelerated frames (and equivalently the gravitation), the general relativity was developed but won't be covered here (an introductory textbook on GR is given in the reference list).

A corollary similar to the one we discussed above of this principle is that *the absolute motions of one inertial frame cannot be detected by the physical experiments carried out within the frame.* Before I only say the mechanical experiment, like juggling balls etc. Now it includes all physical experiments and also chemical and biological experiments as well, since these natural sciences are all governed by the physical laws.

This Relativity Principle sounds highly plausible. It is the belief of Einstein's that there should be unity in physics and a natural extension of the relativity principle of mechanics. It would be indeed strange otherwise if all inertial observers are experiencing same mechanical laws while can tell difference by measuring light.

Postulate 2 (Universality of c): There is a speed that is same in all inertial frames; it is the speed of light.

Comments: It is also stated more succinctly as: the speed of light is same in all inertial frames. Unlike the relativity principle, this speed of light postulate is quite bold, it contradicted in every old classical sense. The root is the measurement of time. As I talked in 11.1, you need this universal speed to determine the time (such as synchronization of clocks) since the detection and determining the speed of absolute motion of inertial frames is impossible.

The reason I split this postulate into two sentences is because from the logic of relativity principle, it requires *one universal velocity*. We shall see later that this universal velocity also sets up a limit to the speed of motion of all inertial frames (Lorentz Transform) and speed of all particles or signals (Cause-Effect argument). The theory only allows ONE universal velocity (KK's 13.4, easy to prove once we learned velocity transformation). Whether this universal velocity is the speed of light or other particles would be a test of experiments. In this sense Einstein acted like god and proclaims as in Genesis "Let there be light". It is possible at least in principle that this universal velocity could be different than the speed of light (has to be larger since it sets the limit of speed). However the speed of light had been subjected to many rigorous tests and found be independent of the frames (such as the one in footnote 119) though always with certain experimental errors. So if any claim that a new speed limit is discovered, it is possible in principle for SR, but it has to be able to explain other experimental facts on the speed of light. For example with improved technique, it may be that the true speed limit may be c + a very small number for some massless particle, the number maybe so small that escaped us by current technique. Of course any such claim has to be sure free of errors itself[124]. In conclusion, it seems that the universal speed of light still holds (agrees with experiments) at present from all the experimental facts.

[124] The reported superluminal speed of neutrino in 2011 turned out to be faulty., it probably caused by the experimental error that a lose connection in the optical fiber generating a 60ns time difference, corresponding to wrong synchronization of clocks in measuring the speed.

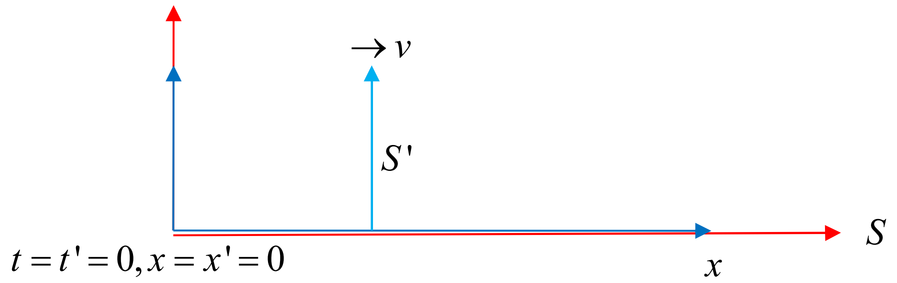
Simple these two postulates may appear on the surface, the conclusions we draw from them would be profound and on the first sight startling. The universality of speed of light is contradicting the Galileo transformation in classical mechanics where the addition of velocity is a conclusion. This implies according to SR, something is wrong in the measuring of velocity, which basically a measurement of distance (space) and time. This is what I shall devote in the next section before deriving the holy grail of SR---Lorentz Transformation.

11.4 Relativity in Measurement of Time and Space Intervals and Simultaneity: Time Dilation, Length Contraction and Simultaneity is Relative.

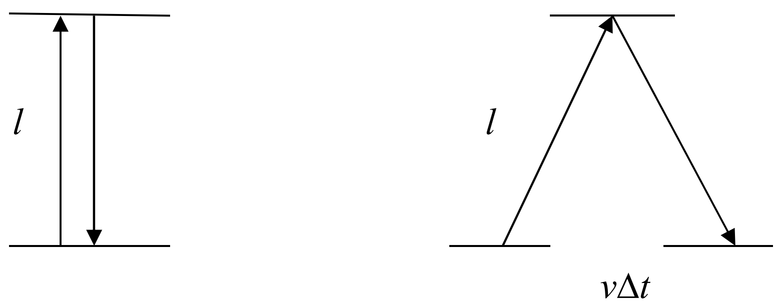
This is a long title and indeed tells you what I am going to discuss in this section. We shall see how we measure the time interval and distance just using the postulates and we have to prove those effects claimed in the title.

11.4-1 Time Dilation

In all the following discussions, I shall only consider inertial frames moving with each other. Say two observers, one on ground, another on a moving train. The one on ground is called S frame and the train S'. I shall choose both the x and x' axes in both frames collinear with the velocity v, the speed of the train, generally the relative velocity between the inertial frames.



At the event that the origin of S and S' overlaps, the time in both S and S' are set to zero, as indicated in the figure, since we stated before that both inertial observers can agree on the simultaneity of events at same place. Imagine that both observers click on the individual stopwatch when the origin crosses and set time to zero. The time will go on but the flow rate of time of the individual watch would depend on the observer. How we define unit of time? Well in relativity we use light clock as below (left):



Let's suppose a light clock in S' (on the train) consists of a pair of mirrors, a light is sending upward and being reflected back. The time interval of this two events (light emitted and light detected) will be used as time unit in S' (Of course the exact copy of this light clock stationed at ground frame will be used as time unit on the ground). The two event when viewed from S' would be (t_1', x_0') for the emission of light, and (t_2', x_0') for the detection of light. Noticed these two events happen in S' at *same location* but with a time interval:

$$\Delta t' = t_2' - t_1' = \frac{2l}{c} \qquad (11\text{-}4)$$

To the ground observer recording same events, the time interval will be different. Because as the figure on the right shows, the light has to travel a longer distance viewed by S (Noticed I assume here that the vertical length l, which is perpendicular to the motion is same for both S and S', the assumption shall be justified later)

The total distance travelled is:

$$d = 2\sqrt{l^2 + (\frac{v\Delta t}{2})^2}$$

$$\Delta t = \frac{d}{c} = 2\sqrt{l^2 + (\frac{v\Delta t}{2})^2} \,/\, c$$

$$c^2(\Delta t)^2 = 4l^2 + v^2(\Delta t)^2$$

$$\Delta t = \frac{2l}{c}\frac{1}{\sqrt{1 - v^2 / c^2}}$$

I only keep the positive root because the order of event has to be same here: the emitting of signal always before the receiving no matter which frame you view them. $\frac{1}{\sqrt{1 - v^2 / c^2}}$ appears a lot in SR, so it deserves a label by itself. We henceforth define:

$\beta \equiv v / c$ (11-5)[125]

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2 / c^2}} = \frac{1}{\sqrt{1 - \beta^2}} \qquad (11\text{-}6)$$

Then the time interval measured by observer on the ground S would be:

$$\Delta t = \gamma \Delta t' \qquad (11\text{-}7)$$

(11-7) is what is called time dilation, and it usually expressed as 'moving clock runs slowly'. We shall explain this further:

A) Noticed that as v<<c, then $\gamma \approx 1$ and $\Delta t = \Delta t'$, we get back the result in Galileo transform. The result from SR postulates is consistent with classical mechanics at low speed limit. (the requirement that the results will be reduce to that of classical mechanics at low speed is correspondence principle)

B) What is (11-7) really tells us? From the argument leading to (11-7), it clearly tells us that for the same events, the time interval observed by S' and S are different but related. What appeared to the S' observer to which the time interval is 1 second, the time interval between the same events is lengthened (dilated) by a factor of γ for observer in S. Since the measurement of time by the light clock is quite general, all the time intervals can be measured by how many loops the light travels in such clock, this means all processes in the moving frame (on the train)

[125] In the advanced treatment, the c will be set as 1. This equals to choose light-second as unit length and β will be the velocity in unit of speed of light.

would appear[126] running slower to the ground observer. Let's be more specific by taking examples, say the train is moving at $v = 0.8c$, then $\gamma = \frac{5}{3}$. If you are taking the final exams on board such train and the time requirement is 1 hour. You certainly want me to be on board too. Since we are in the same frame, the 1 hour interval on my watch will be same as yours. If I were on the ground, the 1 hour time you spent on the exam would appear to me as 5/3 hours, I thus conclude you guys work the problems too slow. Of course I do not really see you in the mundane sense that you are working slowly. It is just at 5/3 hour according to my time, where you are already 4/3 light-hour away from me, but my observer at that location will really sees (in the mundane sense) that your clock only passed 1 hour while his watch is already 5/3 hour..

C) Though I only used light clock in the example, it applies to all clocks. The Relativity Principle guarantees this. For instance, S and S' both use grandfather pendulum clock. Then the time dilation still applies to

[126] Here I use words' appear, observed or see', these words need to be understood as measurement by a network of observers recording events in one frame, as I mentioned earlier in this chapter. It is not same as the common meaning of these words, which means really sees with own eyes by one single observer. I shall explicitly point out if the 'see, observe…' in the notes means the mundane meaning, such as " I see (in the mundane sense)" then you know I am talking about seeing with own eyes. Otherwise, they mean measuring the space-time of events by network of observers (just too long and boring if I type these every time). A mixing of these sometimes cause confusion. For example If I really sees(in mundane sense) what you are doing, it would appear your motion runs faster like in a fast winding video tape if the train is approaching me; and you motion will be slower (seen by me in mundane sense on the ground) if the train is leaving from me. This is the Doppler effect we shall talk about later (there is probably the only occasion that I shall use 'see' in its mundane sense in this course). Time dilation is always there no matter whether the train is approaching or leaving.

the time interval of the pendulum clock (say one cycle of swing of the pendulum). Because the swing cycle of pendulum can be calibrated with the light clock. Say one swing cycle equals 1000 light clock round trips. Then this 1000 would apply both to the pendulum clocks in S and S'; otherwise you will notice the difference in the ticking rate between the pendulum and light clocks on ground or on train, that would give you a method to determine which frame is really moving by experiments within the frame, which is impossible by the relativity principle. The time dilation thus affects all processes including the biological ones. For example you guys are boarding on the train with v=0.8c. You grow 1 years old according to your clock (which is 1 year in your biological clock too); but the ground observer (me) will see this taking 5/3 years during which I aged 5/3 years. So it certainly possible that 1 day in fast spaceship, thousands years past on earth. (However, noticed that these 1 day and thousands years are measured in different clocks, 1 day by the clock on the ship and thousands of years by clock on earth).

D) The above discussion though different from old classical, still quite straightforward. Things get interesting (or puzzling) if we take point of view in S', i.e. suppose we have a light clock stationary on the ground, so that the round trip events are: (x_0,t_1), (x_0,t_2) measured by S, happened at same place but with time interval $\Delta t = t_2 - t_1$.Now what do these events appear for the observer in S'? Well all inertial frames are equivalent from relativity principle, the S' would see the S frame also moves with speed v relative to the S', but in the reversed direction (so the velocity S moves relative to S' is –v). From S' point of view, it is basically the same figure above, while the left stationary one corresponds to the events observed in S, while the right one corresponds to the same events viewed by S', except the direction of velocity is reversed (or just change the direction of arrow point). This means the time interval in the S' frame would be:

$$\Delta t' = t_2' - t_1' = \gamma \Delta t \qquad (11\text{-}8)$$

So for the observer in S', he will conclude that the clocks in S runs slower by the same factor γ. In the exam example above, if you are on board a train and watch me solving problems on ground, you will draw exactly same conclusion as above, that I am working slower (I spent one hour according to my watch, but you would see it is 5/3 hours according to your watch). This is the essence of relativity principle that all inertial frames are equivalent, the observation of symmetrical events in all frames have to be same. If I (on the ground S) accuse that your clock (on the train S') runs slow; you have equal right to claim that according to your observation, my clock runs slow. How could this be? Well this is the results of viewing the symmetrical events from different point of view. An analogy is two people are separating apart in distance, each will see (in mundane sense) the other guy becoming smaller.

Wait a minute, you claim, if you put (11-8) $\Delta t'$ into (11-7), you will get: $\Delta t = \gamma^2 \Delta t \to \gamma = 1 \to v = 0$, but the train is moving with nonzero velocity (0.8c in the example). It is absurd and how I am going to explain this? The problem of doing above is these two equations are for two different sets of events! The (11-7) is for events that are stationary in the S': (t_1', x_0') and (t_2', x_0'), while the (11-8) is for another set of events that are stationary in the S frame: (t_1,x_0), (t_2,x_0). They are symmetrical (I am observing a clock stationary to you and you are observing a clock stationary to me, you and me are in different inertial frames) but not same events! You cannot just plug the results for unrelated events into each other, otherwise it is similar to that my son Bart is 6 years old, and my pet dog is also 6 years old, but you cannot draw any conclusion between the boy and the dog, can you ☺? Only if we are describing the same or related events, we can make such substitution etc. This is a common mistake that causes confusion in the first place, so I think it is worth pointing it out explicitly even with risk of the sacrifice of my son's reputation. Maybe I should be less sloppy and write the (11-7) and (11-8) more clearly as:

$\Delta t\,|_{\Delta x} = \gamma \Delta t'\,|_{\Delta x'=0}$ for (11-7)

$\Delta t'\,|_{\Delta x'} = \gamma \Delta t\,|_{\Delta x=0}$ for (11-8)

And the difference may become clearer in the above expression. The time interval for the events that happened at the ***same place*** has a special name: **proper time**. In the time dilation measurement, in the situation leading to (11-7), $\Delta t'|_{\Delta x'=0}$ is the proper time, it is the reading of *one clock at same location* in S'; while $\Delta t|_{\Delta x}$ is not the proper time because it is essentially the difference between readings of two different clocks in S frame (one at x_1, another at $x_2 = x_1 + v\Delta t$, the watches are synchronized but different Rolex we give to observers at these two locations). For the situation leading to (11-8), you figure out the proper time.

All the relations I am writing are for time intervals between events, if the interval is between one event and the event we used to define the overlapping origins of the coordinate systems, then $(x_1 = 0, t_1 = 0)$ and $(x_1' = 0, t_1' = 0)$ for event 1, then we could drop Δ in the equations. (I preferred Δ, it reminds me the relations are about intervals of events). We usually use τ symbolizing the proper time defined above, then the (11-7), (11-8) can be written in one simple form:

$t = \gamma\tau$ (11-9)

There will be further important roles played by proper time when we come to the 4-vector part in SR.

However, there still seems one puzzling question remaining. From above derivation and discussion, we have seen that in the relative moving frames, A observes B's clock running slow (1 second time interval on B appears as 1.xxx sec. to A); and B observes A's clock running slow; then how to reconcile this one puzzling question: If B observes A's clock running slow, then how come when A using his slow clock measuring B's time and concludes that B's clock running slow? This tricky question will be answered later when we study the simultaneity is relative, and the key point is **simultaneity of events happened at different places** cannot be agreed by observers in different frame, and simultaneity is the key for synchronization between clocks (synchronization means setting different clocks at different locations to a fixed value simultaneously), and synchronization of clocks is crucial in measurement of time. I point this out first here and will talk details in 11.4-3. But first let me show you some examples on time dilation and then another effect called length contraction.

Example 1. Twins 'paradox'

Suppose a pair of twins, let's call them Adam and Bob and A, B for short. A is a clerk that stays on earth, while B is an astronaut boarding a spaceship travels with v=0.8c relative to earth. The spaceship is launched from the earth towards a star F that is 8-light years away from earth (this is distance measured in the earth frame). B took the round trip from earth to F and back. How many years will pass according to B's clock (the watch on B's wrist)?

This is easy, the time the clock ticks on B's wrist is the proper time in B's frame (for B he is not moving and always stays at the same location in his frame, so the time measured by him in his frame is the proper time). In the A's frame the trip takes 10 years for the single trip, with $\gamma = 5/3$, the trip that last 10 years in A's frame will be:

$$\Delta t_A = \gamma \Delta t_B, \ \ \Delta t_A = 10y \to \Delta t_B = 6y$$

So for the single trip that took 10 years according to A only takes 6 years according to B. The round trip would only double the number, so 20 years passed according to A when B returns; while only 12 years passed according to B. This means there will be an age difference between the twins. The one stays on earth (A) is older by 8 years than the astronaut. Not surprising if you recalled in the comment C), that B's biological clock appears running slow to A. This is clear and dandy in A's point of view.

The 'paradox' lies what happens in B's point of view. From the comment D) above according to the relativity principle that should not B also entitle to claim that A's clock running slower than his, so that it is A is younger? It cannot be true that both of them are younger, what is the catch? The simplest explanation (only qualitatively here, and we shall talk about B's point of view in detail in later sections)is that B is not an inertial observer, during the take-off and turn-around (we will see it is mostly due to turn-around), B is in an accelerated frame, he is no longer inertial there. Yes during the constant flight interval when both A and B are inertial, A's clock indeed appears running slower for B. But during the acceleration-deceleration process at the turn-around point at star F, B is not inertial. Cannot it be that from B's point of view, it is not he is accelerating/decelerating but is A who is accelerating/decelerating, since A appears moving faster/slower away from him? No, B cannot make this claim because acceleration is not symmetrical as constant motion. During the process of acceleration, B feels the forces exerted on him, he is pushed and pulled by the inertial force, the blood pressure rise or fall, temporary blindness and even the life can be endangered by the inertial force. If it is A that stays on the ground drinking coffee snugly is accelerating, how on earth that B's life is endangered by the inertial force? You see that indeed during the acceleration B cannot invoke relativity principle to claim that it is A accelerating. There is no real paradox here, it appears so because you use the SR that works for inertial observers to a non-inertial B[127]. So the answer is that B, the astronaut will return to earth with 8 years younger than his twin brother.

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[127] A curious one may ask that during the B's acceleration/deceleration, is A allowed to use time dilation formula? Indeed A can, because for A though B is accelerating./decelerating, A can divide it in many tiny time intervals where within each interval B is moving with certain velocity, this is called instantaneous inertial frames from A's point of view. Of course this will complicate the computation since the v is changing .In the above estimation, let's assume that the acceleration/deceleration is completed in a very short time interval (it may kill B but that's ok in our example), so this will not affect time estimation in A's frame and 10 years is a good approximation. We cannot apply the same to the non-inertial observer B and to explain B's point of view, some knowledge of general relativity will be needed, because that is the correct theory works for B during the acceleration/deceleration. Actually the B's point of view can be worked out from special relativity only and we can see when the strange things happened in B's frame, the general relativity on the other hand tells us why this strange thing happened.

Example 2. Decay of muon measured in the lab.

This is a demonstration of time dilation effect. The muons (another kind of elementary particle which is charged as electron but about 200 times massive) can be produced by cosmic ray (high energy particles from universe) hitting the atmosphere surrounding the earth. The muon has very short life time which is about 2 micro-second ($2\mu s = 2\times 10^{-6} s$). This is the proper time measured (or inferred) in the muon frame (i.e. the frame in which muon is not moving, you have a watch, running as fast as muon so to stay right beside it to measure its lifetime in this frame). So I should put $\tau = 2\mu s$. The lifetime is defined as for every time interval of τ, the number of muons will drop by a factor of e^{-1}. Or: $A(t) = A_0 e^{-\frac{t}{\tau}}$. (in case you are interested that what this muon decays into, it decays into electron and neutrinos) The muons produced by cosmic rays will travel at high speed close to c. Even at this high speed, if there is no time dilation (pretending that you only know classical mechanics) hardly any muon will reach the ground. The earth outer atmosphere is about 20km above, it takes muon about $t \sim 10^{-4} s$, and the fraction of numbers reached to the ground would be negligible according to the classical mechanics: e^{-50}. However muons produced by the cosmic rays had been detected on earth by Rossi and Hall in 1941, and a repeat demonstration of this experiment by Fritch and Smith is what I am showing here[128].

The experiment is detecting a group of muons with selected speed whose $\gamma \sim 9$, meaning v~0.994c. This was achieved by letting the muons passes through certain thickness stopper (to set the low limit of velocity), and stops in a thin layer of scintillator (the electrons released in the decay will make this layer of material luminous) thus sets the high limit of velocity . So a group of muons with certain speed can be detected. The experiment was first carried out in the mountain top which is about 2000 meters above sea-level. The record of selected group of muons is 563/hour (on average). Then the experiment was repeated at sea level, and the record was about 400 counts/hour.

From the classical point of view:

$$A_{sea-level} = A_0 e^{-t/\tau} = A_0 e^{-vt/v\tau} = A_0 e^{-2000m/600m} \approx 20$$

This will give us only 20-counts/hour which is far from the experimental measurement. The reason is time dilation. In the muon frame, its proper lifetime is $2\mu s = 2\times10^{-6}s$, but viewed from the lab clock, the decaying process would last longer, by a factor of $\gamma \sim 9$. This means $\Delta t = \gamma\tau = 18\times10^{-6}$ s, and this number should replace τ in the decay formula, since both times are measured by the lab clocks :

$$A_{sea-level} = A_0 e^{-t/\Delta t} = A_0 e^{-vt/v\Delta t} = A_0 e^{-2000m/5400m} \approx 390$$

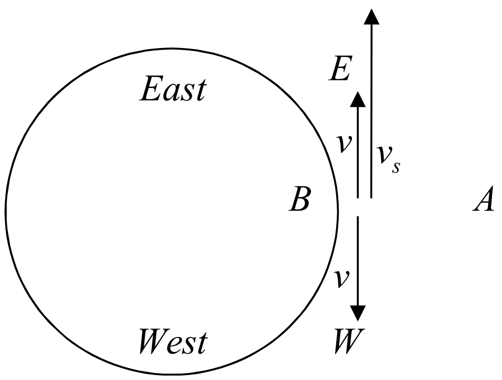
This agrees well with the experimental results. So this time dilation

[128] Fritch and Smith *Am. J. Phys*. **31**. 342 (1963). Also in French's Special Relativity, chap.4

indeed happens for the high velocity moving particles as this kind of experiments demonstrated.

Example3. Hafele-Keating Experiment[129].

In 1971, Hafele and Keating took an experiment that each of them aboard a flight around the world, one from west-to-east (along the earth spin), the other from east-to-west (against the earth spin). Each of the passengers took an atomic clock that can record the time with ns (10^{-9}s) precision. After the round trip each of them compare the time elapsed according to his own clock with the one stays on the ground, the result agrees with the computation using relativity.



As the figure shows, we have to choose an inertial observer, let this be A, an imaginary observer fixed to sun. This one is introduced because the earth frame, the east-bound and west-bound frame are not inertial. With this setup, there are 3 frames moving relative to the inertial observer A. These 3 frames are accelerating but we can treat them as instantaneous

[129] Hafele and Keating, *Science* **177**, 166-168; 168-170. (1972)

inertial frames:

1) The B frame that spins along the earth, with velocity v_s relative to A
2) The East-bound frame E, it is an airplane flying with speed v relative to earth. Its speed relative to A is $v+v_s$ (strictly speaking, I should use relativity velocity addition which I have not talked yet, but in this case the speed is low enough to use the classical result)
3) The West-bound frame whose speed is v_s-v relative to A

To simplify the computation, let's consider a special case that E, W took off and landed at B at same time after a trip around the earth. The time the trip takes in A's frame is time t (setting all t=0 at the take off , so I shall skip writing Δ), in B's frame is τ_B, and τ_E,τ_W respectively for E and W. Noticed that I am using τ_B,τ_E,τ_W, because these are proper time (time interval between events at same place) measured in their frames. Notes that the t in A's frame is not a proper time, though it may appears to you that the planes came back to the same place, why cannot t measured by the A's watch be proper? Well the simplest way to calculate t (to use the simple formula of time dilation)is to imagine a network of observers fixed in space of A's frame, say one on top Indonesia, one on top of Maldives, one on top of Kenya….at t=0. As the flight passes through these observers, they record the time interval. At each interval, it is like the situation shown in the figure above, and flights and earth's spin are linear motion with constant speed for these observers, and A's observer can use time dilation formula. You add all the time intervals recorded by these observers to get t, so t is not a proper time[130]. but this t is related to the proper time in the other frames by:

$$t = \frac{\tau_B}{\sqrt{1 - v_s^2 / c^2}} = \frac{\tau_E}{\sqrt{1 - (v_s + v)^2 / c^2}} = \frac{\tau_W}{\sqrt{1 - (v_s - v)^2 / c^2}}$$

Our job is to compare the time in B,E, W (these are measured data in the experiments by the atomic clocks in the plane and on the ground):

τ_B is the easiest to calculate. It is the time according to ground controller in the airport for the flight around the world. The speed of airplane is: $v = 800 km / hr = 220 m / s$, the flight is around world close to equator, so the total distance is 40000km.

$$\tau_B = \frac{40000}{800} = 50 hrs = 1.8 \times 10^5 s$$

Using this we can compute τ_E, τ_W, with $v_s = \dfrac{40000 km}{24 hr} = 460 m / s$, you can directly plug in the numbers to get τ_E, τ_W and thus gives the difference between the readings $\tau_E - \tau_B$ and $\tau_W - \tau_B$ (try it yourself and you will see the disadvantage of this direct plug in). I shall instead proceed with further simplifications in order to ease the computation:

$$\tau_B \approx t(1 - \frac{1}{2}\frac{v_s^2}{c^2}); \quad \tau_E \approx t(1 - \frac{1}{2}\frac{(v_s + v)^2}{c^2}); \quad \tau_W \approx t(1 - \frac{1}{2}\frac{(v_s - v)^2}{c^2})$$

---

[130] Actually the B,E,W may not end to the same position in solar space as takeoff, so t is really not proper time from this simple argument. However the above shows you why I can use time dilation, though the velocity is changing directions, this is the same trick of instantaneous inertial frames I talked about in the footnote in twin paradox

$$\Delta t_{east} = \tau_E - \tau_B = -\frac{1}{2}\frac{v_s v + v^2}{c^2}t$$

$$\Delta t_{west} = \tau_W - \tau_B = \frac{1}{2}\frac{v_s v - v^2}{c^2}t$$

Calculate t (it is practically just $\tau_B = 1.8\times10^5 s$) and put in the numbers for speed:

$$\Delta t_{east} \approx -250ns$$
$$\Delta t_{west} \approx +150ns$$

This result makes sense, since the eastbound travels faster so its time is dilated more (the clock runs slowest) compare to that on earth; while the west bound travels at the lowest speed and its time is dilated less.

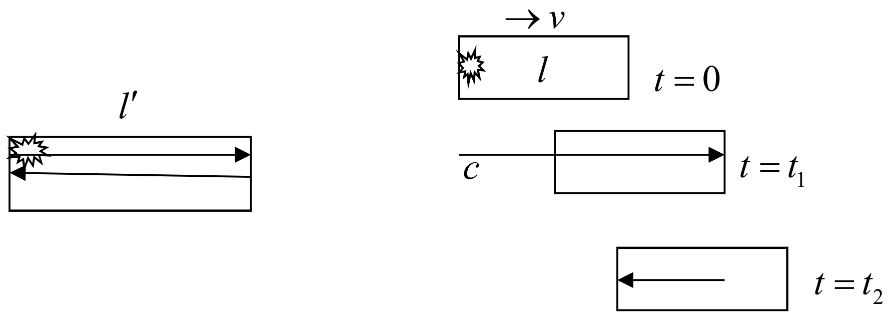
There is another important factor that I skipped here, the gravitational effect on time. The plane is flying high above the ground (say 10000m), the effect of gravitation on time cannot be neglected in this experiment and it can be estimated from general theory of relativity (time dilation due to gravity) which I won't cover here. With all these effect taken into consideration, and computed with the real flight data (height, speed, etc), the computed results from relativity theory agree with measurement.

	nanoseconds gained			
	predicted			measured
	gravitational (general relativity)	kinematic (special relativity)	total	
eastward	144±14	−184 ± 18	−40 ± 23	−59 ± 10
westward	179±18	96±10	275±21	273±7

11.4-2 Length Contraction

Constant speed of light and time measurement will enable us to measure distance (length). Suppose you want to measure the distance between Beijing and Tianjin, it is straightforward to send a light signal and received by a receiver at Tianjin, record the time interval between events of sending and receiving with synchronized clocks located at Beijing and Tianjin, the distance then can be calculated. Or another way is to as the figure below shows to use a round trip of light signal between start and destination, such as put a mirror at Tianjin and reflect the light back along its incoming path to Beijing. The round trip time can be recorded with a single clock at Beijing and the distance can also be computed from this time interval, this way we do not need synchronized clocks. This is all true for the measurement within one inertial frame.

We have seen that time measurement between events by observers in different frames will be different in the last section, and since time interval measurement is important in distance measurement, this will have an impact to the distance measurement, resulting in what is called length contraction.



The figure above shows the arrangement. A train has length l' in its own frame S', i.e. the frame that is moving along with the train so that the train is stationary in this frame. What the length of the train measured by the observer on the ground, i.e. what is l? Suppose a light burst at one end of the train (event 1), and the light will be reflected by the opposite end of the train and take a round trip and back to its starting point (event 2). Within frame S', this gives us:

$$2l' = c\Delta t' = c\tau$$

Because the two events in S' frame happened at same location, so the time interval is recorded by the single clock at the left end of the train, it is proper time. Now how these events appear to observers in S? The figures on the right show what S observes: The total time interval between the events are:

$$\Delta t = t_1 + t_2$$

$$ct_1 = l + vt_1 \rightarrow t_1 = \frac{l}{c-v}$$

$$ct_2 = l - vt_2 \rightarrow t_2 = \frac{l}{c+v}$$

$$\Delta t = \frac{l}{c-v} + \frac{l}{c+v} = l\frac{2c}{c^2 - v^2} = \frac{2l}{c}\frac{1}{1 - v^2/c^2} = \frac{2l}{c}\gamma^2$$

From the time dilation in the previous section, we know the relation:

$$\Delta t = \gamma\tau$$

$$\frac{2l}{c}\gamma^2 = \gamma\frac{2l'}{c}$$

$$l = \frac{l'}{\gamma} \qquad (11\text{-}10)$$

This is the length contraction. It says when an observer (here S) measures the length of a moving object (the train), the length would be smaller (contracted) comparing to the measurement if the moving object is stationary (in S'). I could give discussion like those A),B),C), D) following the time dilation, but I shall skip most of it since they are similar.

Let's look at a symmetrical event: in this case, it is the ground observer (S) has a length stick l, what this length stick measured by the observer on the train? Following the similar arguments, the length measured by S' would be:

$$l' = \frac{l}{\gamma} \qquad (11\text{-}11)$$

Just like you cannot plug (11-7) into (11-8) in the time dilation case, you should not try plug (11-11) into (11-10), because they are referring to different sets of events. Similar to the proper time definition, we can define **proper length** l_p. It is the length measured in a frame in which the object is not moving, another same saying is that it is the length measured in the frame that is co-moving with the object (the proper time is also the time measured in the frame that is co-moving with the particle like in the muon case). With this definition, the length contraction can be summarized as:

$$l = \frac{l_p}{\gamma} \qquad (11\text{-}12)$$

Repeat myself once more: l_p is the proper length measured in a frame that the object is stationary; l is the length measured in a frame that the object is moving.

Example 1. Another way deriving the length contraction

The setup is similar but no light signal. A moving train (S') with v is passing a ground observer S. The two events are the head of train passes the ground observer, and the tail of the train passes the ground observer.

The ground observer will record the time interval between the events as Δt, and since the velocity is v, the ground observer will conclude that the length of the train is: $l = v\Delta t$. According to the S', the time interval will be $\Delta t'$, and the distance travelled by the ground observer is: $l' = v\Delta t'$. There is relation between Δt and $\Delta t'$, question is which is the proper time here?

The proper time here is Δt, the time measured by the single ground observer at same place (the $\Delta t'$ is measured by the head and tail observers in S' which is not at the same place). So we have $\Delta t' = \gamma\Delta t$, and plug this back into $l' = v\Delta t'$ (we can do this because we are talking about same events): $l' = v\gamma\Delta t = \gamma l$, so $l = \dfrac{l'}{\gamma}$, same as (11-10).

Example 2: Muon decay experiment from muon's point of view

Last section, I solved muon decay in the lab frame with time dilation. Now we shall see that how the same result in muon's frame (a frame travels at same speed as muon). In this frame, the lifetime of muon is just the proper time $\tau = 2\times10^{-6}s$. What is the distance between the mountain top and sea-level viewed by muon? This distance 2000m (proper length in lab frame) will be contracted in muon's frame by $\gamma \sim 9$, i.e. $l' = \dfrac{l_p}{\gamma}$

$$A_{sea} = A_{mountain}e^{-t'/\tau} = A_{mountain}e^{-vt'/v\tau} = A_{mountain}e^{-l'/600m} = A_{mountain}e^{-2000m/9\times600m}$$

This is exactly same as before.

There is one important point to be discussed, that is the length perpendicular to the direction of motion. We see that different observers will get different length *along* the direction of motion, and the following argument based on the relativity principle makes sure that different observers have to agree on the length *perpendicular* to the motion.

Suppose we have two meter sticks with equal length measured on the ground, both are 1 m. One of the meter stick was carried on a train and both sticks were hold vertically. Suppose also we attach laser guns on the head of these meter sticks, let them shoot (always parallel with the horizontal plane) each other when they across. If the meter stick aboard the train (y') is contracted according to the ground observer, then the laser gun on y' will hit the meter stick on the ground (y) below 1 m line, while the ground stick will hit the one on train above its 1 m line. However for the observer on the train, it is the ground stick that is moving and according to the relativity principle, the ground stick will be contracted. That means the laser gun on the train (y') will hit the stick on the ground (y) above 1 m line; while the y will hit y' below its 1 m line. These are the same events observed by different observers, the results are contradictory to the different observers if the contraction (or elongation) along perpendicular direction happens. So there will be no change in distance along the perpendicular direction, i.e. y=y' and z=z'. This result is used in the derivation of time dilation where I used same vertical l, and above argument is the justification of doing so.

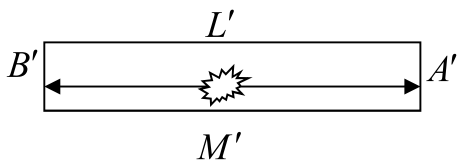
A final remark on the length contraction: The contraction I am talking about is the result of measurement, you use the stopwatch to let the end and tail of a moving rod passes you and computed length from time interval as in example 1; or with the help of the network of observers in your frame, measure the value of x of the head and tail of the moving rod ***simultaneously***. Simultaneous measurement of both head and tail will give you the length in your frame. This length is smaller compared to the value of proper length, the length measured by the observer riding with the moving rod (I realized I repeat myself again, forgive this blibber-blobber). This is not the length you really see with your own eyes (in the mundane sense), that is a perception. Seeing with your own eyes is a different ‘measurement’, i.e. the light scattered from the rod reach your eyes simultaneously. In this case you have to take the travelling of light from the object to your eyes into account. The result is not a contraction of the moving rod, instead the rod appears to your perception as rotated around an axis perpendicular to the rod and your line of view. What I am saying is if you take a photo-picture of the moving rod with a camera, the rod in the picture is not shortened but rather rotated or tilted. The detail analysis will not be discussed here[131], the lesson is again that seeing (in mundane sense) is not same as measurement in relativity. We have stressed this before, but our old habit somehow always makes these two as equivalents. It would be true if the light travels at infinite speed and takes no time to reach our eyes, but that is not the case here in relativity, though it is a pretty good approximation in our daily life to treat such speed as infinite. You probably would not believe that it took 50 years after the birth of special relativity, physicists noticed this difference and Terrell was first to point this out in 1959. Another similarly related topic is for time dilation, it is also the results of measurement, a network of observers read watches and make record of time. It is not one observer really looking at the clock hang on the wall of a moving train with his

[131] If you want to learn more on this issue, please refer to Greiner’s ‘classical mechanics, point particles and relativity’, chap.31. or to the original papers by Terrell, *Phys. Rev.* **116**, 1041 (1959) and Weisskopf, *Phys. Today* **13**, 24 (1960)

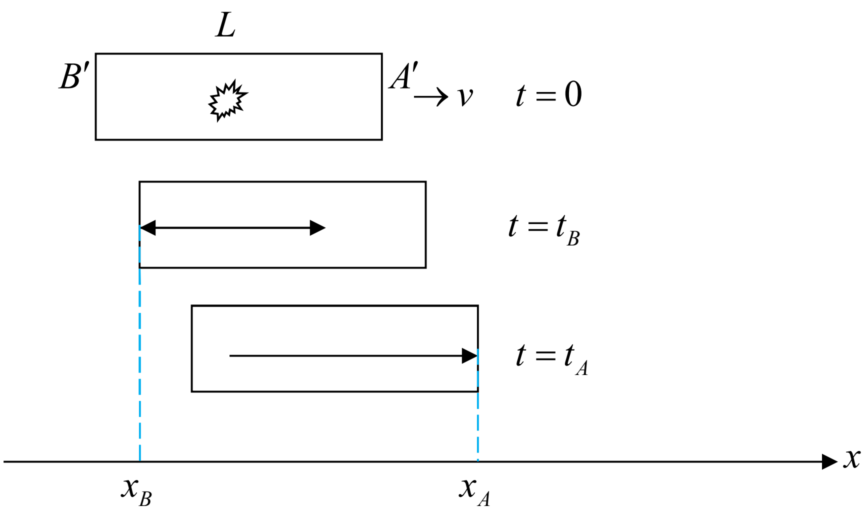
own eyes, that is a different observation and we shall discuss this later in Doppler shift.

11.4-3 Simultaneity is Relative

Here is another fundamental effect from the postulates of SR, we shall see that it is the root of time dilation and length contraction. First we need to define simultaneity, simultaneity of events means just the events happened at the same time. We discussed that no problem for the simultaneity of events happened at same place. But for events happened at different places, it has to be recorded by the clocks at those different places and this requires synchronization of the clocks. How to synchronizing the clocks? Using light signals, for example:



In the frame of a train S', with length L' and the head-tail ends are A',B' with the middle point M', a light signal goes off and sends light towards head and tail, the light will take exactly same time in S' to reach A' and B', so these two events (A', B' receiving the light signal) are simultaneous in S'. It can be used to synchronize the clocks at A' and B', say at the time they receive the signal, both clocks are set to zero (or some other fixed number). However what are the two events (A' and B' receiving the light) when viewed from another frame? Suppose the train S' is moving with certain velocity v w.r.t. ground S. In the train's frame S', A', B' are simultaneous, but for the ground observer in S frame, they are not! The reasoning is straightforward as the figure below describing the events observed by observer on the ground:



Three snap shots are sketched in the figure viewed by S. The event that the tail of the train receiving the signal is *before* the head, the two events are not simultaneous from point of view in S! Similarly if two events are simultaneous in S, then viewing from S', the events are not simultaneous, the event at the tail end will happen before the head events (head, tail are defined along the direction of motion). The reasoning is exactly same just with the direction of motion reversed. This is the meaning of the claim in the title that simultaneity is relative. People in different frame cannot agree upon the simultaneity of events happened at ***different places***.

We see that simultaneity is crucial in synchronizing the clocks, so synchronized clocks in one frame for instance, say the clocks are synchronized in S by the simultaneity of light signal receiving method described above, it will appear for the observer in S’ that they are not synchronized according to S’ clock and this will give rise to time dilation and length contraction in the measurement of time and length between different frames. I shall describe this quantitatively in the following paragraphs, thus explain the puzzle I raised previously: namely how come when A concludes that the B’s clock running slower, however B uses his slower clock concludes that it is A’s clock running slower (same puzzle for the length too).

First let’s calculate for the events that the tail and head of the train receiving signals simultaneously in the train’s frame, what are the measurement results for the ground observer?

The t_B can be calculated from the sketch in the middle:

$vt_B + ct_B = \frac{L}{2}$ here I used the fact that the event 0 (light goes off) is still at the middle point of the train. This true, though there will be length contraction so that $L \neq L'$, but the contraction is uniform so that the middle point in one frame would still be in another.

$$t_B = \frac{L}{2}\frac{1}{c+v}$$

Using the lowest sketch, easy to get:

$$t_A = \frac{L}{2}\frac{1}{c-v}$$

The time difference between event 1(tail receiving light) and 2 (head receiving light) are:

$$\Delta t_{AB} = t_A - t_B = \frac{L}{2}\left(\frac{1}{c-v} - \frac{1}{c+v}\right) = \frac{Lv}{c^2 - v^2} = \frac{v}{c^2}\gamma^2 L$$

So the events viewed in S are not simultaneous but different by this much. I shall further express the above in terms of spatial coordinated difference between events in S:

$\Delta x_{AB} = x_A - x_B = L + v\Delta t_{AB}$ straight from the two sketches in the figure, now put the expression for the time interval into it:

$$\Delta x_{AB} = L + L\frac{v^2}{c^2 - v^2} = \frac{c^2}{c^2 - v^2}L = \gamma^2 L$$ [132]

So rewrite the time interval as:

$$\Delta t_{AB} = \frac{v}{c^2}\Delta x_{AB} \qquad (11\text{-}13)$$

This says that for simultaneous events A', B' in S', the time difference of these events measured in S frame is given by (11-13).

This derivation is closely related to the length contraction as indicated in footnote 132. Here is a detailed account. How we measure the length of an object? We take a record of the position of head and tail and subtract, say $L = x_A - x_B,\ or\ L' = x'_A - x'_B$. If the object is stationary, then you can measure the head and tail at any time you like. However, if the object is moving with respect to you, you have to measure the position of head and

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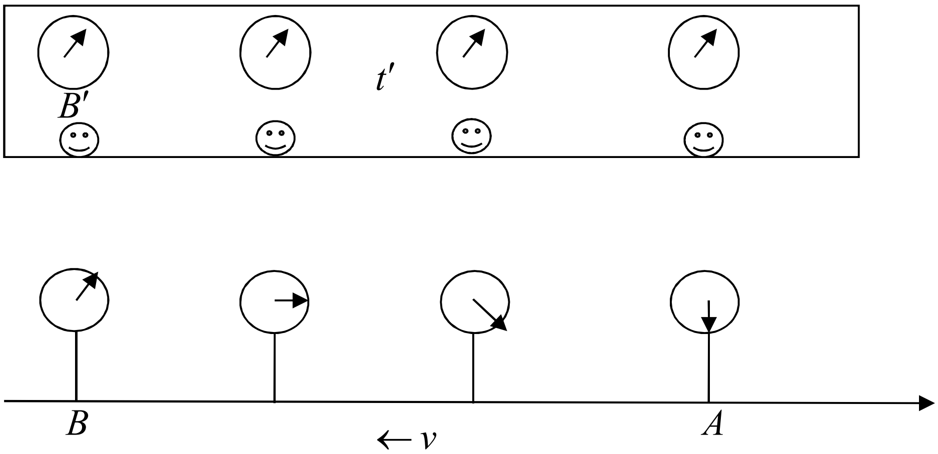
[132] This seeming simple result is not surprising if you brood on it a little while. The L is the measurement of length L' in S' by the S observer, so that $L = L'/\gamma$. The Δx_{AB} is the distance in S but if measured in S' would be just L' (think of the reason for this yourself, I will come to this immediately in the notes). That means: $L' = \frac{\Delta x_{AB}}{\gamma}$, and this will give the same result: $\Delta x_{AB} = \gamma^2 L$

tail *simultaneously*! It does not make any sense if you measure the head of a moving train at one moment, take a nap, and measure the position of the tail, you will certainly get shorter distance. So *simultaneity* is important in the measurement of length of a *moving* object.

But we have seen that simultaneity is a relative thing, different observers in different frames cannot agree upon it, and this cause the difference in the measurement of length by different observers. Back to the above example, for the observer Bob on the train S', he wants to measure some distance. The train itself can act as a ruler with length L'. Bob equipped the train with two laser guns at the head and tail and he will shoot these guns simultaneously in his frame (such as triggered by light signals from the middle), the shooting will strike AnDingMen (A) and BaiTaSi (B) on the ground and thus Bob will conclude that the distance between A, B is L'. Because though AB is moving in Bob's frame he measured it simultaneously (according to him). However from the above derivation we see that the ground observer does not agree on this. What he observed is that Bob's train shoots BaiTaSi first and after a time interval (according to ground clock) given by (11-13), the head of the train shoots at AnDingMen. The distance covered on the ground is just the Δx_{AB} calculated above which is larger than L'. The ground observer thus accuse Bob measured the proper distance (AB is not moving on ground so its distance is proper for the ground observer) with an improper method (same as Bob measured length of moving object head first and after awhile, measure the tail) and thus the result L' disagrees with Δx_{AB}. Similarly if the ground observer Adam tries to measure a proper distance in the moving train, Bob would also see the measurement conducted by Adam that is simultaneous according to Adam is not simultaneous to Bob. Thus the lack of agreement upon simultaneity helps us to understand why the results of measurement on length are different in different frames.

Now comes the time dilation from the lack of simultaneity. This can be understood as problems in synchronization of clocks. Because we use simultaneity events to synchronize clocks, a disagree upon simultaneity means disagree on the synchronization of clocks. Still use the Adam (on ground) and Bob (on train) example, Bob synchronized his clocks but Adam would not agree on this. Here is how:

Equation $\Delta t_{AB} = \frac{v}{c^2}\Delta x_{AB}$ (11-13) tells us exactly this. For a simultaneous events viewed by Bob, say at exactly time t' in Bob's clock, this means all the clock's in Bob's frame points at t' (synchronized). The clocks in Adam's are not, they are different as distance changes. Suppose Bob hires a network of observers and let them watch out (really see in mundane sense, but since the distance between the observers on the train and clocks on the ground can be taken as small as possible in this network observers case, we can neglect the travel time of light from clock to the observer) the window of the train, see the clocks on the ground (Adam ordered his network of observes raised their Rolex for Bob's to see). At that time t' in Bob's frame, the figures below are what the Bob's observers see:



In Bob's view, the clocks in Adams are not synchronized as Bob's. The tail ones (the one on the right, since to Bob, Adam is moving from right to left with v) are ahead of time than the head ones. Suppose a proper time interval τ in Bob's frame such as defined as time intervals between a tick-tock of one of the Bob's clock B', let's see the time interval measurement by Adam's clock in Bob's point of view (we had done this in Adam's point of view in time dilation):

At the event of tick, Adam record with clock at B; at the event of tock, Adam record with clock A (assume now A moves to overlap with B' at the tock). Bob figured that Adam takes the record of time on clock A and B and subtract them to get the time interval of the tick-tock, but this is not proper according to Bob, because the clocks B,A are not synchronized in his view. A is ahead of time, so the subtraction will be more than τ, and

Adam's measurement of tick-tock will be longer, that is why Adam has time dilation. This explains to Bob, how can Adam uses a slower clock, measures Bob's clock rate and concludes that it is Bob's clock running slower.

Now let's work out the same formula for time dilation from Bob's point of view, it is more involved than derivation from Adam's.

Within time interval τ, Adam moves a distance according to Bob by:

$$\Delta x' = v\tau$$

In Adams frame, the actual distance is:

$$\Delta x = \gamma \Delta x' = \gamma v \tau$$

So the difference due to poor synchronization according to Bob is:

$$\Delta t_{AB} = \frac{v}{c^2}\Delta x = \gamma \frac{v^2}{c^2}\tau$$

There is another time need to be considered for the total time Δt, that is during the tick-tock, how much time elapsed on clock A, the Δt_A? This happens at clock A, so Δt_A is proper in this case, i.e.:

$$\gamma \Delta t_A = \tau \quad \text{or} \quad \Delta t_A = \tau / \gamma$$

Then the total time measured by Adam during the tick-tock is:

$$\Delta t = \Delta t_{AB} + \Delta t_A = (\gamma \frac{v^2}{c^2} + \frac{1}{\gamma})\tau = (\frac{\gamma^2 v^2 + c^2}{c^2 \gamma})\tau$$

$$\gamma^2 v^2 + c^2 = \frac{v^2}{1 - v^2 / c^2} + c^2 = \frac{v^2 + c^2 - v^2}{1 - v^2 / c^2} = c^2 \gamma^2, \text{ put this back}$$

$$\Delta t = \gamma \tau$$

This gives back the (11-7) time dilation. The calculation is from Bob's point of view and is more complicated than before.

In solving problems in SR, I strongly recommend you to specify the events and work out the problem using one frame. In the above derivations, I have to switch back between frames which are bad for beginners unless you know what you are doing, because this could cause confusion, such as what is proper length or time etc. So the good strategy for beginner is choosing one frame and stick to it as long as possible, list out the events what you observed from the point of view of the frame you choose and work from there. The reason I am showing you the example above is of course to let you see how time dilation arises from Bob's point of view, still rooted in the disagreement of simultaneity and thus resulting in the problems of synchronization of clocks viewed by observers in different frames.

In SR, there are generally many different ways to solve the problem, at least two, one from each frame of your choice. A wise choice may give you easier solution, the other point of view though may (not always) be complicated but it can offer more physical insight or at least served as double check.

The example I am doing so far only used or at least started from the postulates only, these postulates will give us a powerful tool besides the important effects talked in this long section (11.4). We shall see the relations in this section can be much easier derived by the powerful tool.

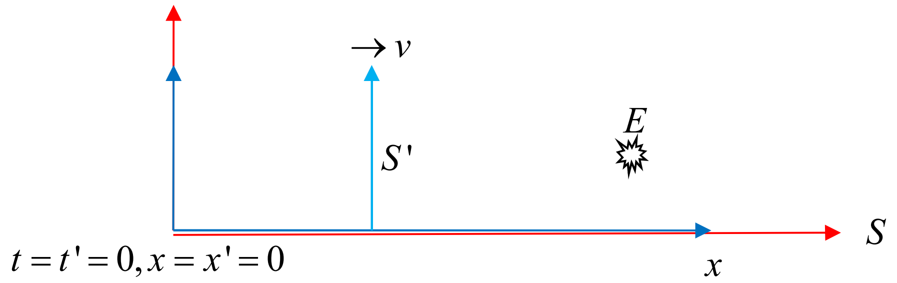
That is Lorentz Transformation we are going to discover.

Also in order to have real physical meanings, the γ has to be a real number, this means no frame is moving faster than speed of light, i.e. $v \le c$, and $\gamma \ge 1$ from its definition.

Chapter 12 Lorentz Transformation and Kinematics in Special Relativity

In this chapter, I shall first derive the most important relations in SR, the Lorentz Transform (LT). Then we shall see how to apply the Transform and see all those fundamental effects we discussed in the last chapter can be worked out directly from the LT. We shall also discover there is a golden combination of space-time that does not change (invariant) in the transform (though we will exploit this feature in much later time). The LT can be expressed nicely in a geometric form---Minkowski diagram and this give us a direct 'picture' sometimes helpful, such as in discussion of cause-effect. Applications and examples of LT will be discussed, such as the important Doppler effect and some interesting 'paradox' problems. The chapter will end with velocity rules in SR which will be important in the development of dynamics in chapter 13.

12.1 Lorentz Transform[133]



The setup of coordinate systems (frames) is shown in the figure above. The x and x' axes are overlapping with the relative velocity between the two frames. This seems like a special case, where most generally, the velocity could be any direction. However, in many cases, we can setup our coordinate system so that x,x' overlaps with v. Suppose an event happened somewhere at sometime. This *same event* E when viewed by S happened at (x,y,z,t); and at (x',y',z',t') by observers in S'. What we are looking for are the relations among the space-time coordinates between the two inertial frames for the **same** event. When you change from one frame to another, the coordinate changes accordingly and this is called transformation (just like when you rotate x-y-z in old days, the coordinates changes accordingly, that is rotational transformation).

(The following argument shows you the physical argument leading to equations (12-1) below, you may skip it and dive to (12-1) directly) The

[133] Strictly speaking, the transform we are considering here which only involves translational motion between frames is called pure Lorentz-Transform (also called Boost LT), it is a subclass of more general LT which includes rotation of the coordinates and is called restricted LT; the restricted LT in turn is a subclass of even more general transform that includes other symmetry operation such as inversion or reflection etc.

transformation will be ***linear***, i.e. it will be in a general form:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

The reason of these linear relations is because of the homogeneity of the space, i.e. you can pick up an arbitrary point as origin and the results should be same. Imagine that if we have non-linear relation such as:

$x' = ax^2$, now suppose we have a rod with ends at x_1, x_2. The length in the S' would be: $x_1' - x_2' = a(x_1^2 - x_2^2)$. Let's change the origin (translate the coordinate by a some fix value), so in this new origin, the two ends are x_1+b,x_2+b: $x_1' - x_2' = a[(x_1 + b)^2 - (x_2 + b)^2]$. This is different from the value before the translation of origin. This means the observation will depend on the origin of choice so that violates the homogeneous space. That is why the transform has to be linear.

Linear as they are, there are still 16 unknown coefficients need to be evaluated. However the isotropic (i.e. rotational symmetry about axes) space would help us get rid most of them.

a) Consider $x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$, now if I rotate around the x (or x') axes by 180 degree (by any degree if you want). This will not change x and x' value as well as t, but the y and z are changed to –y and –z. To keep this equation, the a_{12} and a_{13} have to be 0 (or $a_{12}y + a_{13}z$ combination is 0 which is same thing since y and z are independent, only possible combination make above 0 is coefficients are 0). This leave us: $x' = a_{11}x + a_{14}t$. Note that it is fine for me to rotate the x axis and claim that the transform (all the coefficients) should be same. This is because we are trying to find transformation due to constant motion between frames, meaning the transform coefficients should only depend on the motion velocity. We set the coordinate axis of S overlapping with S': x-x' overlaps and along direction of motion; the y overlaps with y', z with z'. When I rotate the x axis, I do not change the expression of v.

b) $y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$. A similar rotation around x(x') axes 180 degree will leave x,t unchanged, y to –y, z to –z and y' to –y'. This will give us a_{21}, a_{24} are 0. With our choice of coordinates as shown in the figure, the x-z plane always overlaps with x'-z'. So any event happens in the x-z plane in S frame would be observed as in x'-z' plane in s' frame. This just means the y=0 (x-z plane) would transform to y'=0 for any z. Using this fact, then for $y' = a_{22}y + a_{23}z$ then a_{23} should be 0, so that $y = \frac{y'}{a_{22}}$. From relativity principle, the coefficient has to be same independent of which frame you are in, i.e. If Adam observes Bob's rod getting smaller or larger; Bob has equal right to claim same thing to Adam's rod. This means $a_{22} = \frac{1}{a_{22}} \to a_{22} = 1$ (the rejection of $a_{22} = -1$ is obvious, it has to reduce to Galileo transform at low velocity). So we have $y' = y$ and similarly $z' = z$. This is a proof of vertical distance is unaltered where I argued with relativity principle before.

c) $t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$. Similar arguments as above, rotating the x (x') axes only alters y and z so that their coefficients (or combination) are 0. So $t' = a_{41}x + a_{44}t$

Now the 16 coefficient above are reduced to 4 unknowns:

$$x' = a_{11}x + a_{14}t$$
$$t' = a_{41}x + a_{44}t \qquad (12\text{-}1)$$
$$y' = y \quad z' = z$$

There are quite a few ways to work out their expressions. The basic method is to pick out some events such as the events chosen in KK (table 11.1). Since we have already worked out some fundamental effect from the postulates directly, we can apply those here to find out the coefficients, i.e. the toil we spent there makes this derivation easier. (The events chosen below to derive the Lorentz Transform is not unique, you may choose other events deriving the same thing.)

A) The origin of the S' (x'=0) in the S frame is moving with velocity v to the right and this will give us:

$$x' = a_{11}(x - vt)$$

B) Using length contraction, for a proper length x' is S', if we measure its length we have to do it simultaneously, i.e. t=0 and then $x = \frac{x'}{\gamma}$. From the above relation, this leads to:

$a_{11} = \gamma$

C) Using time dilation, for a proper time in S, i.e. time interval t at x=0, we have $t' = \gamma t$, and this leads to:

$a_{44} = \gamma$

D) Now only one coefficient to be evaluated a_{41}. I shall use the relativity principle on the time dilation, i.e. a proper time t' is S' (x'=0) would be:

$t = \gamma t'$

The x needs to be replaced by x' with $x' = \gamma(x - vt) \to x = \frac{x'}{\gamma} + vt$,

$t' = a_{41}x + \gamma t = a_{41}(\frac{x'}{\gamma} + vt) + \gamma t$, for the proper time in S', x'=0, then:

$t' = (a_{41}v + \gamma)t$, so compare with $t = \gamma t'$,

$$(a_{41}v + \gamma) = \frac{1}{\gamma} \to a_{41} = \frac{1}{v}\frac{1-\gamma^2}{\gamma} = \frac{1}{v\gamma}(\frac{-v^2/c^2}{1-v^2/c^2}) = -\frac{v}{c^2}\frac{1}{\gamma}\gamma^2 = -\gamma\frac{v}{c^2}$$

Now we have all the coefficients and the transform is:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y, \quad z' = z \qquad (12\text{-}2) \\ t' &= \gamma(t - \frac{v}{c^2}x) \end{aligned}$$

This is the famous Lorentz Transform, once again it relates the space-time coordinate of **same event** in different inertial frames that is moving with v relative to each other. Actually it is intervals of space-time coordinate (above the event 1 is taken as the overlap of the two origins

$t = t' = 0, x = x' = 0$), so for any two events when viewed from S and S', their space-time intervals are related by:

$$\begin{aligned}
&\Delta x' = \gamma(\Delta x - v\Delta t) \\
&\Delta y' = \Delta y,\ \ \Delta z' = \Delta z \qquad (12\text{-}3) \\
&\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)
\end{aligned}$$

The form is not quite symmetrical between x and t, this is because we are using units in which $c \neq 1$. If we use unit c=1 (i.e. distance is measured by light-second while time is still second) or using ct and ct' (they will be in dimension of distance), the transform will be more symmetrical:

$$\begin{aligned}
&x' = \gamma(x - \frac{v}{c}(ct)) = \gamma x - \gamma\beta(ct) \\
&ct' = \gamma(ct) - \gamma\beta x
\end{aligned} \qquad (12\text{-}4)$$

If we ask knowing the x',t' of events in S', what are the x,t in S? The relation should be same from relativity principle, only the v changes to –v:

$$\begin{aligned}
&x = \gamma(x' + vt') \\
&y = y',\ \ z = z' \\
&t = \gamma(t' + \frac{v}{c^2}x')
\end{aligned} \quad \text{or} \quad \begin{aligned}
&\Delta x = \gamma(\Delta x' + v\Delta t') \\
&\Delta y = \Delta y',\ \ \Delta z = \Delta z' \\
&\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x')
\end{aligned} \quad \text{or} \quad \begin{aligned}
&x = \gamma x' + \gamma\beta(ct') \\
&ct = \gamma(ct') + \gamma\beta x'
\end{aligned} \qquad (12\text{-}5)$$

12.2 Fundamental Effects from LT

In this section, I will show that we can get the fundamental effects, i.e. time dilation, length contraction and relative simultaneity, from Lorentz

Transform. Of course during the derivation I already used some of these, it may appear like arguing in circle. The fact is that the LT can be derived without invoking the effects directly from relativity principle and constant c. This will provide an easier and nice way to remember those effects.

(1) Time dilation

The proper time in S is the events happened at same place with time interval, this means: $\Delta x = 0, \Delta t = \tau$, then from (12-2) or (12-3):

$$\Delta t' = \gamma\tau$$

This is the formula for time dilation effect we talked before. If the proper time is in S', then you work out the result.

(2) Length Contraction

For a measurement of proper length in S', i.e. $\Delta x' = l_p$ (it is stationary in S'), we have to measure it at the same time in S, i.e. $\Delta t = 0$, (12-3) will give us:

$$\Delta x' = l_p = \gamma \Delta x = \gamma l$$
$$l = \frac{l_p}{\gamma}$$

This is the length contraction result before. Notice that I have explicitly specified events in the time dilation and length contraction to avoid confusion. For the measurement of proper length in S, you figure it out yourself.

(3) Relative Simultaneity

For a simultaneous events in S', say all the clock on the train all point to 12 on the train, $\Delta t' = 0$. The readings of the clock on the ground (S) will be different depending on its position, from (12-3):

$$0 = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$

$$\Delta t = \frac{v}{c^2}\Delta x$$

This is the (11-13) that I worked out with more effort before. As always, I will ask you to figure out for simultaneous events in S (ground clock all point to one reading for the ground observer), what are the clocks on the train viewed by the ground observers?

Mathematically, starting from LT and work out the effects are simplest, however the reason I goes other way around is that I hope you will appreciate the physical reasoning from the points of postulates more.

12.3 Space-Time Interval: an Invariant under LT

The LT above clearly shows the relations between the space-time coordinates of events viewed by different inertial observers. The spatial distance, the time interval between events will appear differently for different observers, it seems like the slang: nobody agrees on nothing. But there is something invariant, means unchanged, upon transformation. We shall see what this will be.

Actually the constant c offers a clue on the expression of the invariant:

Suppose a light goes off, a photon flies with same c in all frames, this means the relations:

$c\Delta t = \Delta r$ in S and in S' we also have $c\Delta t' = \Delta r'$,we find that the value:

$c^2(\Delta t)^2 - (\Delta r)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ will have same value in all frames (of course you plug in the coordinates measured in the frames), it is always 0 for the light in all frames. This should provoke you to think that besides light, for any other events, do we have a similar invariant property of the above relation? i.e. for any events when viewed in different frames, such as a particle flies across space between staring point A and end point B. The space interval and time interval measured by different observers in different frame will be different, but do we have:

$$c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \quad (12\text{-}5)$$

It may not be nice value 0 as for light, but its value is unchanged upon LT. Indeed the (12-5) is true and the proof is direct from LT (I won't consider y and z below for obvious reason):

$$(c\Delta t')^2 = c^2\gamma^2(\Delta t - \frac{v}{c^2}\Delta x)^2$$

$$(\Delta x')^2 = \gamma^2(\Delta x - v\Delta t)^2$$

$$\begin{aligned} c^2(\Delta t')^2 - (\Delta x')^2 &= c^2\gamma^2(\Delta t)^2 + \frac{v^2\gamma^2}{c^2}(\Delta x)^2 - v^2\gamma^2(\Delta t)^2 - \gamma^2(\Delta x)^2 \\ &= (\frac{c^4}{c^2 - v^2} - \frac{v^2c^2}{c^2 - v^2})(\Delta t)^2 - (\frac{c^2}{c^2 - v^2} - \frac{v^2}{c^2 - v^2})(\Delta x)^2 \\ &= c^2(\Delta t)^2 - (\Delta x)^2 \end{aligned}$$

QED.

Sometimes we may drop the Δ symbol if the starting event is the overlap between origins defined before, the (12-5) will be:

$$(ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2 \qquad (12\text{-}6)$$

The invariant value $c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ deserves a unique name, the space-time interval, s^2:

$$s^2 \equiv (ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2 \qquad (12\text{-}7)$$

Note that it's a bookmark to write space-time interval as s^2, it is not some squared number, so its value could be positive or negative corresponding to time-like (s^2>0), space-like (s^2<0) or light-like (s^2=0) events when we talk about causality.

The relation (12-7) states that the space-time interval defined is an invariant upon transformation. The analogy is the magnitude (length) of a vector upon rotation. There as the coordinate system rotates, the same vector will have different coordinates but its magnitude is same for all. (12-7) is not for some length in 3-D or 4-D in the Euclid space, which will be defined as summation of squares instead of time squared minus the space squared. The time and space are intertwined in Lorentz transform (change of x' depend on both x an t and vice versa), i.e. time and space are related but not exactly equivalent. The feature of invariant space-time interval and its analogy to 3-D vector length will fully be exploited in the 4-vector formalism later, and that is a beautiful theory (an advertisement here).

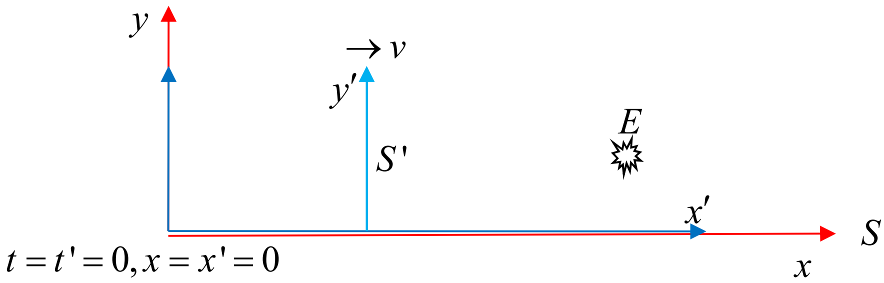
Since s^2 is same for all observers, what is its expression for the observer travels with the particle? Here, the particle appear stationary, always stays at same location in such frame, and the time interval is the previously defined proper time! So we have equation:

$s^2 = c^2\tau^2$ (12-8)

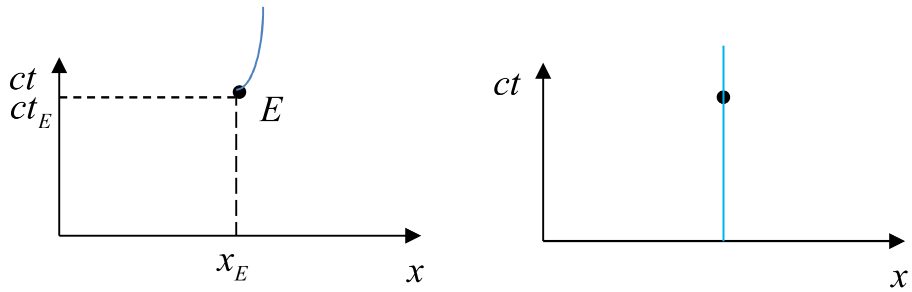
Different observers in different cars when observe something happened on a moving train, their results of measurement are not same numbers. But if we ask the question another way, what happened using the data measured by different observers in the car and infers from those to the measurement conducted on the train, the data should give the same results. Put it more plainly that Bob may measure muon's lifetime one value, Adam in a different car will measure the lifetime to be some other value, but if we ask to Bob and Adam, what is the muon's lifetime in muon's frame, they all agree that it is $2\mu s$. That is what (12-8) tells us.

12.4 Minkowski Diagram

This is a geometric representation of the Lorentz Transform and gives you a sort of 'direct picture' of the transform. The inertial frames are still same as before as in the figure below:



For any event E with space-time coordinate given by (t, x, y, z), the event can be represented by a point in space-time diagram, the Minkowski diag.



For a single event E, it is space-time coordinate in one inertial frame is indicated by the dashed lines in the figure and the coordinate $(x_{E,} ct_E)$. Noticed that it is conventional to choose ***ct*** be the vertical[134] axis in the diagram, this makes the H-V axes with same dimension (both in length). So any event will be represented by points in this diagram and a sequence of events, such as particle flying through space over time tracing out a trajectory, will be represented as curves or lines (as the blue one in the figures). Such line representing a sequence of event is called world line, the blue vertical one in the right figure represents particle E is motionless in this frame: as time passes, no change of position.

For the Lorentz Transform:

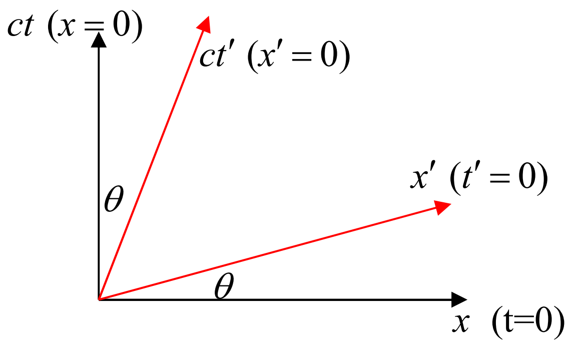
---

[134] To avoid unnecessary confusion, I shall use V (vertical) for the y axis in Cartesian, it represents ct in the diagram. The horizontal axis (H) represents space position (1-D along the motion, usually choose to be x).

$$ct' = \gamma(ct) - \gamma\beta x$$
$$x' = -\gamma\beta(ct) + \gamma x$$

The world line $t' = 0$ is S' frame, and $x' = 0$ (these are the H-V axes in S') in the S frame are:

$$\gamma(ct) - \gamma\beta x = 0 \rightarrow \frac{ct}{x} = \beta \quad (\text{for } t' = 0,\ x' \text{ axis})$$
$$-\gamma\beta(ct) + \gamma x = 0 \rightarrow \frac{ct}{x} = \frac{1}{\beta} \quad (\text{for } x' = 0,\ ct' \text{ axis})$$

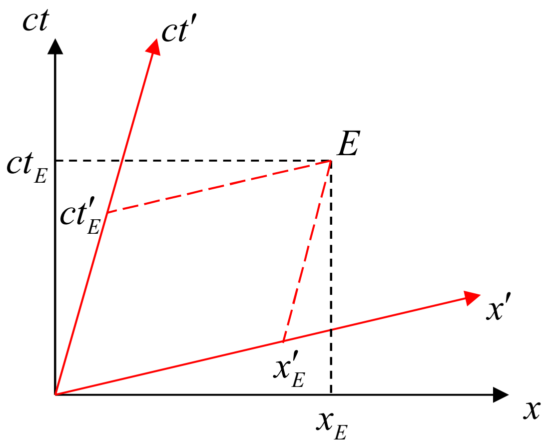


The red lines representing the ct' axis (the world line x'=0 in the S', the object is not moving in S' would appear moving with β viewed in S) and the x' axis (events at different location happened at same time t'=0, a simultaneous events in S' would not be so in S) are shown in the figure above. They symmetrically span an angle θ with respect the ct and x axes, and:

$$\tan\theta = \beta \equiv \frac{v}{c} \qquad (12\text{-}9)$$

Comment: The tilted axes of S' are the how events been observed in S, this does not mean that in S' coordinate system, the x' axis of the Cartesian coordinate have an angle with x axes of S. They are overlapping as shown in the first figure of this section.

Our next question is how an event is represented by (x', ct'), i.e. how to read the space-time coordinate of an event in S'? For the events happened simultaneously in S', $\Delta t' = 0$ or $\frac{c\Delta t}{\Delta x} = \beta$ will exactly trace out a line parallel to that of x' axes (line of t'=0); and for events happened at same location in S', i.e. $\Delta x' = 0$ or $\frac{c\Delta t}{\Delta x} = \frac{1}{\beta}$ will exactly trace out a line parallel to ct' axis (line of x'=0). So the space-time coordinate of an event happened at $(x_{E,} ct_E)$ in S is (x'_E, ct'_E) in S' and the graphic representation is:

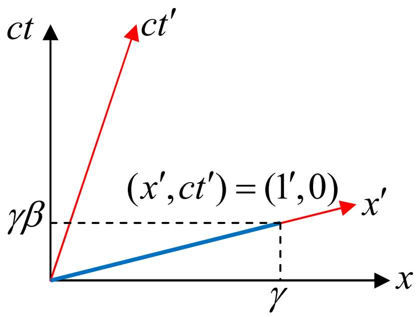


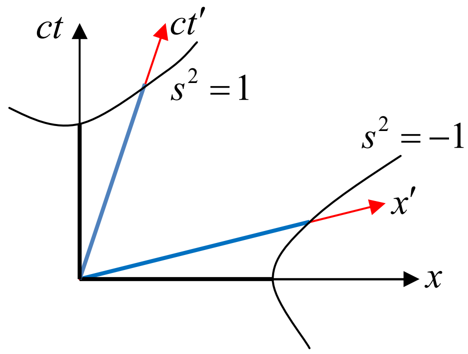
The space-time coordinate of event E is read off by drawing parallelogram with respect to ct' and x' axis.

There is one important catch in this diagram, in the figure above the unit length is S' are different from unit length in S! This is best illustrated with Lorentz Transform or the invariant space-time interval s^2. First using Lorentz transform, for unit length along the x' axis, that is (x'=1, t'=0) in S'. In the S frame, using:

$$x = \gamma x' + \gamma\beta(ct')$$
$$ct = \gamma(ct') + \gamma\beta x'$$

This gives $x = \gamma$, $ct = \gamma\beta$





The unit length (1',0) measured in S' is not unit anymore to S, it is:

$$1' = \sqrt{\gamma^2 + \gamma^2\beta^2} = \sqrt{\frac{1+\beta^2}{1-\beta^2}} \qquad (12\text{-}10)$$

Similarly the argument would apply to the time unit (0,1') in S' corresponds to ($\gamma\beta$, γ) in S and we have a same expression for time unit in S'.

These could also be viewed from invariant s^2, this number is same for both S and S'. For the $s^2 = -1$ world line (all events satisfying this condition along the line), the curve in the S frame is a hyperbola:

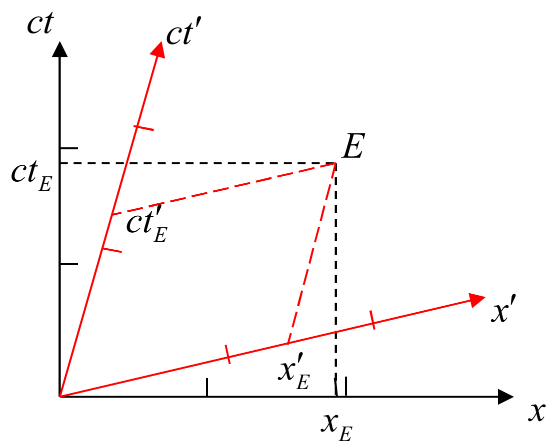
$$s^2 = (ct)^2 - x^2 = -1$$

This curve will intersect the x axis (t=0) to give unit length in x, (1,0) for S. But s^2 could also be expressed as space-time coordinates in S', i.e. the same hyperbola curve when viewed in S', represents events satisfying:

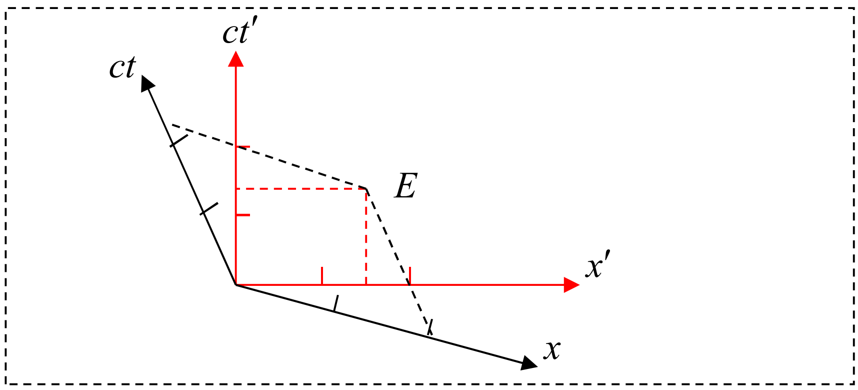
$$s^2 = (ct')^2 - x'^2 = -1$$

This will intersect at x' axis (t'=0) to give unit length in x' (which is different than that in x). $s^2 = -1$ acts as a calibration curve for the unit length in both frames.

Combined all this, we have a complete Minkowski diagram:

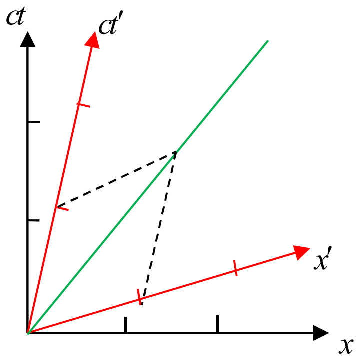


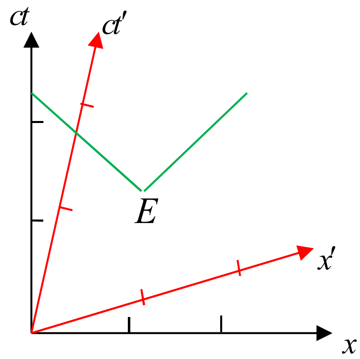
The above is to choose the S frame orthogonal in Minkowski diagram, we certainly can choose S' as orthogonal and let S be moving towards –x' direction. The diagram would be:



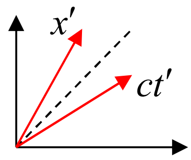
I have told you all the basics on Minkowski diagram, let's now put it in use to see some examples:

Example 1: Light propagation



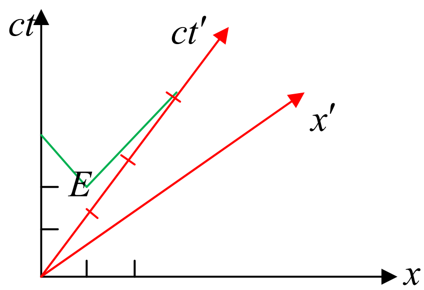


The figure above left represents a light signal generated at the origin and time=0. Its world line is a 45 degree line in the S, like a time-path of a particle moving with speed 1 ($x/ct=1$); viewed in S', the light signal also travels with speed 1, i.e. the light line also has same angle with respect to x' and ct' axes (the angle is just $\frac{\pi}{4}-\theta$). The light bisects all the inertial frames. As the relative speed v between the S and S' increases, the S' axes will be tilted more approaching the light line, but since $v \le c$, the ct' and x' axes never flips across the light line, the following cannot happen:

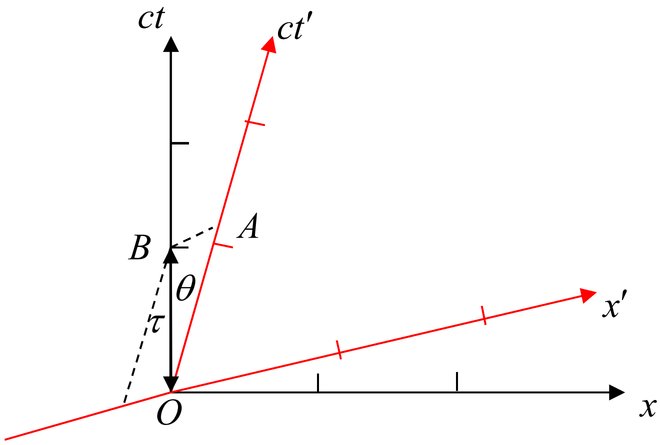


The figure on the right is an event that light signal goes off at certain space-time coordinate E (arbitrary), the light propagates along the +x(x') (+45 world line) and –x (x') (-45 world line) direction. The time taking the light to be seen (in mundane sense) by the observers staying at the origin of the S and S' are the intersects at the ct and ct' axes. This is clearly different from the event when it is happened, i.e. recording coordinates of events is different from seeing in mundane sense, a point I talked a couple time already, here is the direct picture.

Of course if I asked a question, a fire cracker explodes at E, (1 ls (light second), 2 sec;) according to S, and v=0.8c between frames; calculate the time when the light signal really seen by observers at origin of S and S'. You can use the diagram directly read off the result (if you draw the diagram and place the event correctly), but in this course I believe most of you will calculate it from the LT transform which is probably faster than drawing the diagram. I trust that you can work out this one by yourself, do it yourself and check the answer (The diagram is shown below): observer at S origin will see light at time=2+1=3 sec, observer at S' origin will see light at time=2+1=3 sec too in this special case. If the event happens at (1ls, 1s) in S instead, than for light to reach S: time=1+1=2s; and for S', time=1/3+1/3=2/3 s. Though you may not use the diagram in the calculation, a sketch of it at least acts as a check for reasonableness of the answer.



Example 2: Time Dilation



For a proper time interval measured in S, represented by the double head arrow OB at x=0, it has time interval τ. This will be measured in S' by the length OA (or t'). You already notice the time dilation from the sketch but let me show the detail calculation. It is a straightforward geometry to show relation between OB (τ) and OA (t') when expressed *in unit of* ***S***:

$$t'\cos\theta - t'\sin\theta\tan\theta = \tau, \quad \tan\theta = \beta$$

$$\cos\theta = \frac{1}{\sqrt{1+\beta^2}}, \quad \sin\theta = \frac{\beta}{\sqrt{1+\beta^2}}$$

$$t'\left(\frac{1-\beta^2}{\sqrt{1+\beta^2}}\right) = \tau$$

$$t' = \frac{\sqrt{1+\beta^2}}{1-\beta^2}\tau$$

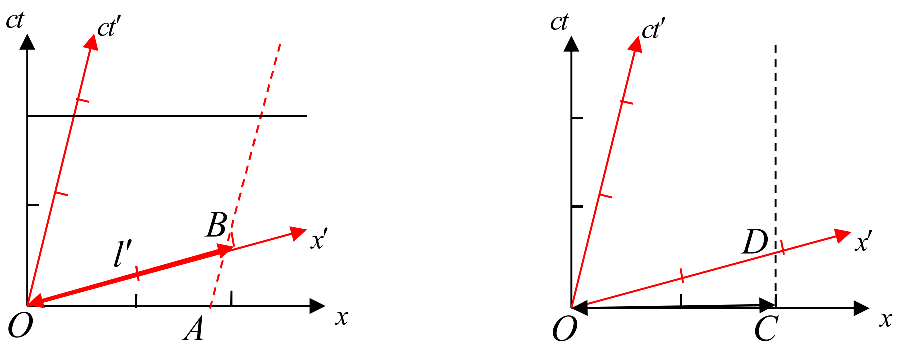
You notice that it is different from the familiar $t' = \gamma\tau = \dfrac{1}{\sqrt{1-\beta^2}}\tau$, this is because the unit is not corrected yet in the above derivation, that is put in the unit of S, it should be calibrated to that of S', i.e. (12-10) $1' = \sqrt{\gamma^2 + \gamma^2\beta^2} = \sqrt{\dfrac{1+\beta^2}{1-\beta^2}}$, t' put in this unit will give:

$t'/1' = \gamma\tau$ exactly as expected.

Comment: Noticed that this computation (solving geometric problem) may be more complicated than the algebra method with LT, the advantage of the diagram is its illustrating power, that it helps you to 'visualize' the time dilation (just like the regular algebra and geometry, prove two lines are perpendicular may be easily done by a dot product using vector algebra rather than prove it geometrically, but a drawing of perpendicular lines does help us understanding it)

For the case of proper time in S', it is left for you to 'visualize' it using the above diagram.

Example 3: Length Contraction



The figure on the left represents a rod rest in S', the world lines of its two end trace out two straight lines, one is just ct' axis, the other is the red dotted one. (the rod is stationary in S' but moving with v in S). When S observer tries to measure its length, he has to do it simultaneously according to him, so OA is the result (he measured the head and tail of the rod at t=0 in his clock, of course he could choose other time t=2 etc., same result). Geometry leads:

$$OB\cos\theta - OB\sin\theta\tan\theta = OA$$

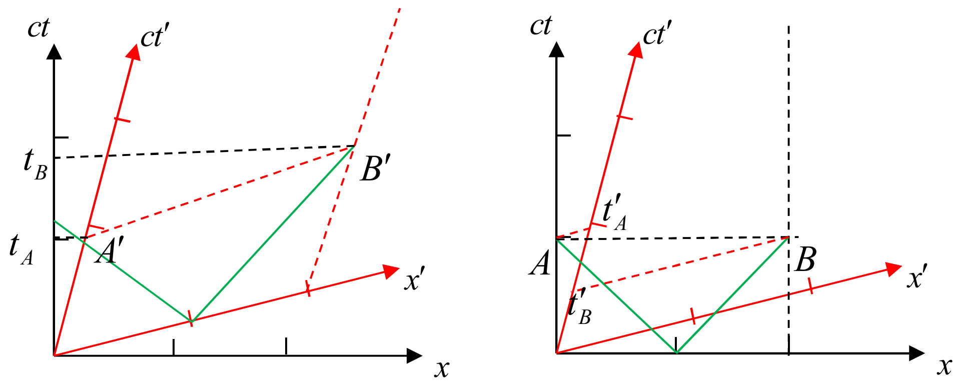
OB is unit of S' is l', since we like to know the length measured in S, so better express it in unit of S, i.e.:

$OB = l'\sqrt{\dfrac{1+\beta^2}{1-\beta^2}}$, $OA = l$ similar calculation as example 2 will give us:

$$l' = \gamma l \quad \text{or} \quad l = \frac{l'}{\gamma}$$

For the measurement of a proper length in S (the rod is stationary in S), its end trace out world lines parallel to ct axis. S' has to measure it simultaneously according to him and the computation is left for you to finish.

Example 4: Simultaneity



The two graphs I hope clearly demonstrate the simultaneity is a relative thing. The figure on the left shows the simultaneous situation in S': The light signal emitting from the middle of train, reach the head and tail simultaneously. So for S' the two events A', B' happened as same time which is also nicely shown that the A'B' line parallel with the x' axis (all the events happened along this line is at the same time in S’). But the A’, B’ in point of view of S is clearly not simultaneous, A’ occurs before B’ (the tail-end happens first, ahead of the head-end). The calculation using geometry to show that the time delay in S is the one given in (11-13), i.e.:

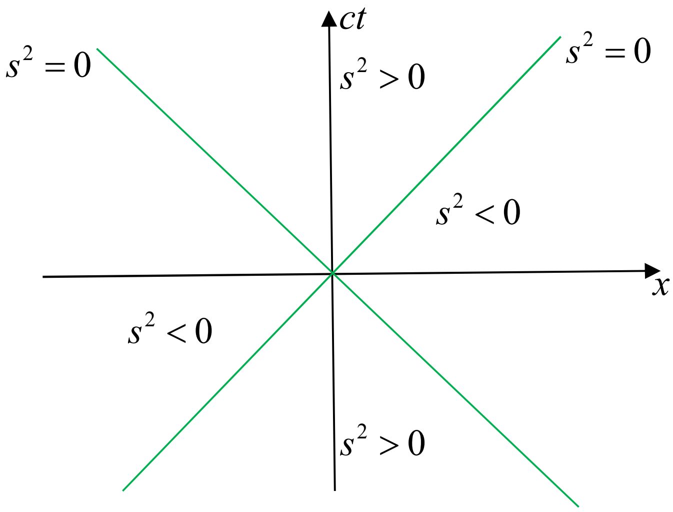
$\Delta t = \frac{v}{c^2}\Delta x$ is left for you to prove.

The figure on the right shows you the simultaneous event in S, A,B when viewed by S’, are not simultaneous anymore. The tail end B (AB is moving to the left for S’) happened first, earlier than the head-end A. The time delay in S’ can be easily found out using LT, and if interested, please calculate it using the geometric of the diagram.

We have discussed the basics on Lorentz Transform and in the following sections, we shall apply the stuffs we learned so far to see some implications and applications.

12.5 Cause-Effect (Causality) and Speed Limit on Particles and Signals

Using the invariant space-time interval s^2 (12-7), we can divide the Minkowski diagram into different zones:



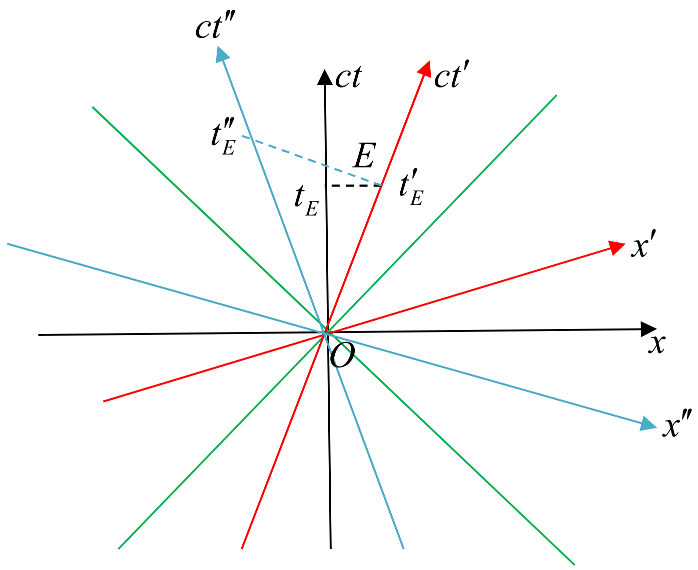
The light divides the space-time into light cones. The one in which $s^2 > 0$ is called time-like zone; the $s^2 < 0$ is called space-like zone and $s^2 = 0$ lines are light-like.

(1) $s^2 > 0$ time like zone

The time orders of events in this zone never changes. The event1 is always chosen as origin (if it is not , always can shift the origin to overlap), say another event happed some time later at some place within the time like zone, i.e. If the (x,t) are the coordinates of the second event: $s^2 = c^2t^2 - x^2 > 0 \to |\frac{x}{t}| < c$. Viewed in another frame that is moving with v and overlapping origins, the event happens at:

$$t' = \gamma(t - \frac{vx}{c^2}) = \gamma t(1 - \frac{vx}{tc^2})$$

Because $v \le c, \frac{x}{t} < c$, so what is in the parenthesis will always be positive, t' will always have same sign as t, i.e. the time order of events in the time-like zone never changes. What happens later will be later for all inertial observers, what happened before will be before for all observers. That is why the top-cone in the time-like zone is called absolute future, because any event happened there will be future to all observers at origins in different frames; the bottom half is called absolute past for the same reason. This can also be directly shown with Minkowski diagram:



An event E happened at the future to O, E will be future (t, t', t'' all >0) for all inertial observer in S, S' and S''. Actually for a certain observer (the S' in the figure) the event E and O happens at the same place but later time. This is the name time-like comes from: you can choose a frame so that the two events (O and E here) happened at the same place with time interval.

What does this to do with cause-effect? Well cause-effect is just a sequence of ***related*** events, the event O (call it cause) is related to event E (call it effect), and the time order between them cannot change no matter which frame you view them! This time order requirement is called Causality Principle, it is common sense. For example, cause: a bullet is fired; Effect: I was killed. You can watch this show boarding on different trains with different high velocities, but all you will agree that the bullet was fired first (at earlier time) and I was killed later. It is going to be absurd if for some viewers I was killed first for no obvious reason and later a bullet was fired (this also applies to ancestor-descendant joke or it is impossible to have scenes like that in the movie Terminator). If the Causality Principle could be violated, i.e. the time order between related events could be changed, the world will go crazy. But fortunately the SR tells you that the events happened in the time-like zones will never change the order of time, so if two related events happened, they are within this zone (say the cause is O, the effect event E has to be inside the light cone). I do not claim all events inside this zone are related, but could be in principle if I choose to. For some observers in a frame the two events could happen at same place and different time, so the earlier event is possible to be the cause of later event.

Rephrase myself again as summary on this part: The Casualty principle requires the time order of related events never flips, the time-like zone satisfies this. The related events have to happen in this zone. (Of course I have to prove yet that the space-like zone do not satisfy this, that is coming right away)

(2) $s^2 < 0$ space-like zone

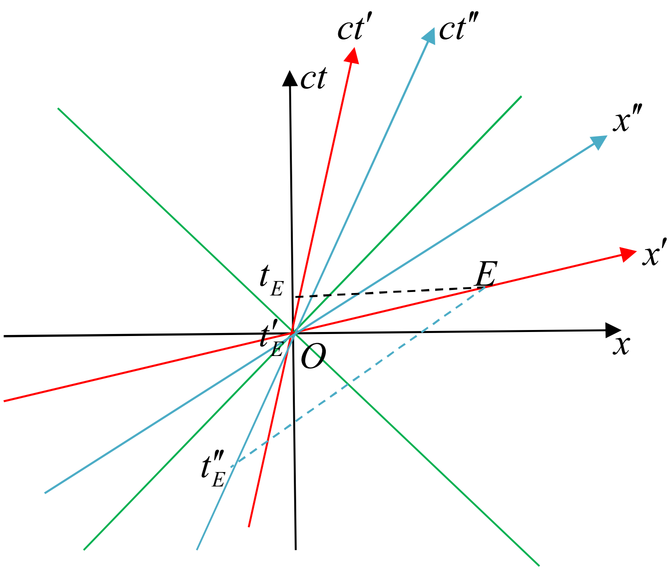
Similar setup for events O, E as above: for event E happened at (x,t) in space-like zone:

$$s^2 = c^2t^2 - x^2 < 0 \to |\frac{x}{t}| > c$$

Viewed in another frame:

$$t' = \gamma(t - \frac{vx}{c^2}) = \gamma t(1 - \frac{vx}{tc^2})$$

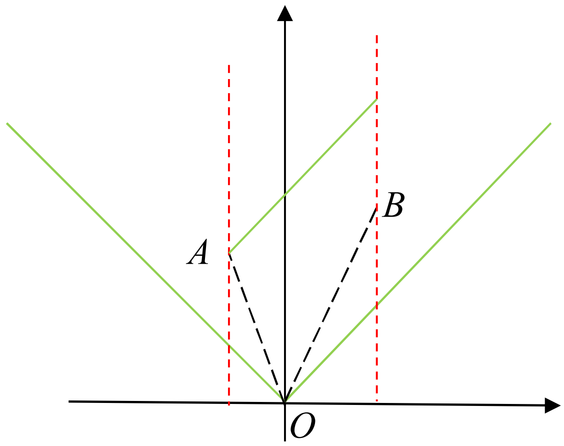
Now because of $|\frac{x}{t}| > c$, though v<c, it is possible that $1 - \frac{vx}{tc^2} < 0$ in this case. So the time order of events can be flipped (O before E for S observer but E before O for S'). This could also neatly shown using the Minkowski diagram:



As shown in the figure, the O, E happened at different place and time for S (O before E); but for S' the two events happened at same time but different places; for S", the E happened before O. Noticed that we can choose an inertial frame to make the events happened in this zone simultaneously with the origin, i.e. no time interval only space separation and that is the name space-like zone comes from.

The event O and any event E in this zone cannot be related due to the flips of time order and requirement of Causality principle. This zone thus is also called absolute elsewhere or absolute alibi, because in some frame a murder happened in New York simultaneously to I am writing this lecture here in Tsinghua, I could not possibly be the cause thus the suspect of the murder.

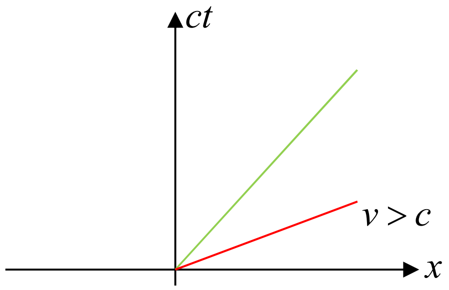
A relaxing example: Suppose you propose to two girls to marry you (you sleazy grifter), A is Alice, B is Brenda (for the feminist, do switch the name and sex). Event O is you called the poor girls (well you cannot call them same time, so let your twin brother be accomplice, or you send messages through cellular), the event A and B are the two girls accept your proposal (in your dream). This is depicted in the diagram below:



The A,B are inside your (O) light cone and you (the cause) propose and A,B accepted (the effect). Could A,B tell that they have been cheated as events A,B happened? They cannot, because the B is outside of the light cone of A. This tells you when and where to pick the opportunity to make the proposal. If you call A too early, then by the time you call B, she already learned you proposed to A. That is one advantage using Minkowski diagram ☺. But be warned, you cannot expect cheat could last: suppose Alice and Brenda do not move (they won't meet probably because of you) so their world lines are red vertical ones. Alice could send a radio signal and Brenda will receive it sometime later as indicated in the figure, and the con shall be exposed.

(3) Speed Limit to Particles and Signals

We have seen that the event in the space-like zone (absolute alibi) cannot be related to some event at origin. Now suppose we have a travelling particle or signal (signal will be made of some particles), whose speed is larger than c. Clearly the world line of this particle will be cause-effect related. It is just the time-path of this particle, I shall call it bullet fired by somebody. If the speed is larger than c, where the world line of this particle will end? The answer is in the space-like zone:



If the particle travels with velocity greater than c, the world line is the red one in the figure. Notice that the way we setup the axes, the smaller slope means higher speed. This bullet will end in the space-like zone, and it is cause-effect related to the origin (the firing of the gun), but we have seen that this will violate Causality principle, that the order of events can be flipped in such zone. This means in accordance with the Causality principle, no particle and signal can travel faster than c, the c is the speed limit imposed by the nature according to SR. So this becomes a popular test for the correctness of the SR. Till now, no violation had been observed or confirmed. A educational purpose movie "Ultimate Speed" shows the speed of accelerated electron will approach but never exceed the limit[135].

You may have heard of some of superluminal phenomena, such as two

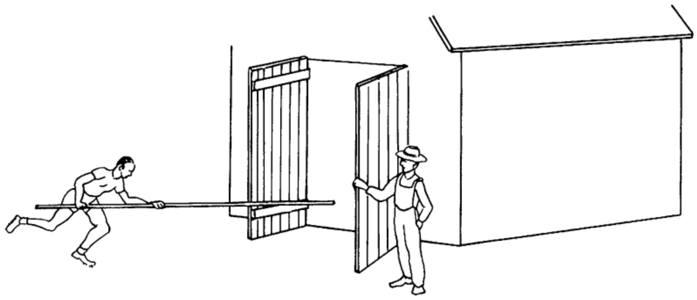
[135] The film "Ultimate Speed" by Bertozzi can be found on internet. The description is in: Bertozzi *Am. J. Phys*. **32**, 551 (1964).

rockets fly apart and each with 0.8 c to the ground observer. For the ground observer, one rocket flies 0.8c to right and the other 0.8 c to left. From the point of view of the ground observer, the two rockets are separating extremely fast, the relative speed according to ground is 1.6c. This is just apparent speed which you cannot to use to transmit signal to change the world, and we will see that the real speed for one rocket relative to another never exceeds c. Another example is later we shall see that the phase velocity of light could exceed c but that velocity also has no way to be applied in transmitting the signal.

12.6 Interesting "Paradox"

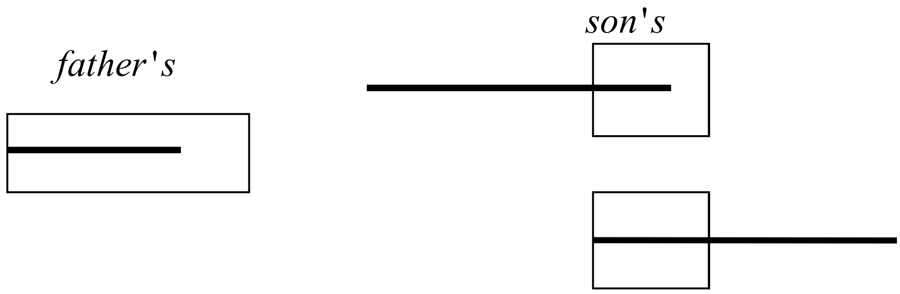
There are many interesting "paradox" in SR. The reason I put them in quotation is because these are not real paradox. It arises mostly from our old habit of thinking (inertia of mind) or sloppy description causes misunderstanding. I shall study a few typical ones in this section, they are fun and they are pedagogical (means good for education) and I hope by studying them, it helps your understanding on SR.

Example 1. The garage and car (or barn-yard) paradox



As the figure shows (it shows the barn-yard as in KK's problem 12.10, but I shall use garage and car all the values will be same as in the problem), a farmer has a garage with length 3/4 L_0, he also owns a car whose length is L_0. So the car does not fit in the garage. One day the old farmer read something about relativity, and it occurred to him that he can use length contraction to fit the car inside the garage. He asked his son to drive the car to speed of $\frac{\sqrt{3}}{2}c$, thus the car would be measured as $L_0/2$ in the farmer's (on ground) frame and will be fit into the garage. He will slam the door closed right at the moment that the tail-end of the car passes the door and trap the car inside. His son on the other hand, also knows something about SR, and he argued that it could not be done. Because if he drives the car and from his point of view, the garage will be measured shorter, contracted to 3/8 L_0, and won't fit the car even worse. The question is who is correct? And if indeed such thing is carried out, what will be the result? Maybe you would like to ponder on these questions (closed this notes) before read on.

The answer to the first question is that they are both correct. How could that be? You exclaim, the car is either fit or not fit in the garage. Yes within one frame the car is either fit or not, but in different frames the observation can be different! This is not a true paradox as someone is killed or not, but an observation depending on your point of view just analogous to stone falling from the sail of the boat, the boat observer will see a free fall and bank observer will see a projectile. More details will follow to illustrate this better.



The father's and son's point of view is illustrated by sketch above. To the father the tail-end of the car at the door and the head-end at some point inside the garage are simultaneous events. But we have learned that ***simultaneity is relative*** so it is not surprise that the son does not agree. To the son, the simultaneous event with the head-end inside garage (at certain location inside the garage) is the tail is way out side of the garage. Or the simultaneous event with tail at the door is the head already outside of the garage wall. These are examples of simultaneous event for the son, the head and end tail both inside the garage cannot be simultaneous from son's view, while it is simultaneous in father's.

Instead of raise the clock and argue the simultaneity as we did before in chapter 11, we now can use LT to solve the puzzle.

The key is which event happened first:

event 1: tail-end at the door, event 2: head-end hits the wall.

For the father event 1 happened before event 2; for the son event 2 happened before event 1. LT will predict exactly this.

In the father's frame the event 1 can be defined as (0,0), then the event will happen at $3/4L_0$, at a time of:

$$t=(\frac{3}{4}L_0-\frac{L_0}{2})/v=\frac{L_0}{2\sqrt{3}c}=\frac{\sqrt{3}}{6}\frac{L_0}{c}$$

The difference between the events are: $\Delta x=\frac{3}{4}L_0, \Delta t=\frac{\sqrt{3}}{6}\frac{L_0}{c}$, now use the LT to see how the son measures this difference, we only care about time order here, so:

$$\Delta t'=\gamma(\Delta t-\frac{v\Delta x}{c^2})=2(\frac{\sqrt{3}}{6}-\frac{3\sqrt{3}}{8})\frac{L_0}{c}=-\frac{5\sqrt{3}}{12}\frac{L_0}{c}$$

Indeed the event 2 happened before event 1 from son's point of view. Here the event 1 and event 2 are not related. If the event 1 is the head-end passes the door and event 2 is head-end hits the wall the two events are related, here is different, so no problem for flip of time order.

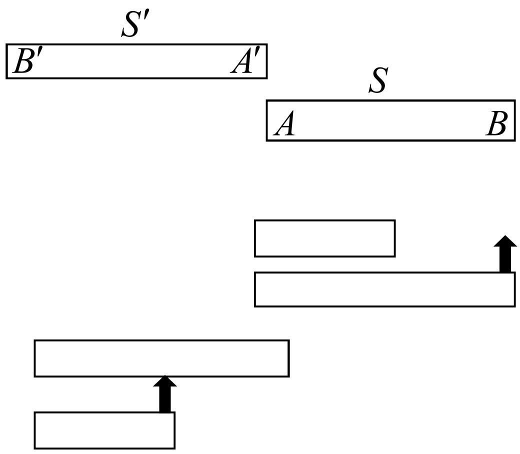
You may further argue if the head-end hits the wall first in son's point of view, how can the car still pushing forward so that the tail-end passes the door? Though the two observers will not agree the time order of the event 1 (the tail passes the door), they all agree that the tail passes door at some time. Will this be a contradiction? This is related to the second question of when the farmer slams the door closed at event 1, what is the result, i.e.

he opened the door later what he will see? From father's point of view, there is no problem for tail passes the door since it happened first. When he slams the door, he wants the car to be stopped. As the car stops, its length increases measured by the farmer, and it will hit the wall and if the wall is made of paper, the car will poke through and if the wall made of steel, the car will be crashed, so either the car or the garage will be damaged.

From the son's point of view, the head will hit the wall first and if it is made of paper, the head will poke through it while the tail passes the door. If the wall is made of steel, the head of the car will be stopped, however this interaction needs time to propagate through the car to the tail to slow down the motion there (***no real rigid body in SR, all interactions takes time)***. So before this interaction reaches the tail, the tail will still push forward passing the door though the head was stopped by the wall. The son probably will be severely hurt while was cursing his old fool. You see that either the car or the garage will be damaged in son's point of view as well.

Example 2. Star War paradox

Two spaceships each with proper length L and approach each other with a slight shift in vertical to avoid collision, the relative velocity is v.



They plan a war exercise. The plan is when the tail of ship S', the B' end overlaps with the head (A) end of ship S, the captain on ship S fires a photon torpedo from its end (B) simultaneously. From Captain S's view, it is harmless, the torpedo will miss the ship S' because due to length contraction, the S' will be shortened as shown in the middle sketch. But the captain S' objects this plan, he argued that from his point of view it is the S ship got shortened and if the S fires torpedo simultaneous when the B' and A end meets, his ship (S') will be hit as bottom sketch shows. What is going on? Whose description is correct and will the ship S' get hit or not?

The catch is the middle sketch and bottom sketch are describing different sets of event. The middle one is a pair of simultaneous events according to S; while the bottom one is a pair of simultaneous events according to S', they are not the same. If the war plan is carried out by the middle one, no ship will be hit, period. So there is no need for the fuss by the captain S'. Let's do the problem more quantitatively. It is easy to understand no hit from S point of view, the ship is hit by torpedo will be an event (if happened) that has to be agreed upon by observers in all frames, so if the observer in one frame (S) did not see the hit, then hit never happens. The S' has to reconcile with this fact instead of using the wrong intuitive picture (the bottom sketch) where the simultaneity is in S' not in S. Let's work out these using point of view of S'.

First need to define events:

Event 1: A,A' meet. This is x=0, t=0 and x'=0,t'=0

Event 2: A,B' meet.

Event 3: torpedo fired at B

We shall work out the coordinates of events in S then LT to get them in S'.

In S, the event 2 happened at $x_2=0$, with a time $t_2 = \frac{L}{\gamma v}$; event 3 happened at same time as event 2 but at the tail of S, so $x_3 = L, t_3 = \frac{L}{\gamma v}$.

In S', the event 2 happens at:

$x_2' = \gamma(x_2 - vt_2) = -L$ this is exactly expected

$t_2' = \gamma(t - \frac{v}{c^2}x_2) = \frac{L}{v}$ also expected from S' point of view.

Event 3 happens at:

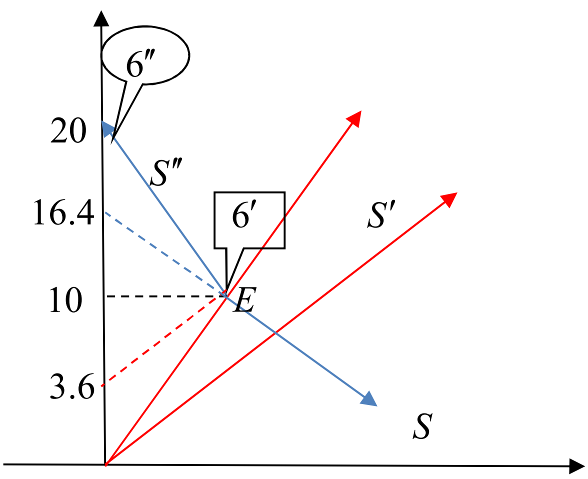
$$x_3' = \gamma(x_3 - vt_3) = \gamma(L - \frac{L}{\gamma}) > 0 = x_{A'}'$$

$$t_3' = \gamma(t_3 - \frac{v}{c^2}x_3) = \gamma(\frac{L}{\gamma v} - \frac{Lv}{c^2}) = \frac{L}{v} - \frac{\gamma Lv}{c^2} < t_2'$$

This means the torpedo will fire before event 2, and it fires ahead of the head-end of the S’ ship. If the war plan were carried out, the S’ will observe that the S ship fires a torpedo from its end B before the head A meets tail of S’, and will miss his ship. There is no paradox here only a misunderstanding of the war plans.

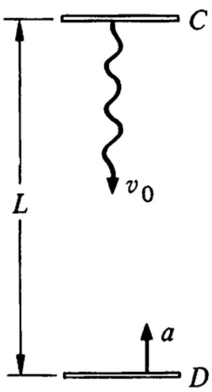
Example 3. Twins Paradox revisited

Same as before: Adam and Bob are twins. Bob took a trip travelling with spaceship v=0.8c ($\gamma = 5/3$)to a star E which is 8 ly away from earth. The round trip take 20 years in Adam’s time and only 12 years in Bob’s, so Bob is 8 years younger when he is back. The above conclusion is drawn from Adam’s point of view as shown before. We shall investigate this here from Bob’s point of view:



Actually all the calculation using SR had been summarized in the diagram and only a brief explanation will be necessary. For astronaut Bob, *Event 1* is taking off from earth, *event 2* is reaching star E. This will take 6 years (indicated as 6’ in the figure) according to Bob’s watch, the time elapse on earth would appear only 3.6 years ($\frac{6}{\gamma}$ a measurement of earth clock by one of observers in S' simultaneous with event 2: Bob reaches E). This is just time dilation effect from point of view of S', so Bob would think only 3.6 years passed on earth looking the record of his network of observers. Then *event 3* happened: Bob quickly go through deceleration and acceleration and end up in the spaceship traveling back with same speed. Something peculiar happened here, the measurement of earth clock by one of observers in S" simultaneous with event 3 will see 16.4 years on earth clock while all clocks in S" point to 6 years (assuming acceleration turn-around time is negligible). This means that if Bob checks the record data of his network of observers he would notice that before the turn-around the earth clock points to 3.6 and after turn around the earth clock points to 16.4, a time jump of 12.8 years in earth's time. This jump of time could be explained by the synchronization difference between the frames (change from S' to S", if both clocks in S' and S" are set to 6 years when they meet at event 3, while the clocks of S and S' are set to zero at the event 1, there will be a time difference in S when you switch the frames from S' to S"). However to fully understand this 12.8 years jump (the physics behind) we will have to talk about time dilatation effect due to gravity. After turn-around, it takes another 6 years in S" (6" in the figure) to be back to earth, that is another 3.6 years on earth time. So total time elapse on earth from Bob's point of view who deduce it from the record data of the observers is 3.6 (trip to E)+12.8 (jump at turn-around)+3.6 (return trip)=20 years. It is same as Adam's.

Now back to the mysterious time jump during the turn-around, it is due to the gravitational red shift (or time dilation) whose formula is derived in KK's pg 370 and 482:



In the accelerated frame (or equivalently in a gravitational field that points downward, in the reversed direction to acceleration), the times elapsed at the high end (C) and low end (D) are related by:

$$T_C = T_D(1+\frac{aL}{c^2})$$

In the case of our problem, the earth is at the high end (with large gravitational potential due to the acceleration frame), the time T_D is the time elapsed at low end, where Bob is making turn-around from velocity 0.8c to -0.8c, so:

$$T_D = \frac{2v}{a}$$

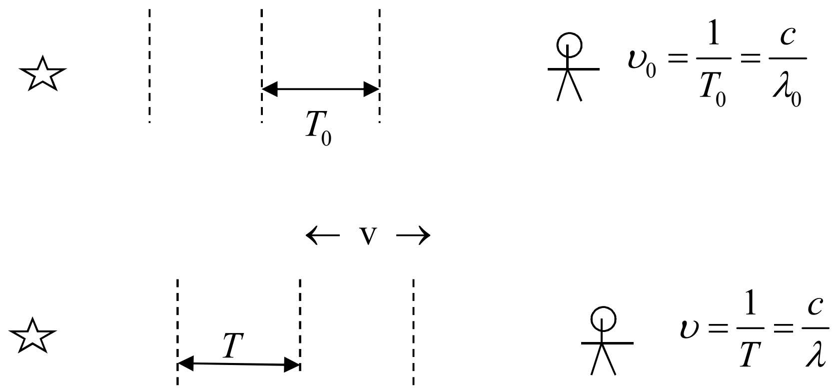
$$T_C = T_D(1+\frac{aL}{c^2}) = T_D + \frac{2vL}{c^2}$$

Assume the turn-around takes little time (a is huge) and this means the time elapsed on earth during the turn-around is approximately:

$$T_C \approx \frac{2vL}{c^2} = \frac{2 \times 0.8c \times 8ly}{c^2} = 12.8year$$

This is exactly the time jump observed by Bob during the turn-around and I showed you here that it is essentially caused by acceleration (or equivalently by gravitation).

12.7 Doppler Effect



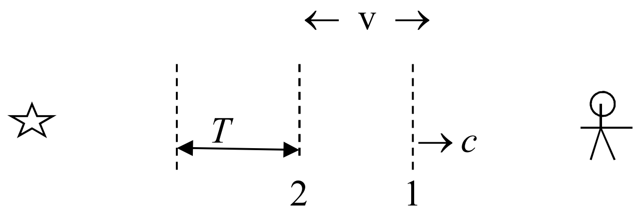
Doppler shift is the frequency (or wavelength) change when there is relative motion between the wave source and observer. It is easy to understand with the figure above. The star is a wave source, it generates light indicated as dashed vertical lines: The line indicate the peak (or valley) of the cosine wave and is called a wave front, or just treat it like the star sends out flash light pulses periodically, the lines are those light pulses. If the star and the observer are stationary as in the top sketch, the time interval between the pulses, the period, is T_0, and the frequency and wavelength can be calculated from the period.

If there is relative motion between the source and observer, the time interval between the pulses detected by the observer will be different from T_0. If the source and observer are approaching each other, it will take less time to detect the adjacent pulses and thus less T ($T<T_0$), (T is the time interval of detecting the adjacent pulses in the observer's frame) larger frequency υ and shorter wavelength λ, this is called 'blue' Doppler shift (since blue light has shorter wavelength in visible light spectrum); If the source and the observer are separating apart, moving away from each other, it will take more time to detect the adjacent light pulses and thus longer T, smaller frequency υ and longer wavelength λ. This is called 'red' Doppler shift.

Doppler shift of light finds wide applications in spectroscopy (Doppler broadening of spectral lines), it is a very important tool in Astronomy (detect the motion of stars by the shift, the expanding Universe and discovery of celestial bodies that do not emit light are found this way), and it is also used in our daily life, such as Doppler radar detecting the speed of a car.

The relation that we are looking for is the change of frequency (or wavelength) due to the relative motion v. The relevant events here are detection of light by a single observer, such as you really see the light with your eyes and count the time interval of received light pulses. As I mentioned before here is a case we really see the events in mundane sense, and we shall find that Doppler effect of light comes from two factors: *a) Relativity effect due to motion; and b)Difference in time for the light signal to be received due to the motion.*

(1) 1-D Doppler shift (Longitudal)

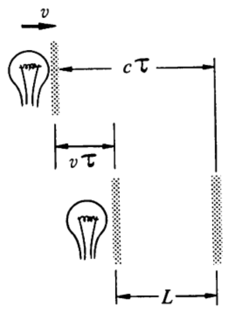


$$\upsilon = \frac{1}{T} = \frac{c}{\lambda}$$

The formula between frequency and velocity can be derived many different ways (we shall see another using energy-momentum conservation later), I will work out this in two ways: from the perspective of the observer and the source, it serves as example to apply what we have learned so far.

First from the perspective of the observer: I shall only consider the case that the source and observer are approaching (blue shift). From the observer's perspective, it is the source that is moving with v towards him. The time interval between the detection of pulse 1 and 2 is just:

$$T = \frac{L}{c}$$



The calculation of L needs to take those factors a), b) into account. As the figure shows, the interval that light is emitted by the source when viewed in this frame (light source is moving) is τ, during the emission period τ, the light source traveled and the distance between the pulses is: $L=(c-v)\tau$ (this is factor b, the motion of source shorten the arrival times of the pulses, like a chasing problem in classical mechanics). There is another factor, the time dilation effect of SR: in the frame of source, the light is emitted during the period of proper time $\tau_0 = T_0$ (the period observed by a stationary observer relative to the source, this is proper time because…you reason it please). So observed τ with the light source is moving (just like measuring the light clock on a train) is related to $\tau_0 = T_0$: $\tau = \gamma\tau_0$. (this is factor a) Put all these together, we have:

$$T=\frac{L}{c}=\frac{(c-v)}{c}\gamma\tau_0=(1-\beta)\frac{T_0}{\sqrt{1-\beta^2}}=\sqrt{\frac{1-\beta}{1+\beta}}T_0$$

$$\upsilon=\frac{1}{T}=\sqrt{\frac{1+\beta}{1-\beta}}\upsilon_0 \qquad (12\text{-}11)$$

This is the blue shift formula, and if the source is moving away from the observer, then the red shift formula will be (just change the –v to v):

$$\upsilon=\frac{1}{T}=\sqrt{\frac{1-\beta}{1+\beta}}\upsilon_0 \qquad (12\text{-}12)$$

At low speed limit, i.e. $\beta<<1$, the above formula will reduce to:

$$\upsilon=\sqrt{\frac{1+\beta}{1-\beta}}\upsilon_0=(1+\beta)^{\frac{1}{2}}(1-\beta)^{-\frac{1}{2}}\upsilon_0\approx(1+\frac{\beta}{2})(1+\frac{\beta}{2})\upsilon_0\approx(1+\beta)\upsilon_0$$

Define the frequency shift as: $\Delta\upsilon\equiv\upsilon-\upsilon_0$, then above equation is:

$$\frac{\Delta\upsilon}{\upsilon_0}=\beta=\frac{\mathrm{v}}{c} \qquad (12\text{-}13)$$

For the red shift case:

$$\frac{\Delta\upsilon}{\upsilon_0}=-\frac{\mathrm{v}}{c} \qquad (12\text{-}14)$$

(12-13) and (12-14) are popular in the application to estimate the Doppler shift, you seldom invoke (12-12) at low speed cases. Of course under high speed limit, such as light emitted by a high energy particle, then you have to use the exact formula.

Sometimes (especially for astronomer) people use wavelength shift:

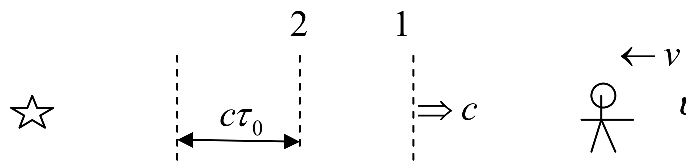
$\Delta\lambda=\lambda-\lambda_0$

When the shift is small ($\beta<<1$), we can use approximation from:

$\upsilon\lambda=c\rightarrow\lambda\Delta\upsilon+\upsilon\Delta\lambda=0\rightarrow\frac{\Delta\upsilon}{\upsilon}=-\frac{\Delta\lambda}{\lambda}$, and (12-13) or (12-14) can also apply to the wavelength shift.

Derivation of (12-12) from perspective of source:

Now the source is stationary, and emit out light at period of $\tau_0=T_0$, and the observer is moving with –v towards source:



$$\upsilon = \frac{1}{T} = \frac{c}{\lambda}$$

Event 1 is when pulse 1 hits the observer, say at (x_1, t_1) in frame of source S.

Event 2 is when pulse 2 hits the observer at (x_2, t_2). The time interval and space interval between events 1 and 2 in S can be easily computed:

$$\Delta x = x_2 - x_1 = -v\Delta t,\ \Delta t = t_2 - t_1$$
$$(c+v)\Delta t = c\tau_0$$

This gives:

$$\Delta t = \frac{c}{c+v}\tau_0;\ \ \Delta x = \frac{-vc}{c+v}\tau_0$$

These events when viewed in the S' frame in which the observer is stationary (we only care about time interval here):

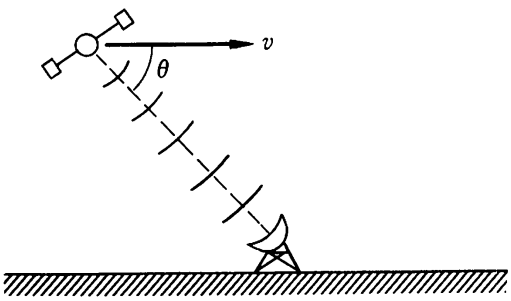
$$\Delta t' = \gamma(\Delta t + \frac{v}{c^2}\Delta x) = \gamma\tau_0(\frac{c}{c+v} - \frac{v^2 c}{c^2(c+v)}) = \gamma\tau_0\frac{c^2 - v^2}{c(c+v)} = \gamma\tau_0(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}\tau_0$$

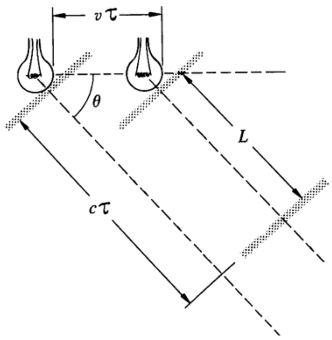
Or you may use the fact that in this case $\Delta t'$ is the proper time and $\Delta t = \gamma\Delta t'$ to get the same result quickly.

$\Delta t'$ is the T that we want, the time interval of detecting the adjacent two pulses by the observer in his own frame, and this will lead us exactly to (12-12).

(2) 2-D Doppler shift

The more general case of relative motion between source and observer is depicted in the figure below: Here I shall choose the frame that the observer is stationary (the solution by choosing source as stationary will be delayed after we learned velocity transformation). The direction of motion of the source and forms an angle θ with respect to the source-observer line (once again I stress that this angle is measured in frame S, observer is not moving)





The derivation of Doppler shift here is essentially same as above (from perspective of the observer, method 1 above):

The time interval between detection of adjacent pulses in O is:

$$\Delta t = \frac{L}{c}$$

$L = (c - v\cos\theta)\tau$ (factor b), and $\tau = \gamma\tau_0$ (factor a)

$$T = \Delta t = \frac{c - v\cos\theta}{c}\gamma\tau_0$$

$$\upsilon = \frac{1}{T} = \frac{\upsilon_0}{(1-\beta\cos\theta)\gamma} \qquad (12\text{-}15)$$

Notice that this will reduce to 1-D formula if $\theta = 0$ for blue shift and $\theta = \pi$ for red shift. It is also interesting to see that when $\theta = \frac{\pi}{2}$, which is called transverse Doppler effect, in this case:

$$\upsilon = \frac{\upsilon_0}{\gamma} \approx \upsilon_0 (1 - \frac{1}{2}\beta^2) \qquad (12\text{-}16)$$

It is a much smaller effect than the longitudal one which is first order in v/c; It is also interesting to see that (12-16) is just another way of writing the time dilation $\Delta t = \gamma \tau_0$ because in this case, only factor *a* (the SR effect) plays a role while factor b (time difference in receiving signals due to relative motion) does not.

12.8 Velocity Transformation

The basic question here is if the velocity of a particle in one frame (S) is u and if viewed by another observer in a moving frame (S') whose velocity is v relative to S, what is the particle's velocity u'. Noticed I shall use v to specify the velocity between moving frames and u for velocity of particles within one frame. Some examples are for a ground observer, two particles travel head-to-head, each with velocity 0.8c w.r.t. the ground observer. Then what is the relative velocity between the particles? A bullet is fired from the gun on a moving train, then what is velocity of the bullet to the ground observer? These are the basic problems we are going to deal with here.

A reminder for the symbols, because we have two kinds of velocities involved, the one specifying the velocity between frames and the velocity of particles within one frame. The γ, β are reserved for the v, the speed between frames; $u/c \equiv \beta_u, \gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}}$ are for speed of particles.

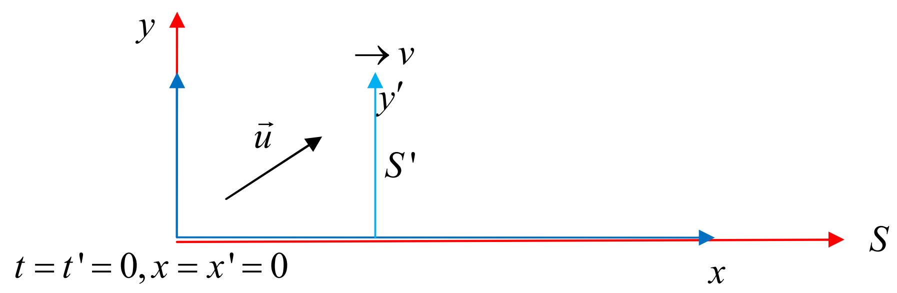
For the velocity of a particle within one frame, its definition is same as in mechanics:

$$u_x \equiv \frac{dx}{dt};\ u_y \equiv \frac{dy}{dt}$$

Its form in another frame will be:

$$u'_x \equiv \frac{dx'}{dt'};\ u'_y \equiv \frac{dy'}{dt'}$$

The basic question is what are the relations between these velocities, i.e. if we know the velocity in one frame, and what is the velocity in another frame that is traveling with v relative to the original one? Suppose we know the velocity in S, as the figure shows;



We use Lorentz Transform and definition of velocity:

$$u'_x \equiv \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx/c^2)} = \frac{dx/dt - v}{1 - \frac{v}{c}\frac{dx}{cdt}} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} = \frac{u_x - v}{1 - \beta\beta_{u_x}} \quad (12\text{-}17)$$

$$u'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - vdx/c^2)} = \frac{dy/dt}{\gamma(1 - \frac{v}{c}\frac{dx}{cdt})} = \frac{u_y/\gamma}{1 - \beta\beta_{u_x}} \qquad (12\text{-}18)$$

If the question is reversed that knowing the velocity in S', the u', then the velocity in S is:

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c}\frac{u'_x}{c}} = \frac{u'_x + v}{1 + \beta\beta_{u'_x}} \qquad (12\text{-}19)$$

$$u_y = \frac{u'_y / \gamma}{1 + \frac{v}{c}\frac{u'_x}{c}} = \frac{u'_y / \gamma}{1 + \beta\beta_{u'_x}} \qquad (12\text{-}20)$$

Comments: 1) The formula will reduce to the familiar simple velocity addition of Galileo Transform if both v and u << c, i.e. no high speed is involved. 2) The x and y components are different. Even though there is no change of the vertical positions upon transform, but the change in time during the transformation shows its effect in the vertical velocity components.

The derivations are simple enough, and let’s do some examples.

Example 1:

$$A \overset{\rightarrow 0.8c}{\bigcirc} \qquad\qquad \overset{\leftarrow 0.8c}{\bigcirc}\ B$$

What is the relative speed between A and B?

Let’s choose A to be our S’ frame, i.e. v=0.8c relative to ground. The particle B’s velocity is $u_x = -0.8c$ in S. In S’, the A is stationary, and B’s velocity is:

$$u'_{Bx} = \frac{-0.8c - 0.8c}{1 - \frac{0.8c}{c}(\frac{-0.8c}{c})} = \frac{-1.6c}{1 + 0.64} = -0.976c$$

The speed of B relative to A is less than c.

Example 2: Bullet-Gun model for light emission. The light is emitted by a high velocity particle (v relative to ground). If it emits a light along the x direction, what is the velocity of light observed on the ground? If it emits light towards direction perpendicular to the motion of particle, what is the velocity observed on ground?

Let S' to be the particle frame, and $u'_x = c$ in the first case, and $u'_y = c$ in the second.

(1) $u_x = \dfrac{c+v}{1+\frac{v}{c}\frac{c}{c}} = \dfrac{c+v}{1+\frac{v}{c}} = c$, $u_y = 0$

The light still travels with same velocity c for ground observer. If the light is –c (travel in reversed direction in S'), you will find it will be –c too in S. Actually as long as one of the velocity (v or u_x, the two are really symmetrical in the formula) equals c, the result will be c.

(2) $u'_y = c, u'_x = 0,$

$$u_x = \frac{0+v}{1+\beta 0} = v$$

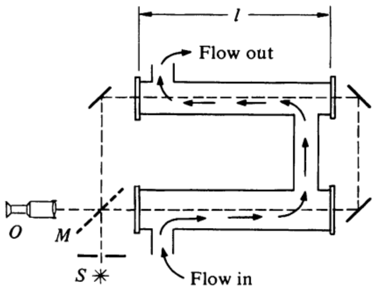
$$u_y = \frac{c/\gamma}{1} = \frac{c}{\gamma}$$

$$u^2 = u_x^2 + u_y^2 = v^2 + c^2(1-\frac{v^2}{c^2}) = c^2$$

So the total speed will be c too for ground observer, but the there will be an angle change for the direction of propagation of light, and we shall see it again in explaining the stellar aberration. For the more general case that the light is propagating along θ' direction is S', the x and y component for ground observer will be left for you to calculate and you will see the total speed will be c too.

Example 3 Explanation on Fizeau Experiment:

We had discussed in Chap.11 the result of Fizeau experiment:



The observed fringe shift of the interference pattern is:

$$N = \frac{4n^2 l}{\lambda c} f v \quad f = (1 - \frac{1}{n^2})$$

This is different from the result of simple addition of light velocity in the flowing water. Now we can compute the velocity of light in th eflowing water with velocity formula derived from LT:

The velocity of light travelling along the water in the frame S is: (c/n is the velocity of light in the water frame, i.e. water is stationary in S')

$$u_{along} = \frac{c/n + v}{1 + \dfrac{v}{nc}} = \frac{c^2 + ncv}{nc + v}$$

If v<<c, the above can be approxiamted as:

$$u_{along} = \frac{c/n+v}{1+\frac{v}{nc}} \approx (\frac{c}{n}+v)(1-\frac{v}{nc}) = \frac{c}{n}+v-\frac{v}{n^2}-\frac{v^2}{nc} \approx \frac{c}{n}+v(1-\frac{1}{n^2})$$

The velocity is not simply c/n+v.

For the light traveling against water flow:

$$u_{against} = \frac{c/n-v}{1-v/nc} = \frac{c^2-ncv}{nc-v}$$

The phase difference bewteen the two paths are:

$$\Delta\phi = \omega\Delta t = \frac{2\pi c}{\lambda}\Delta t$$

Number of fringes are:

$$N = \frac{\Delta\phi}{2\pi} = \frac{c}{\lambda}\Delta t = \frac{c}{\lambda}(\frac{2L}{u_{against}} - \frac{2L}{u_{along}}) = \frac{2Lc}{\lambda}(\frac{nc-v}{c(c-nv)} - \frac{nc+v}{c(c+nv)})$$
$$= \frac{2L}{\lambda}(\frac{2n^2cv-2vc}{c^2-n^2v^2}) = \frac{4Lv}{\lambda c}(\frac{n^2-1}{1-n^2\beta^2}) \approx \frac{4Lv}{\lambda c}(n^2-1)$$

Same as the Fizeau's observation.

Example 4. Velocity direction in S and S' and stellar aberation

For a particle travels with velocity $\vec{u}$ in one frame (S), it has angle with x-axis θ. Then the $u_x = |u|\cos\theta, u_y = |u|\sin\theta$ (refer to figure on pg 480). What is the particle's direction of travel (direction of velicity) when viewed by S'?

This is straightforward, let's call the angle with x'axis in S'is θ':

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{\dfrac{u_y/\gamma}{1-\beta\beta_{u_x}}}{\dfrac{u_x - v}{1-\beta\beta_{u_x}}} = \frac{u_y/\gamma}{u_x - v} = \frac{|u|\sin\theta}{\gamma(|u|\cos\theta - v)} \qquad (12\text{-}21)$$

Reversely, i.e. knowing the direction of velocity of a particle in S' and its direction viewed in S is (you can use velocity formula but just from symmetrical point of view, repalce v with –v):

$$\tan\theta = \frac{u_y}{u_x} = \frac{u'_y/\gamma}{u'_x + v} = \frac{|u'|\sin\theta'}{\gamma(|u'|\cos\theta' + v)} \qquad (12\text{-}22)$$

For the special case of light, I shall choose the source as S' frame, and ground to be S frame for consistency of symbols in the following discussions (of course there is nothing sacred of which is called S or S').

For light the total speed |u| or |u'| is always c (you should verify this as exercise in example 2 above, and of course this has to be from the postulate), so if light propagate along θ in S:

$$\cos\theta' = \frac{u'_x}{c} = \frac{(c\cos\theta - v)/c}{1-\beta\cos\theta} = \frac{\cos\theta - \beta}{1-\beta\cos\theta} \qquad (12\text{-}23)$$

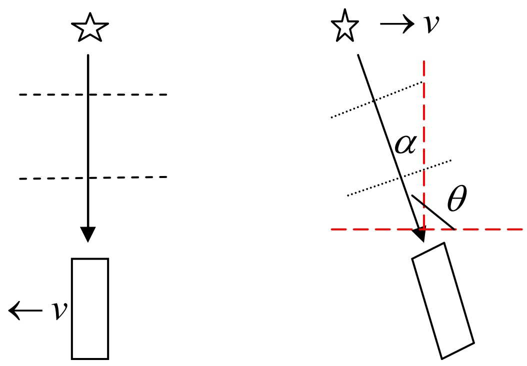
$$\sin\theta' = \frac{u'_y}{c} = \frac{\sin\theta}{\gamma(1-\beta\cos\theta)}$$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \qquad (12\text{-}24)$$

For reverse relations:

$$\cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'} \qquad (12\text{-}25)$$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)} \qquad (12\text{-}26)$$



With the above relations, we can explain the stellar aberation with SR. In the S' frame (star is not moving), the light emitted by the star is approxiamted by a plane wave (dashed line represent wave front), with angle $\theta' = -\frac{\pi}{2}$. Viewed by the ground observer (S' is moving with v), the angle of incoming light is given by (12-26):

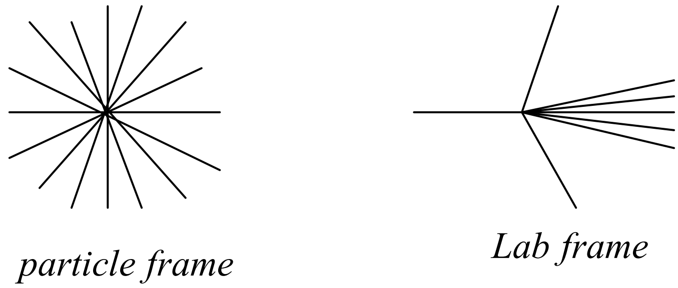
$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)} = -\frac{1}{\gamma\beta}$$

The aberation angle α is:

$$\tan\alpha = \cot\theta = -\gamma\beta \approx -\beta = -\frac{v}{c}$$

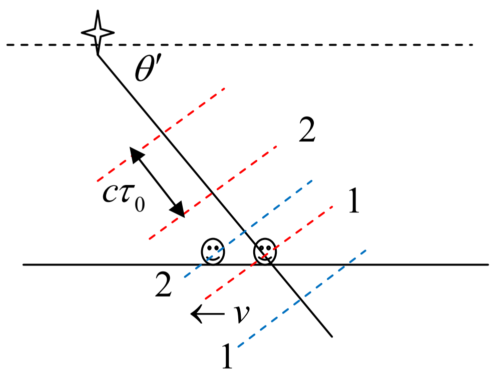
This is the aberation angle in chapter 11.

Another interesting result of (12-25), (12-26) is the search light effect for light emitted by particles travelling with high speed. For example the light is emitted by particle isotropically in its own frame S', but viewed in S, if the $\beta \to 1$, γ will be large, from (12-25) or (12-26), it is straightforward to see that $\cos\theta \to 1, \tan\theta$ small for different θ'. This means viewd in lab frame S, the light emitted would be strongly concentrated along the direction of the particles motion:



This is observed in sychrotron where laser light is generated by fast moving particles in a circular accelerator, and speed of particle can reach 0.9999c and the light emitted by it will be highly concentrated into forward direction in lab frame. (Please do some computation for v=0.9999c case and choose some θ' to see what are the θ, use Matlab if possible)

Example 4, Prove 2-D Doppler shift from perspective of source:



The events viewed in the S'frame (the light source is stationary, the observer is moving) is shown in the sketch. Event 1 is when the wave front 1 meets the observer (Smily), at ceratin (x_1', t_1'), event 2 happens, the wave front 2 meets Smily after certain interval:

$\Delta x' = -v\Delta t'$

$c\Delta t' + v\Delta t' \cos\theta' = c\tau_0$

So: $\Delta t' = \dfrac{c\tau_0}{c + v\cos\theta'}, \Delta x' = -\dfrac{vc\tau_0}{c + v\cos\theta'}$

These intervals will give us the time interval of the events in frame S, which is what we wanted:

$$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x') = \frac{\gamma c\tau_0}{c + v\cos\theta'} - \frac{\gamma v^2 \tau_0}{c(c + v\cos\theta')} = \gamma\tau_0 \frac{c^2 - v^2}{c(c + v\cos\theta')}$$
$$= \gamma\tau_0 \frac{1-\beta^2}{1+\beta\cos\theta'} = \frac{\tau_0}{\gamma}\frac{1}{1+\beta\cos\theta'}$$

$$\upsilon = \frac{1}{\Delta t} = \gamma\upsilon_0(1+\beta\cos\theta')$$

This formular compared with (12-15), you will notice the difference. This is of course just because the above euqation is expressed in terms of angle in S' frame, while (12-15) is viewed from angle in S, apply the relation (12-23), change the θ' into θ:

$$\upsilon = \gamma\upsilon_0(1+\beta\cos\theta') = \gamma\upsilon_0(1+\beta\frac{\cos\theta-\beta}{1-\beta\cos\theta}) = \gamma\upsilon_0\frac{1-\beta^2}{1-\beta\cos\theta} = \frac{\upsilon_0}{\gamma(1-\beta\cos\theta)}$$

This is exactly (12-15) expressed in angle from observer's perspective.

Example 5 One universal velocity

In SR, we state that the speed of light is a universal one, same in all frames. In this example we shall see that SR only has room for ONE universal velocity. Let's formulate problem like this: Suppose we have another speed d besides c which is universal same in all frames, what will happen?

Using the velocity formula, say in one frame S, c is c, d is d. In another frame S' moving with v, we have seen that c'=c in S', what is d'?

$$d' = \frac{d-v}{1-\beta\beta_d} = \frac{c^2(d-v)}{c^2-vd}$$

Since d'=d if it is universal, then:

$$\frac{c^2(d-v)}{c^2-vd} = d \to c^2d - c^2v = c^2d - vd^2 \to vc^2 = vd^2 \to |c| = |d|$$

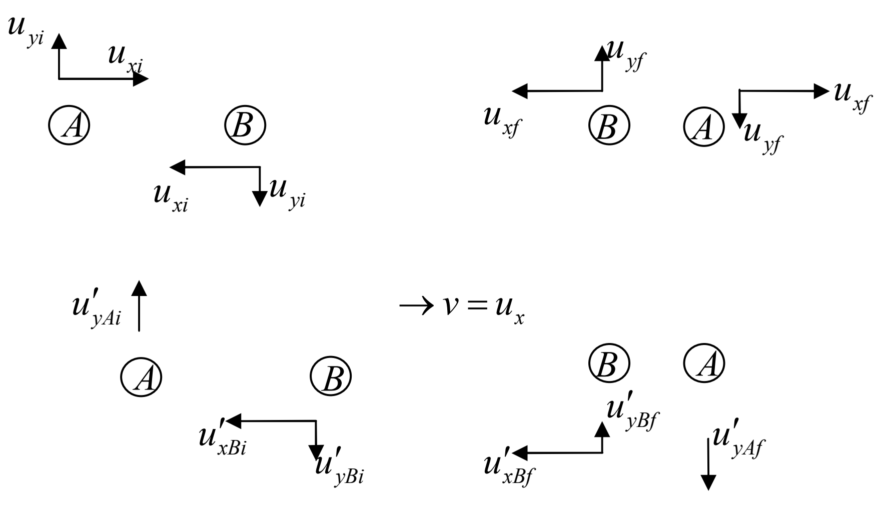
So this d has to be same value as c. The fundamental postulate of SR requires one universal velocity, and this velocity is that of light had be tested to be right by numerous experiemnts.

Chapter 13 Energy and Momentum and 4-Vectors

In this chapter we are going to study the dynamical process in SR. Instead of working on equation of motion (relation between force and acceleration), I shall adopt conservation as working horse because this is the easiest approach. First then I need to define what are momentum and energy in relativity sense, what are their formulas expressed by terms already defined? Another important issue is for the momentum and energy defined in such way, do they satisfy the relativity principle, i.e. If the energy and momentum is conserved in one frame, do they conserve in other frames too? And also the correspondence principle, i.e. at low speed limit, do the relativistic energy and momentum defined reduce to old friends in Newtonian mechanics?

Actually these important questions (relativity principle and correspondence principle) will be my guideline in the first ad hoc approach to the energy-momentum, starting from assumptions that their forms take some similarity with classical ones and find out what their relativistic formula are. Then we shall work some dynamical examples to get familiar with these relativistic formulas for energy and momentum. Finally, a powerful concept and definition of vector in 4-D space-time, called 4-vectors will be introduced, and we shall see that the SR can be formulated quite elegantly out of this approach. Both Lorentz Transform and relativistic energy-momentum shall be re-derived from this.

13.1 Momentum



Let's consider a special elastic collision in Newtonian mechanics (in the special relativity, all collisions are energy conserved, so here elastic means the kinetic energy is conserved). One simple one is illustrated by the sketch: $|u_{xAi}|=|u_{xB_i}|=|u_{xAf}|=|u_{xBf}|=u_x; |u_{yAi}|=|u_{yB_i}|=|u_{yAf}|=|u_{yBf}|=u_y$ in lab frame S. (u_{yAi} means y component of velocity of particle A initially before the collision), A and B have same mass. The collision is a glancing type, where the x-component of velocity does not change after collision while the y component of velocities flipped between A and B. This simple collision is certainly possible in classical mechanics and satisfies both momentum and energy conservation.

Now we know SR, and we know relativity principle. So naturally if I asked what happened if I view this collision in another frame, say the frame S' that move along x-direction with A, so $v=u_x$. In this frame the momentum conservation (as well as energy conservation, we shall focus on momentum first) should also hold. But we shall see that if we take the classical form of momentum p=mv, the momentum will not be conserved in S' though it does in S. This suggests the classical form of momentum is not appropriate in SR, it is a low speed limit of the correct formula (since it works fine in Newtonian-mechanics at low speed). Our job in this section is to find the correct formula of momentum in SR.

First thing is first, so let me show you the classical definition is not appropriate. The strategy is I shall use p=mv and show that if momentum of this type is conserved in S (top sketch), it is not in S' (bottom sketch).

The setup clearly shows the total momentum is zero before and after collision in S. In S', I shall only focus along y' direction (for simple computation). Using the velocity formula derived last chapter, it is straightforward to express the u' in terms of u:

$$u'_{yAi} = \frac{u_y / \gamma}{1 - \beta\beta_{u_{xAi}}} = \frac{u_y / \gamma}{1 - u_x^2 / c^2}$$

$$u'_{yBi} = \frac{-u_y / \gamma}{1 - \beta\beta_{u_{xBi}}} = -\frac{u_y / \gamma}{1 + u_x^2 / c^2}$$

$$p'_{yi} = mu'_{yAi} + mu'_{yBi} = \frac{mu_y}{\gamma}\left(\frac{1}{1 - u_x^2 / c^2} - \frac{1}{1 + u_x^2 / c^2}\right) > 0$$

$$u'_{yAf} = \frac{-u_y / \gamma}{1 - \beta\beta_{u_{xAf}}} = -\frac{u_y / \gamma}{1 - u_x^2 / c^2}$$

$$u'_{yBf} = \frac{u_y / \gamma}{1 - \beta\beta_{u_{xBf}}} = \frac{u_y / \gamma}{1 + u_x^2 / c^2}$$

$$p'_{yf} = mu'_{yAf} + mu'_{yBf} = \frac{mu_y}{\gamma}\left(\frac{1}{1 + u_x^2 / c^2} - \frac{1}{1 - u_x^2 / c^2}\right) < 0$$

The momentum defined classically as mv is not conserved in S'.

What we need is finding a formula of momentum which will be conserved in all inertial frames, i.e. if the total momentum is conserved in one frame, it is also conserved in other inertial frames in accordance to relativity principle.

Comment: Conserved is not same as invariant. Total momentum of a closed system is conserved, say during the collision process, before and after the collision, the total momentum is A in one frame. Observing the process in another frame, the total momentum could be B, it is still conserved if before and after the collision the value is B (which could be different from A). Invariant value is unchanged upon change of frames, such as space-time interval of an event. Conserved physical property and invariant value are both important because they offer powerful tools in analysis of physical processes.

In order to find the correct formula for momentum, we start from the requirement that the total momentum needs to be conserved in all inertial frame. There is another clue from correspondence principle that the momentum is mv at low speed limit. So we shall assume the correct formula for momentum is still p=mv, but m is not a constant as in classical mechanics, m here may be a function depending on v. Strictly speaking, I should write m(v) or f(v)m , but I shall keep this in mind and just using m in the following arguments.

Let's consider a case of collision between particles in one dimension for simplicity. The total momentum is 0 (this is not as specific as it seems, we can actually choose an inertial frame in which the total momentum is zero) in frame S:

$P = m_1 u_1 + m_2 u_2 = 0$

m, u are mass (not a constant number here) and velocity of individual particles. We can define the total mass (an assumption that relativistic mass is additive):

$M = m_1 + m_2$ and like the center of mass frame in classical mechanics, we can call this frame S as our center of mass frame in SR in which total momentum is zero.

$P = MV$ and V=0 in C.M frame. This is like putting a "black box" over the two particles (or observing the two-particle system from far way so that we cannot tell the detail), the total momentum of the system is equivalent to that of the "black box".

Now let's view the collision process from a frame (S') that is moving with v w.r.t. the S. The mass and velocity of particles are: m'_1, m'_2, u'_1, u'_2, and total mass is: $M' = m'_1 + m'_2$, the total momentum need to be conserved too in S' and it should be same as M' traveling with –v viewing in S', i.e.

$$P' = -M'v = m'_1 u'_1 + m'_2 u'_2 \to -(m'_1 + m'_2)v = m'_1 u'_1 + m'_2 u'_2$$

$$\frac{m'_1}{m'_2} = -\frac{v + u'_2}{v + u'_1}$$

From the velocity transformation:

$$u'_1 = \frac{u_1 - v}{1 - u_1 v / c^2}; u'_2 = \frac{u_2 - v}{1 - u_2 v / c^2}$$

$$\frac{m'_1}{m'_2} = -\frac{v + u'_2}{v + u'_1} = -\frac{\dfrac{v - u_2 v^2 / c^2 + u_2 - v}{1 - u_2 v / c^2}}{\dfrac{v - u_1 v^2 / c^2 + u_1 - v}{1 - u_1 v / c^2}} = -\frac{u_2}{u_1}\frac{1 - u_1 v / c^2}{1 - u_2 v / c^2} = \frac{m_1}{m_2}\frac{1 - u_1 v / c^2}{1 - u_2 v / c^2}$$

I used $P = m_1 u_1 + m_2 u_2 = 0$ above.

$$\frac{m'_1 / m_1}{m'_2 / m_2} = \frac{1 - u_1 v / c^2}{1 - u_2 v / c^2}$$

We appear stuck with this relation, and I shall use a 'trick' (the reason I know this is because I sort of cheated since I know the final answer) and derive a useful relation:

$$\beta \equiv v/c, \gamma \equiv 1/\sqrt{1-\beta^2}; \beta_u \equiv u/c, \gamma_u \equiv 1/\sqrt{1-\beta_u{}^2}; \beta_{u'} \equiv u'/c, \gamma_{u'} \equiv 1/\sqrt{1-\beta_{u'}^2}$$

what are the relation between them?, the v, u, u' are related by the velocity transformation.

$$\beta_{u'} \equiv \frac{u'}{c} = \frac{\beta_u - \beta}{1-\beta_u\beta}$$

$$1-\beta_{u'}^2 = \frac{1+\beta_u^2\beta^2 - 2\beta_u\beta - \beta_u^2 - \beta^2 + 2\beta_u\beta}{(1-\beta_u\beta)^2} = \frac{(1-\beta_u^2)(1-\beta^2)}{(1-\beta_u\beta)^2}$$

$$\gamma_{u'} = \gamma\gamma_u(1-\beta_u\beta) \qquad (13\text{-}1)$$

(13-1) is a very useful relation, it will pay off if you remember it in this chapter.

Then:

$$\frac{m_1'/m_1}{m_2'/m_2} = \frac{1-u_1v/c^2}{1-u_2v/c^2} = \frac{\gamma_{u_1'}/\gamma_{u_1}\gamma}{\gamma_{u_2'}/\gamma_{u_2}\gamma} = \frac{\gamma_{u_1'}/\gamma_{u_1}}{\gamma_{u_2'}/\gamma_{u_2}}$$

The masses have to satisfy this relationship in order to have momentum conserved in all frames. If we have:

$\frac{m'}{m} = \frac{\gamma_{u'}}{\gamma_u}$, this will satisfy the above requirement[136]. If the u=0, the mass of the stationary particle is called **rest mass**, and is assigned to symbol m_0, and for the particle under motion we have:

[136] Actually m'/m could only be proportional to the ratio of gammas. We can use correspondence requirement that the mass are reduced to classical form at low speed limit to see that the proportional coefficients is 1. The whole purpose of these argument to get the correct formula of momentum in SR is not a water tight proof and I do not intend to. It served to show you the physical reasoning why the momentum in SR has to be in the formula (13-3), a better and more elegant proof (or definition of momentum) will be given in 4-vector section.

$m(u) = \gamma_u m_0$ (13-2)

The momentum of the moving particle with velocity u is:

$p = \gamma_u m_0 u$ (13-3)

The mass in (13-2) is called relativistic mass, while the mass we used in Newtonian mechanics is the rest mass (the low speed limit). I shall adopt the convention that uses m_0 as much as possible and include the SR in the γ_u, this way when speaking of mass, it only means rest mass because the relativity mass has another equivalent term for it that we shall see later (it is energy), and I shall use m_0 as reminder. However in case I get sloppy and careless, and use m(u) in the formula and relativity mass in statement, please forgive me.

The (13-3) comes from argument in 1-D. It can be extended to higher dimension straight forward:

$$\vec{p} = \gamma_u m_0 \vec{u}, \ \ \gamma_u = \frac{1}{\sqrt{1 - u^2 / c^2}} = \frac{1}{\sqrt{1 - (u_x^2 + u_y^2 + u_z^2) / c^2}} \qquad (13\text{-}4)$$

Once again I shall stress that the usefulness to express momentum as (13-3) or (13-4) is this definition will have momentum conservation in all frames. To build up your confidence, may be you try these formula for momentum and rework the example in the beginning (elastic collision between identical particles) example to see that if the momentum is conserved in S, it is conserved in S' too, it not a bad practice.

Example: Rocket velocity in relativity:

We have worked rocket velocity in classical mechanics using conservation of momentum before (chap. 5). There for a rocket with mass M, and dm fuel is ejected with velocity u_0 relative to the rocket during some time interval, we have:

$Mdu = -u_0 dM$ and $u_f = u_0 \ln \frac{M_i}{M_f}$ all M are rest masses.

In SR, I will choose the rocket as an instantaneous inertial frame S'. i.e. viewed from ground, the rocket will travel at certain speed u at an instant. In this S' the rocket will start from 0 velocity and increases speed, and we have (classical result applies to the S'):

$M'du' = -u_0 dM'$ M' equals the rest mass (since in S', the initial u'=0) , u_0 is the thrust velocity of fuel and du' is the increase of speed, all in S'. For the ground observer the velocity of the rocket will become:

$$u + du = \frac{u + du'}{1 + udu'/c^2} \approx (u + du')(1 - udu'/c^2) = u + du' - \beta^2 du'$$

(the higher order of small term is neglected above)

$du = (1 - \beta^2)du'$, put this into the mass-speed change relation:

$$\int du' = \int -u_0 \frac{dM'}{M'}$$

Note it is not legitimate to integrate this as it is, because the S' is an instantaneous rest frame, you cannot do the integration within S', there will be many S's in the processes with different instant speed u. However we can do the integral in the ground frame:

$$\int_0^{u_f} \frac{du}{1 - u^2/c^2} = -u_0 \int_{M_i}^{M_f} \frac{dM'}{M'}$$

$$\frac{c}{2}\ln\frac{1+u_f/c}{1-u_f/c}=u_0\ln\frac{M_i}{M_f} \qquad (13\text{-}5)$$

u_f is the final speed viewed from ground, u_0 is the fuel speed relative to rocket, and M's are rest masses. This is the formula for rocket speed in SR. If we want to have higher final velocity, we need to have large u_0 and huge load of fuel (bigger mass ratio). We cannot accelerate the rocket to speed of c, that would require final mass down to null. Also the most efficient way to propel rocket is with photons because this will give us biggest thrust speed $u_0 = c$.

13.2 Energy

We shall not consider the gravity in SR and start from kinetic energy of motion. I will adopt an ad hoc approach here (making assumptions rooted from Newtonian mechanics to get the formula for energy) for now and leave the elegant method (4-vectors) for later. In classical mechanics, kinetic energy is introduced by work-energy theorem: $\Delta K = Work = \int \vec{F} \bullet d\vec{r}$ In SR, the relation between force and motion is no longer F=ma, but let's assume that it is in the form of:

$$\vec{F}=\frac{d\vec{P}}{dt} \qquad (13\text{-}6)$$

This cannot be proved like the 2nd law and will be treated as definition of force[137] in relativity. With the force defined as above and relativistic momentum in last section, we can now find out energy using work-energy theorem (assume it still applies). I will write the work-energy theorem with differential form (with assumption that m_0 does not change over time):

$$dK = \vec{F} \cdot d\vec{r} = \frac{d\vec{P}}{dt} \cdot d\vec{r} = d\vec{P} \cdot \vec{u} \qquad (13\text{-}7)$$

$$d\vec{P} \cdot \vec{u} = d(\gamma m_0 \vec{u}) \cdot \vec{u} = m_0 \gamma d\vec{u} \cdot \vec{u} + m_0 \vec{u} \cdot \vec{u} d\gamma$$

$\gamma = \dfrac{1}{\sqrt{1 - u^2 / c^2}} \rightarrow c^2 \gamma^2 - \gamma^2 u^2 = c^2$ take differentials on both sides:

$$2c^2 \gamma d\gamma - \gamma^2 d(\vec{u} \cdot \vec{u}) - \vec{u} \cdot \vec{u} d\gamma^2 = 0$$
$$2c^2 \gamma d\gamma - 2\gamma^2 \vec{u} \cdot d\vec{u} - 2\gamma \vec{u} \cdot \vec{u} d\gamma = 0 \rightarrow c^2 d\gamma = \gamma \vec{u} \cdot d\vec{u} + \vec{u} \cdot \vec{u} d\gamma$$

This is just what the expression on the R.H.S. in equation (13-7):

$$dK = m_0 c^2 d\gamma = d(\gamma m_0 c^2) \qquad (13\text{-}8)$$

Following our convention so far, I should use the symbol γ_u above and I will do that in the following formula. If we start from u=0, stationary (where kinetic energy is 0 by convention) and reach certain u finally, the kinetic energy change of the process is:

$$\Delta K = \int_i^f dK = \int_0^u d(\gamma_u m_0 c^2) = \gamma_u m_0 c^2 - m_0 c^2 \qquad (13\text{-}9)$$

If we do not start from 0 but u_1 to u_2, then the kinetic energy change is:

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137 I realized that I may contradict my own statement in chapter 4 that F=dP/dt does not define the force. What I mean here is that the momentum change tells us there will be interaction which we will call force, and change of momentum can be a measure of how big such interaction (force) is. It cannot reveal the nature of the interaction (due to gravitation, electro-magnetic, or strong interaction of nuclear force?) which has to be investigated separately. So the better statement should be the dp/dt is a measure of force, but I am a little sloppy here and as in many other books. For example, refer to Goldstein's Classical Mechanics (2nd edition) section 7-6.

$$\Delta K = \gamma_{u_1} m_0 c^2 - \gamma_{u_2} m_0 c^2$$

First thing is that at low speed limit the above relation will just reduce to the 1/2mu^2 in Newtonian mechanics (prove yourself from 13-9 by expansion of γ_u). More important the above relation suggests we can express the general energy term:

$$E = \gamma_u m_0 c^2 = m_u c^2 \qquad (13\text{-}10)$$

The kinetic energy is just the energy difference between the two states of motion. m_u is the relativistic mass defined in (13-2).

13.2-1 Equivalence between energy and mass

The famous equation (13-10) clearly shows the equivalence between energy and mass. The relativistic mass is related with energy just by a constant factor c^2, knowing one is equivalent to knowing the other. That is why I stated earlier that I shall use mass referring to the rest mass as much as possible in this course, and relativistic mass is just energy.

The energy in form of (13-10) includes all forms of energy, kinetic, heat, nuclear etc. The conservation of energy in relativity will be conservation of total energy of a closed system and is same as relativistic mass conservation. For example in a complete inelastic collision between two particles, two particles collide to from one particle. In previous Newtonian treatment of this kind inelastic collision, the mechanical energy is not conserved. In SR, the kinetic energy is still not conserved, but the total energy is conserved (if the system is closed), so the E in form of (13-10) is conserved.

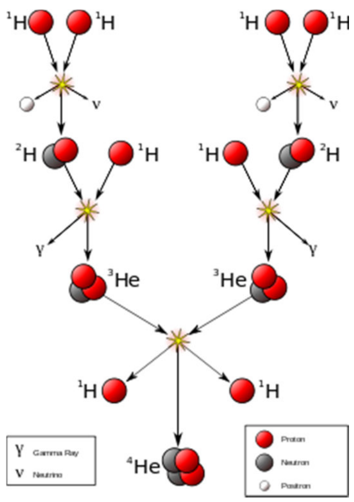
I shall rewrite (13-9) in form of:

$$\gamma_u m_0 c^2 = K + m_0 c^2 \rightarrow E = K + m_0 c^2 \quad (13\text{-}11)$$

This tells us the total energy of a moving particle can be seen consisting of two parts: kinetic energy related to the motion and internal energy (heat, internal potential etc) of the particle. The internal energy $m_0 c^2$ is just the total energy of a motionless particle and thus is called rest energy, and this rest energy is equivalent to rest mass in the same sense as total energy is equivalent to relativistic mass.

In the example of inelastic collision above, the loss of kinetic energy will change into other energy forms, such as heat and/or internal potential increase, such as electron being excited to higher energy states, and this will be equivalent to increase of rest mass of the final particle. More drastic examples will be creation of new particles through collision between high energy particles in accelerator. When two high energy particles collide, their energy can be transformed into new particles with different rest mass, a famous example will be collision of γ photons with other particles (a nuclei or another photon) to create electron-positron pair. (we will see some of example in following sections).

Another example will be increase (or decrease) of rest mass of a stationary object when change its internal energy (rest mass and internal energy is equivalent). If you heat up an object and you weigh the object on balance, its rest mass will increase (of course the object needs to be isolated after heating). In practice, such energy change is so small comparing to the original m_0c^2 because of large factor c^2. The object need to be heated to absurdly high (which will probably melt everything it touches) to noticeable difference. Another example is formation of hydrogen atom by proton (rest mass=930MeV)[138] with an electron (rest mass 0.5MeV), during the process 13.6 eV energy will be lost. So the final rest mass of hydrogen will be slightly smaller than the sum of rest mass of electron and proton but the difference ratio is on the order of 10^{-8}. An opposite example is nuclear weapons in which during the nuclear fission (atomic bomb) or fusion (hydrogen bomb), the fraction of mass change during the reaction will release humongous amount of energy:



[138] In relativity, the mass is usually expressed in terms of energy. So a rest mass of electron is 500keV=0.5Mev and proton mass is about 1800 times electron, which is about 900MeV. Muon is about 200 times that of electron. You should be able to compute this from conventional units, with c^2 factor and 1ev=$1.6X10^{-19}$J

This is a chain reaction in Sun of hydrogen fusion[139], the last step is two Helium3 combined to from Helium4 and 2 protons:

$${}_{2}^{3}He+{}_{2}^{3}He\rightarrow{}_{2}^{4}He+2p^{+}+12.9MeV$$

The released energy (the $12.9MeV$) can be estimated from rest masses:

$1au=931MeV$ (atomic unit of mass)

$$M_{{}^{3}He}=3.0160293au;M_{{}^{4}He}=4.002602au;M_{p^{+}}=1.007276au$$

Though only a small percent of mass changes after the nuclear reaction, the energy released is larger than the chemical reactions (Mev comparing to eV)

The first experimental demonstration of relativistic mass is by Bucherer in 1909[140]. It is essentially a measurement of charge/mass ratio of electrons. The electrons with certain speed are selected and pass through a magnetic field perpendicular to the electron's motion. The magnetic Lorentz force will bend the electron and by measuring the radius of this bending, the e/m ratio can be determined (it is a standard high school practice, but with modification on the mass; the detail of the motion in SR can be derived after learning chap. 14, here only gives the result):

$$euB=\gamma_{u}m_{0}u^{2}/R\rightarrow\frac{e}{\gamma_{u}m_{0}}=\frac{e}{m_{u}}=\frac{u}{RB}$$

At different speed u the e/m ratio can be computed from meausred R,B, and it is not a constant as in Newtonian mechanics, and the plotted ratio

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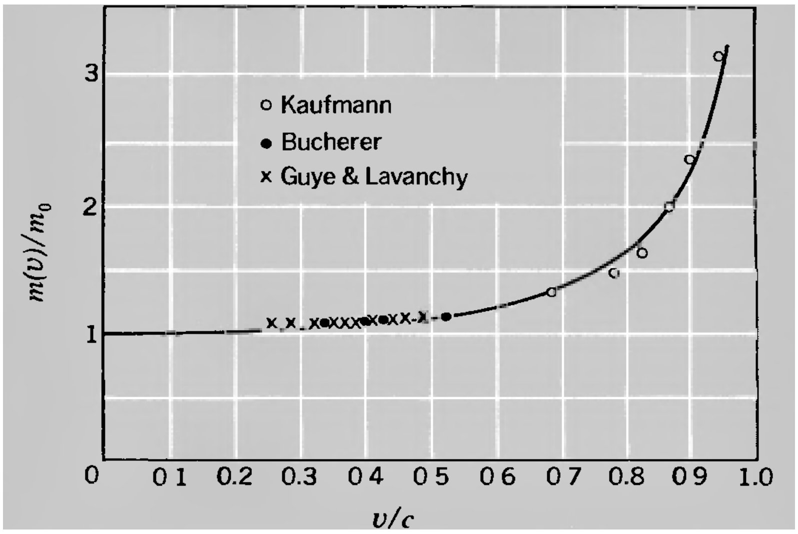
[139] Taken from Wiki under "nuclear fusion".

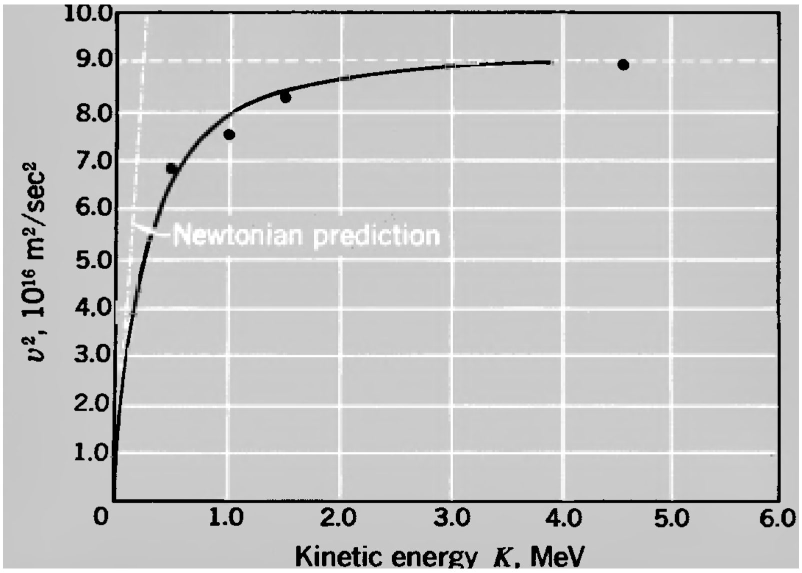
[140] A.H. Bucherer, *Ann. Physik* **28**, 513 (1909)

m/m_0 does show $\gamma_u = 1/\sqrt{1-u^2/c^2}$:



Comment: this experimental proof of relativistic mass is based on one assumption that the charge is invariant upon motion. This is reasonable considering the otherwise situation if charge is dependent on the motion then the neutrality of matter will depend on motion. In one frame an atom may appear neutral while in another moving frame, the atom may be charged then this will violate the relativity principle (conservation of charge due to gauge symmetry in EM) .

One experimental demonstration of the (13-9) is the 'ultimate speed' by Bertozzi mentioned before:



The electron under different acceleration will approach speed limit c but cannot exceed it. Though the increase of speed is small (only a few percent between the last two points in the figure), the increase of kinetic energy is tripled. This can be understood due to increase of relativistic mass. This kind of experiment proved the energy form (13-10) and also demonstrated that for massive particles (rest mass >0), it cannot reach speed limit c, because this will make its relativistic mass to infinity, equivalent to infinite amount of energy.

13.2-2 Relation between Energy and Momentum and Massless Particle

In Newtonian mechanics, we have $K=2P^2/m$. Now in SR, the relation is between total energy and momentum. This can be derived from the formula for momentum and energy:

$$P = \gamma_u m_0 u$$
$$E = \gamma_u m_0 c^2$$

γ_u is also related to u, and we can get rid off u from the above equations:

$$P^2 = \frac{1}{1-u^2/c^2} m_0^2 u^2 \to P^2c^2 - P^2u^2 = m_0^2 u^2 c^2 \to u^2 = \frac{P^2c^2}{P^2 + m_0^2 c^2}$$

$$E^2 = \frac{1}{1-u^2/c^2} m_0^2 c^4 \to E^2c^2 - E^2u^2 = m_0^2 c^6$$

$$E^2(c^2 - \frac{P^2c^2}{P^2 + m_0^2c^2}) = E^2 \frac{m_0^2c^4}{P^2 + m_0^2c^2} = m_0^2 c^6$$

$$E^2 = P^2c^2 + m_0^2c^4 \qquad (13\text{-}12)$$

The importance of (13-12) lies in two aspects:

(1) Invariant upon LT

Let me rewrite it in forms of:

$$\frac{E^2}{c^2} - P^2 = m_0^2 c^2$$

The RHS is a scalar that is independent of LT. While the momentum and energy do depend on the motion speed u, thus is frame dependent, a combination like above however is invariant. Put it more plainly, different observers in different frames will see the particle travels at different speed thus has different energy and momentum, but the above combination (observer 1 put in the values measured in his frame and observer 2 put in his) always comes out same for all observers. This is strikingly similar to the space-time interval s^2 we talked about before, and this is no coincidence as we shall see later in 4-vetcor.

(2) Works for massless particle

The massless particle is referring to particles whose rest mass m_0=0. It appears their momentum and energy will be zero according to the formula, unless the speed of the particle reaches c, then $\gamma \to \infty$, and the particle will have momentum and energy. We apply (13-12) to such massless particle, we get the relation between E and P:

$|P| = E / c$ for massless (13-13)

Such massless particle will travel with speed c (and as we discussed that since it requires infinite energy for massive particle to reach speed limit c, so the massless particles are the only ones that can reach c, they are the Ferrari in physics) and they travel at c only, because if the speed is less than c, their energy and momentum will be zero, which means they will disappear (being absorbed or annihilated) in the process.

The known detectable massless particle is photon, the question of whether the photon travels with speed limit is same as asking whether the rest mass of photon is indeed zero. The test of this is through vacuum dispersion of light, i.e. light with different frequencies travel in vacuum, there should be no dispersion (dependence of travelling speed on frequency), all frequency of light will have same speed in vacuum.

For the general particle differentiate (13-12) with respect to P:

$$2EdE = 2Pc^2 dP \to \frac{dE}{dP} = \frac{P}{E}c^2$$

From definition of momentum and energy:

$$\frac{P}{E} = \frac{\gamma_u m_0 u}{\gamma_u m_0 c^2} = \frac{u}{c^2} \qquad (13\text{-}14)$$

$$\frac{dE}{dP} = u \qquad (13\text{-}15)$$

The relation dE/dP is called dispersion relation[141]. For the massive particles, it could be any speed, but for massless particle, (13-13) tells that it could only be c. The measurement of vacuum dispersion set the high limit of photon mass to be less than 10^{-40}kg, which says if the photon has mass, it is going to be less than this value (Details in KK example 13.9 and 13.10). Other candidates for massless particles are possibly the graviton (the particle responsible for interaction in gravity) which is not detected in lab yet; and neutrino which may have a very tiny mass and travels very close to speed c.

You probably heard of the quantization of photon energy, the famous relation:

$$E = h\upsilon = \frac{h}{2\pi}\omega = \hbar\omega \qquad (13\text{-}16)$$

Put this into E/c=P:

$$P = \frac{h\upsilon}{c} = \frac{h}{\lambda} = \hbar\frac{2\pi}{\lambda} = \hbar k \qquad (13\text{-}17)$$

This is the famous de Broglie relation. Indeed de Broglie started from this and made hypotheses that it applies to all other particles, not

[141] This is equivalent to the conventional form $d\omega / dk$ for wave once we learned the relation between the frequency and energy, and wave vector with momentum.

limited to light. We shall see this in quantum again.

13.3 Examples Applying Energy and Momentum in SR

We have so far worked out the formula for energy and momentum in SR using a ad hoc method, i.e. starting from limit of Newtonian mechanics (correspondence principle) and make certain assumptions, such as momentum is still in form of mv and work-energy theorem works for kinetic energy etc. The usefulness we express the momentum and energy in those formula relies on the fact that they shall be conserved in all inertial frames, satisfying relativity principle. I had started from this to get the formula of momentum and the fact that energy by (13-10) will be conserved in all frames will be left for next section to prove. This ad hoc method though less satisfactory from theoretical point of view, it does clearly show the evolution and bond from Newtonian to SR. Before starting the elegant approach in SR, I shall work out some examples with the knowledge we learned so far. For convenience, I list the formula below first:

$P = \gamma_u m_0 u;\ \ E = \gamma_u m_0 c^2;\ \ m = m_0 \gamma_u$

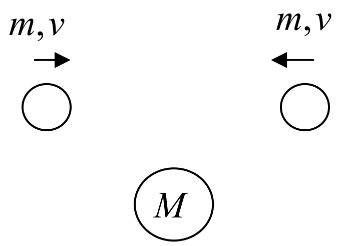
$E^2 = P^2c^2 + m_0^2c^4;\ \ E = Pc$ for massless particle

$E = K + m_0c^2;\ K = E - E_0 = m_0c^2(\gamma_u - 1)$

$P / E = u / c^2;\ \ dE / dP = u$

These are all the formula (some are redundant) we shall use and with the conservation of momentum and energy, we will work out the following examples.

Example 1: Inelastic collision



This is a fairly simple one, I used it basically to show that indeed the energy is conserved in different frames (since we know this is true for momentum already) and the rest mass is not conserved in the process. The two identical particles travel with same speed and stick together to form one particle finally.

In frame S (the lab frame, particles are moving), the M will be stationary with u=0, which is easy to see from conservation of momentum (total momentum before and after collision are both 0). The mass M can be computed from energy conservation (though inelastic, only kinetic energy is not conserved but the total energy is still conserved):

$E_i = 2\gamma mc^2, \quad \gamma = 1/\sqrt{1-v^2/c^2}$

$E_f = \gamma_u Mc^2 = Mc^2$ in S (only have rest energy)

$M = 2\gamma m$

The final mass is larger than 2m by factor γ, this is because kinetic energy is transformed into internal energy.

Now work the problem in S' frame, in which the left particle m is stationary, the final M will travel with velocity $u' = -v$ in this frame. The other m will travel with velocity:

$$u'_m = \frac{-2v}{1+v^2/c^2}, \quad \gamma_{u'_m} = \gamma\gamma_{u_m}(1-\beta\beta_{u_m}) = \frac{1+v^2/c^2}{1-v^2/c^2}$$

I used the trick (13-1) to get the $\gamma_{u'_m}$, surely you can also work this out by definition. For the momentum:

$$P_i = 0 + \gamma_{u'_m} m u'_m = \frac{1+v^2/c^2}{1-v^2/c^2} m \frac{-2v}{1+v^2/c^2} = -2\gamma^2 mv$$

$$P_f = \gamma_{u'_M} M u'_M = \gamma M(-v)$$

$P_i = P_f \rightarrow M = 2\gamma m$ same as above, which should be since the M is rest mass.

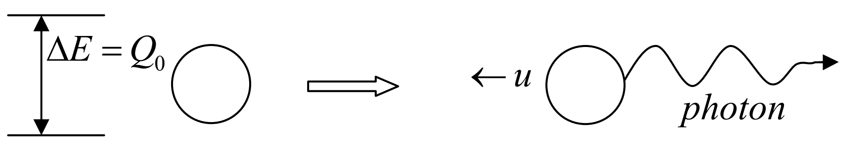
Take a look at energy:

$$E_i = mc^2 + \gamma_{u'_m} mc^2 = mc^2(1+\frac{1+\beta^2}{1-\beta^2}) = 2mc^2\gamma^2$$

$$E_f = \gamma_{u'_M} Mc^2 = 2\gamma^2 mc^2$$

The energy indeed is also conserved in S'.

Example 2: Emission of photon by a moving particle:



The particle (an atom or molecule) is at rest at beginning, with electron at higher energy level (upstairs), then the electron is relaxed to the lower energy state (downstairs) and a photon is emitted. The question is what is the photon's energy? Is it Q_0?

The answer is not exactly, the photon's energy will be slightly different from that of Q_0. This is because the recoil of the particle after emission, it starts moving in the opposite direction of the photon due to conservation of momentum, thus the energy Q_0 will be sum of the two. We'd better work it out in detail from conservation laws:

Initial: $P = 0, E = M_0 c^2$

Final: For photon: $E_p, P_p = E_p / c$; For particle: $P', E'^2 = P'^2 c^2 + M'^2 c^4$

Note the final rest mass will be different than the initial one because of the internal energy change (the final state will have less internal energy and thus less rest mass), actually we have:

$Q_0 \equiv M_0 c^2 - M' c^2$

Conservation of P:

$P' + P_p = 0$

Conservation of energy:

$E' + E_p = M_0 c^2$

E_p is what we wanted, and we can get it using the energy-momentum relation. (Noticed that generally if we are not asked to calculate the velocity, we do not want to put velocity in the equations since those γ's a little bit messy to work with. We will use energy-momentum relation instead)

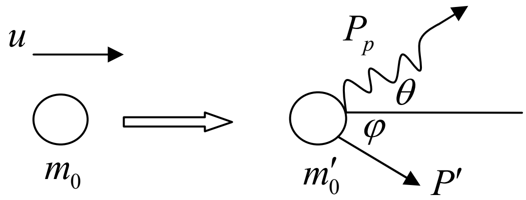
$$E'^2 = M_0^2c^4 + E_p^2 - 2E_pM_0c^2$$
$$P'^2c^2 + M'^2c^4 = M_0^2c^4 + E_p^2 - 2E_pM_0c^2,\ |P'| = |P_p| = E_p / c$$
$$E_p^2 + M'^2c^4 = M_0^2c^4 + E_p^2 - 2E_pM_0c^2$$
$$(M_0c^2 - Q_0)^2 = M_0^2c^4 - 2E_pM_0c^2$$
$$Q_0^2 - 2M_0c^2Q_0 = -2E_pM_0c^2$$
$$E_p = Q_0(1 - \frac{Q_0}{2M_0c^2})$$

If we use the quantum relation between frequency and energy for photon $E_p = h\upsilon$, and define $Q_0 = h\upsilon_O$, then the emitted light frequency by a stationary particle is:

$$\upsilon_p = \upsilon_O(1 - \frac{Q_0}{2M_0c^2}) \qquad (13\text{-}18)$$

It is slightly different from that of internal energy difference Q_0, but in many cases (light in atomic physics, Q_0 is on order of eV, M_0c^2 is on order of GeV) the difference is small that we neglect it say that the photons energy is same as internal energy difference. However in case of high energy photon resulting from larger internal energy change (such as nuclear reaction, large Q_0 give rise to gamma rays) the recoil effect cannot be neglected. One special case is the Mossbauer effect in gamma radiation, where the whole lattice of crystal acting like a giant molecule (billions of atoms) and the M_0 is huge, than the recoil effect is completely negligible and this is used in the famous experiment by Pound et al to test the gravitational red shift of light.

Example 3: Doppler Shift from conservation laws.



I choose the lab frame to do the calculation (all angles are w.r.t. this frame).

Initial: $P_i = \gamma_u m_0 u;\ E_i = \gamma_u m_0 c^2$

Final: P', E' for particle and P_p, E_p for photon.

$P_i = P'\cos\varphi + P_p\cos\theta$, and $P'\sin\varphi = P_p\sin\theta$ We do not need particle angle so we cancel it out by:

$$P'^2 = P_i^2 - 2P_iP_p\cos\theta + P_p^2$$

Considering energy:

$$E_i = E' + E_p$$

$$E'^2 = (E_i - E_p)^2 = P'^2c^2 + (m_0'c^2)^2$$
$$E_i^2 + E_p^2 - 2E_iE_p - (m_0'c^2)^2 = P_i^2c^2 - 2c^2P_iP_p\cos\theta + P_p^2c^2$$
$$(m_0c^2)^2 - (m_0'c^2)^2 = 2(E_iE_p - cP_iE_p\cos\theta)$$

E_p is the one we wanted (related to frequency by $h\upsilon$)

The R.H.S of the equation is:

$$2(E_iE_p - cP_iE_p\cos\theta) = 2E_p(\gamma_u m_0c^2 - \gamma_u m_0c^2\beta_u\cos\theta) = 2\gamma_u m_0c^2E_p(1-\beta_u\cos\theta)$$

The L.H.S. of the equation is:

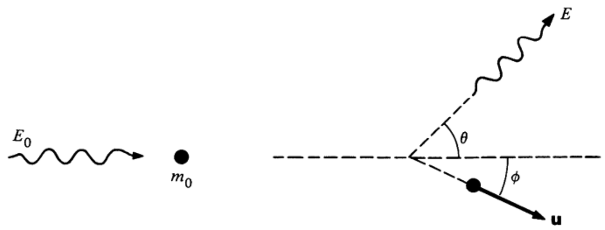
$$(m_0c^2)^2 - (m_0'c^2)^2 = (m_0c^2)^2 - (m_0c^2 - Q_0)^2 = 2m_0c^2Q_0 - Q_0^2$$
$$= 2m_0c^2[Q_0(1-\frac{Q_0}{2m_0c^2})] = 2m_0c^2h\upsilon_0$$

I used fact in the previous example that the frequency of light emitted by stationary particle ($h\upsilon_0$, it is the υ in the last example) is the term in bracket.

$$2\gamma_u m_0 c^2 E_p (1-\beta_u \cos\theta) = 2m_0 c^2 h\upsilon_0$$

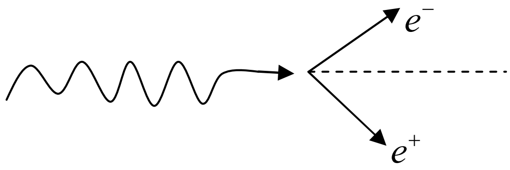
$$h\upsilon = \frac{h\upsilon_0}{\gamma_u (1-\beta_u \cos\theta)}$$ This is same as before.

Another very similar example would be the Compton's effect, scattering between photon and electrons:



In which the electron can be approximated as stationary at beginning, and a photon with certain frequency (energy) comes in, interact with electron and both particles are scattered afterwards, there will be a frequency (or equivalently wavelength) shift of the scattered photon depending on the scattering angle. This is called Compton's effect after Arthur Compton who carried out the experiment in 1920's to test the particle nature of photon. The dependence of frequency shift on angle will be left as an exercise for you to work out (this is exactly similar to the worked example here, and the answer can be found in standard textbooks).

Example 4 Pair production



A high energy gamma ray (photon) is the incoming particle, could it produce a pair of particle and antiparticle, such as the electro-positron pair?

The answer is NO. The charge is conserved, both the electron and positron have same mass, so it may appear if the gamma photon is energetic enough (say > 2mc^2, which is about 1MeV), this process is possible. But following the detailed analysis from conservation, we shall see that it is impossible to satisfy both momentum and energy conservation for this simple process!

$$\vec{P}_p = \vec{P}_- + \vec{P}_+$$

$$E_p = E_- + E_+$$

$$E_-^2 = P_-^2 c^2 + (m_0 c^2)^2, E_+^2 = P_+^2 c^2 + (m_0 c^2)^2$$

$$E_-^2 = (E_p - E_+)^2 = E_p^2 + E_+^2 - 2E_p E_+$$

$$(\vec{P}_p - \vec{P}_+)^2 c^2 = E_p^2 + P_+^2 c^2 - 2E_p E_+$$

$$P_p^2 c^2 + P_+^2 c^2 - 2\vec{P}_p \cdot \vec{P}_+ c^2 = P_p^2 c^2 + P_+^2 c^2 - 2|P_p| c E_+$$

This will give us the energy of positron is:

$$E_+ = \frac{\vec{P}_p \cdot \vec{P}_+}{|P_p|} c < |P_+| c$$

But this is not correct, because it violates the energy-momentum relation. This contradictory exists because the conservation of energy and momentum cannot be satisfied both. So a single photon cannot produce a pair of particle+antiparticle. It requires a 4^{th} party (such as another photon or nuclei) to make the process possible.

Another quicker method is choosing the total zero momentum frame for the product (assume the process is possible and the frame chosen is the total momentum of positron and electron is zero), so in such frame the final momentum is zero, but the initial momentum is not in the single photon case, which means our assumption that the process is possible is wrong.

Example 5: General Inelastic collision

Like in the example 1, we here consider the general case where the two particles may have different mass and speed, say initially, the two particles are: m_a, v_a, P_a and m_b, v_b, P_b, what is the final object's mass and velocity?

$P_a + P_b = P_f$ and $E_a + E_b = E_f$

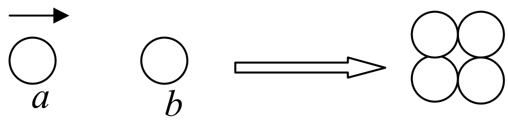
$$E_f^2 = E_a^2 + E_b^2 + 2E_a E_b = (M_f c^2)^2 + P_f^2 c^2 = (P_a^2 + P_b^2 + 2P_a \cdot P_b)c^2 + (M_f c^2)^2$$

$$(E_a^2 - P_a^2 c^2) + (E_b^2 - P_b^2 c^2) + 2E_a E_b - 2\vec{P}_a \cdot \vec{P}_b c^2 = (M_f c^2)^2$$

$$M_a^2 c^4 + M_b^2 c^4 + 2E_a E_b - 2\vec{P}_a \cdot \vec{P}_b c^2 = (M_f c^2)^2$$

Given the initial conditions, such as m_a, v_a, m_b, v_b, the energy and momentum of a, b will be known, and the final mass M can be computed. The velocity will be just: $u_f / c^2 = P_f / E_f$.

Example 6: Particle creation from collision



Considering the process of the figure above, a particle a collides with a stationary particle (in lab frame) b, and 4 particles are created, this process happens such as $p+p \rightarrow p+p+p+\overline{p}$, 2 protons collide to from 3 protons and 1 anti-proton. What is the minimum energy requirement for the moving particle *a*? (the particles here all have same rest mass m_0)

First I need to answer what is the possible minimum energy of the products. Then from energy conservation, that will give us the minimum required initial energy. The 4 product particles (for simplicity, I just use proton) can moves with respect to each other or stick together like a snowball. The smallest energy is when they stick together and moves like a snowball (this is the case of inelastic collision considered above). In old days, we learned Konig theorem (still remember) that the energy of multi-particles are sum of energy of CM and energy relative to CM. If the protons move like snowball, then there will be no kinetic energy in C.M. frame and the energy will be smallest. In SR, the analogous of Konig theorem will be derived when we learned 4-vector theorem, and it tells us for multi-particles: $E_{total}^2 - P_{total}^2 c^2$ will be invariant to LT. The CM frame is defined as a frame that $P_{total} = 0$. So in our case the lab frame: $E_{total}^2 - P_{total}^2 c^2 = E^2{}_{total(in\ CM)} - P^2{}_{total(inCM)} c^2 = E^2{}_{total(in\ CM)}$. The minimum energy in CM is just the rest energy (the snowball case), and P_{total} is some conserved value, this will lead to the possible minimum energy in lab frame too.

The initial condition: $P_b = 0, E_b = m_0 c^2; P_a, E_a$ unknown

Final condition: $M_f = 4m_0, P_f, E_f$ unknown but would be irrelevant.

Since this is just like the inelastic collision of previous example, I just skip the derivation and use the result above directly:

$$E_a^2 + E_b^2 + 2E_a E_b = (P_a^2 + P_b^2 + 2P_a \cdot P_b)c^2 + (M_f c^2)^2$$

$$E_a^2 + m_0^2 c^4 + 2m_0 c^2 E_a = P_a^2 c^2 + 16 m_0^2 c^4$$

$$2m_0 c^2 E_a = 15 m_0^2 c^4 - (E_a^2 - P_a^2 c^2) = 14 m_0^2 c^4$$

$$E_a = 7 m_0 c^2$$

From this you can calculate the γ factor and the velocity of a. The energy required is considerably larger than the $4m_0c^2$ as the naïve thinking would give. This is because in this setup, the final snowball is moving and having kinetic energy so that part of the input is wasted for this (useless from particle creation point of view). The more efficient way to create new particles would be head-head collision with particles traveling against each other with opposite momentum.

Alternatively you can work out the above example in the CM frame, and find out the energy for particle *a* in CM ($E'_{a(CM)} = 2m_0 c^2$, this part is easy), then transform back to the lab frame to get the answer, this would require knowledge of how energy and momentum transform with change of frames. That is the topic of next section.

13.4 Transform of Energy and Momentum in Different Frames

The basic question is a particle with certain momentum and energy, viewed by different observers in different frames, say Adam in S, and Bob in S', there is relative v between S and S'. The energy-momentum measured by Adam is E and P, and E',P' by Bob. Then what is the relation between (E,P) and (E',P')? Just like Lorentz transform exists between space-time coordinate of event, there is also a relation between energy-momentum, and we shall find out (very strikingly) that this relation is same as that space-time coordinates, or the same Lorentz transform exists between energy-momentum too.

For single particle with mass m_0, it moves in S with velocity u, then the energy and momentum is just:

$$E = \gamma_u m_0 c^2, P = \gamma_u m_0 u$$

In S' that moves with velocity v to the S, the velocity of the particle would be u', and energy, momentum in S' are:

$$E' = \gamma_{u'} m_0 c^2, P' = \gamma_{u'} m_0 u$$

For the simplicity of calculation, I shall choose a special case where v and u are parallel, only has x component. i.e. u=(u,0,0); v=(v,0,0). In the more general case, where v=(v,0,0), u=(u_x,u_y,u_z), where $u=\sqrt{u_x^2+u_y^2+u_z^2}$ the calculation would be more involving but will give same conclusion and you are encouraged to derive it as exercise[142].

We already learned the relation between u,u' (only x component here):

$$u'=\frac{u-v}{1-\beta_u\beta}$$

$$P_y=P'_y=0, P_z=P'_z=0$$

The troublemaker is $\gamma_{u'}$ and fortunately I already work it out in (13-1):

$$\gamma_{u'}=\gamma\gamma_u(1-\beta_u\beta)$$

Now it is straightforward to find relation between (E', P'x) and (E,Px)

$$E'=\gamma\gamma_u(1-\beta_u\beta)m_0c^2=\gamma(\gamma_u m_0c^2-\beta c\gamma_u m_0u)=\gamma(E-\beta cP_x)$$

$$P'_x=\gamma\gamma_u(1-\beta_u\beta)m_0\frac{u-v}{1-\beta_u\beta}=\gamma(\gamma_u m_0u-\gamma_u m_0v)=\gamma(P_x-\frac{v}{c^2}E)$$

You already see the striking similarity in the transform, where E is transforms analogous to t and Px is analogous to x. I shall rewrite the above in a more symmetrical way:

$$E'/c=\gamma(E/c-\beta P_x)$$

$$P'_x=\gamma(P_x-\beta E/c) \qquad (13\text{-}19)$$

This is important, we see that the (E/c, P_x, P_y, P_z) just transform same as (ct, x, y, z) between frames, obey Lorentz Transform. Just as space-time interval $s^2=(ct)^2-r^2$ is invariant upon LT, we see that $(E/c)^2-P^2$ would be also invariant upon LT. In one special frame this combination is

[142] Or read it in other textbooks, e.g. French's "Special Relativity".

easiest to compute and that is the rest frame of the particle, in which P=0, and $E=m_0c^2$. Since this combination has same value in all frames, you see we derived the famous energy-momentum relation from this invariance point of view.

The analogous between energy-momentum and space-time is no coincidence, we will see starting from next section an elegant formalism of SR, and the energy-momentum defined there have same formula as we are using here. Both space-time and energy-momentum are physical quantity that is called 4-vector, whose transformation between inertial frames obeys LT.

13.5 4-Vectors and Lorentz Transform as Hyper-rotation[143]

We have seen the similarity between the transformations (LT) from one frame to anther of variables (ct, x, y, z) and (E/c, P_x, P_y, P_z). We shall call such combination 4-vectors in analogous to the 3-D vector we defined before.

Define:

$$\underset{\rightarrow}{X} \equiv (ct, x, y, z) \equiv (x^0, x^1, x^2, x^3) \equiv (x^\mu), \mu = 0,1,2,3 \equiv (ct, \vec{r}) \qquad (13\text{-}20)$$

I used the symbol as in KK's book: an arrow below the character for

[143] A review on Chapter 3 on vectors and its transformation under rotation would be helpful, please read it again if you need it. I hope by the time we reach here, your liner algebra course already taught you the basics on transformation.

4-vector, and the arrow above character is reserved for the ordinary 3-D vectors. The x^{μ} are the components of the 4-vectors, and the convention I adopt is to write the time component as 0th component. There is another convention (like that in KK) to express time as 4th component. Such choices of convention is not important as long as we stick to one convention consistently (it is more like a habit rather than necessity), and the results would be same in all cases. The reason I used the superscript for label is the convention in tensor analysis[144]. An immediate question is that are any 4 components listed as above can be called 4-vectors? The answer is NO. Same as definition of vector under rotational transformation, the 4-vectors has to satisfy the Lorentz Transformation from one inertial frame to another. For example: $(c, \vec{v})$ is not a 4-vector because we know that velocities do not transform as LT. Since I had proved that the (ct, x, y, z) satisfy this and that is why I say here it is a 4-vector. Similarly for the displacement 4-vectors:

$\Delta \underset{\rightarrow}{X} \equiv (c\Delta t, \Delta x, \Delta y, \Delta z) \equiv (c\Delta t, \Delta \vec{r})$ and the energy-momentum 4-vector.

Now we could understand the meaning of the golden combination that is invariant under LT (I shall use the displacement 4-vector as model for 4-vector, same as the 3-D displacement vector is the prefect of vector in 3-D):

$(\Delta s)^2 = (c\Delta t)^2 - (\Delta r)^2$ is same in all frames. It is analogous to a 3-D

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[144] These components are called contravariant components (another jargon).

vector's module (it is also called norm or length) which is invariant upon unitary transformation. But here the "length" of the 4-vector is not in Pythagoras form. The square of time component has to have different sign with respect to the spatial components. Mathematically, this is because the LT is not a generalized rotational transform in 4-D (but a hyper one as we shall see later); the physical meaning of this is that though the space-time are related in relativity, time and space are not equivalent with reason unclear at present. Below I shall only use 2-D (y,z are not changed so only consider transform of ct and x)

R(XY→X'Y')= $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (transformation matrix in 2-D rotation)

While the LT matrix is (from 12-4):

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

$$LT = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \qquad (13\text{-}21)$$

It is similar but different from rotational matrix, notably it is (the determinant of matrix):

$\gamma^2 - (-\gamma\beta)^2 = \gamma^2(1-\beta^2) = 1$ instead of summation equals 1. There is no way you can express the matrix element in LT to some cosine/sine terms, so to express the LT matrix as some rotation matrix. It is just this $\gamma^2 - (-\gamma\beta)^2 = \gamma^2(1-\beta^2) = 1$ demands that the subtraction of square instead of summation in the space-time interval.

Above I used the worked out formula of LT to show the difference and similarity between LT and rotation. Actually it is illustrative to derive the LT from linear transformation with requirement of $(\Delta s)^2 = (c\Delta t)^2 - (\Delta r)^2$ is invariant. I shall work this out in detailed steps below (again only consider the simplified version in 2-D, only involves t,x).

To avoid introducing metric tensor in defining the inner product of 4-vectors, I shall adopt the trick to represent either x or t as a pure complex number (of course the time and space we measured are real numbers, here is the mathematical trick to express one as complex so that we can apply some powerful tools in math). The convention is to choose t as complex, though this will give us x^2-$(ct)^2$ as module but it does not matter. This *mathematical* complex time, real position space is called Minkowski Space. To state the problem in matrix language, I shall use column matrix representing vector (in Minkowski Sapce):

$\begin{bmatrix} ict \\ x \end{bmatrix}$ for input and $\begin{bmatrix} ict' \\ x' \end{bmatrix}$ as output. The 'length' (norm, module) of the vector is defined as inner product (dot product) between themselves as usual which is reason we use complex(in matrix language is $A^T A$):

$$s^2 = \underset{\rightarrow}{X}_\mu \cdot \underset{\rightarrow}{X}^\mu = [ict \;\; x]\begin{bmatrix} ict \\ x \end{bmatrix} = x^2 - (ct)^2$$ [145]

This trick will make the length is still defined as the sum of square of components like in regular vector, i.e. the Pythagoras theorem works in

[145] Noticed here there is a digression from usual linear algebra language, where the conjugate for regular column vectors (like that of 3-D case) with complex number will be a row vector which is transposed and complex conjugate (C.C)of the original column. Here only transposed but no C.C. This will apply to the matrix operation too, see below.

Minkowski space. We are looking for a transformation:

$\begin{bmatrix} ict' \\ x' \end{bmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} ict \\ x \end{bmatrix}$ which will keep the length defined above unchanged (a,b,c,d are generally complex numbers) Let's write out the matrix expression explicitly:

$$ict' = a(ict) + b(x)$$
$$x' = c(ict) + d(x)$$

So we see that if a, d are pure real number and b,c are pure imaginary number the above can be satisfied (t,x are all real numbers of course), so I shall write the transform as:

$L = \begin{pmatrix} a & ib \\ ic & d \end{pmatrix}$ (here a,b,c,d are real numbers)

To keep the length invariant tells us this L transformation is a Unitary Transformation (refer to section 3.5 for details) where $L^{-1} = L^T$:

$$L^T = \begin{pmatrix} a & ic \\ ib & d \end{pmatrix}$$

$$L^{-1} = \frac{1}{\det(L)} \begin{pmatrix} d & -ib \\ -ic & a \end{pmatrix} = \frac{1}{da+bc} \begin{pmatrix} d & -ib \\ -ic & a \end{pmatrix}$$ [146]

The Unitary Transformation requirement will put restrictions to the matrix elements:

$$\det(L) = ad + bc = 1$$

$$a = d, b = -c$$

So the matrix for LT is:

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[146] For those unfamiliar with inverse matrix formula, just try matrix production of L^{-1} with L to see whether you will get identity matrix.

$L = \begin{pmatrix} a & ib \\ -ib & a \end{pmatrix}$ with $a^2 - b^2 = 1$ (while $a^2+b^2=1$ for regular rotation)

Actually there is a function form satisfies this which is called hyper-sinusoidal functions which is defined as:

$$\cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}$$

These hyper-sinusoidal functions are usually called just hyperbolic functions. They have many similar properties as their cousins of sinusoidal functions (I would not list them, please check math handbooks or Wiki them if this is your first time seeing them, actually these functions can be worked out from Euler formula by allowing imaginary angles). The property we need these function here is:

$$\cosh^2\theta - \sinh^2\theta = 1$$

With the hyperbolic function, the matrix can be expressed with only one undetermined parameter:

$$L = \begin{pmatrix} a & ib \\ -ib & a \end{pmatrix} = \begin{pmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{pmatrix}$$

Summarizing above, I have derived from point of view of unitary transformation that the linear transform above will guarantee the invariance of space-time interval.

Now I do not want those imaginary symbol *i* dangling around anymore (I want to switch back from imaginary Minkowski space to real time-space):

$$ict' = \cosh\theta(ict) + i\sinh\theta(x) \rightarrow ct' = \cosh\theta(ct) + \sinh\theta(x)$$
$$x' = -i\sinh\theta(ict) + \cosh\theta(x) \rightarrow x' = \sinh\theta(ct) + \cosh\theta(x)$$ or just:
$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

The one parameter θ can be determined by the motion between frames: S' is moving with velocity v relative to S:

(x'=0 is moving with v viewed by S)

$$0 = \sinh\theta(ct) + \cosh\theta(x)$$
$$\tanh(\theta) \equiv \frac{\sinh(\theta)}{\cosh(\theta)} = -\frac{x}{ct} = -\frac{v}{c} \equiv -\beta$$

(Throw in another jargon that this angle is also called rapidity in SR)

Knowing the tanh and finding out cosh and sinh is exactly similar to the sinusoidal case:

$$\sinh\theta = -\beta\cosh\theta$$
$$\cosh^2\theta - \sinh^2\theta = 1$$
$$\cosh^2\theta = \frac{1}{1-\beta^2} \equiv \gamma^2 \qquad (13\text{-}22)$$ [147]
$$\cosh\theta = \gamma, \sinh\theta = -\gamma\beta$$

The final LT between (ct', x') and (ct,x) will be exactly same as (13-21), but this has been derived from quite different (more mathematical, requiring "length invariant" and Unitary transformation) point of view. To put all in matrix form, the LT between 4-vectors is:

[147] The reason that the cosh only takes the positive gamma is we want the transformation reduces to Galileo under low speed limit where gamma=1. (Correspondence principle)

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma \quad -\gamma\beta\text{.........} \\ -\gamma\beta \quad \gamma\text{..........} \\ \text{..............}1\text{......} \\ \text{.......................}1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \qquad \text{(13-23) (0 in place of …)}$$

We have thus finished proof of LT using unitary transformation and now back to the business of 4-vectors. ***The 4-vectors are defined to satisfy the transformation in (13-23) between different frames***[148], and their 'length' defined as:

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \qquad \text{(13-24)}$$[149]

This is invariant upon LT.

Important properties of 4-vector (in analogous to 3-D vector)

1) Linear Combination: Any linear combination of 4-vectors will give us another 4-vector, i.e. if A, B are 4-vectors:

   $a\underset{\rightarrow}{A} + b\underset{\rightarrow}{B}$ will be also a 4-vector (a,b are constant, they are invariant upon transform, so they are scalars).

   This is the most important property and can be easily proved because the LT is a linear transformation. We shall see that we can generate more 4-vectors from our prefect displacement 4 vectors! (this is exactly analogous to common 3-vector)

[148] Actually I could throw in the 3-D rotational matrix (by extension to 4-D with time unaltered upon regular rotation) to make the transformation also include the rotation with messier final matrix. But here I shall only concentrate on the so called boost transformation (another jargon for the transformation between translational moving frames).

[149] Sorry for the awkward symbol, I trust you can distinguish what is the indices label for components and what is the power (squared).

2) If a physical equation between 4-vectors in one frame, it will be true in all frames. e.g.:

$\underset{\to}{A} = \underset{\to}{B}$ in S, then in the S' frame we still have: $\underset{\to}{A}' = \underset{\to}{B}'$.

This is also related to the LT is a linear transform:

Proof: $\underset{\to}{A} - \underset{\to}{B} = 0,$ 0 is a null matrix which is same in all frames:

$\hat{L}(\underset{\to}{A} - \underset{\to}{B}) = 0 \to \hat{L}\underset{\to}{A} - \hat{L}\underset{\to}{B} = 0 \to \underset{\to}{A}' = \underset{\to}{B}'$

We have seen that the (E/c, P) (E,P in SR formula)satisfy the LT transform and is a 4-vector, so if the energy conservation and momentum conservation is true in one inertial frame (which is nothing but equivalence between 4-vectors), it will be true for all inertial frames, I just proved this. (Of course what I mean the two 4-vectors are same means their components are same)

One extension of this will be if the physical laws relating the 4-vectors are true in one frame, they will be true in all frames, provided the coefficients on the 4-vectors are tensors (scalar is the 0^{th} rank tensor). Actually since vector is just another special case in tensor (1^{st} order), I should say that in order for physical laws to be same in all frames (relativity principle), the laws need to be expressed in tensors. I won't push this too far since tensor analysis is required for higher order tensors. (we do not need much here in this course is because most laws here only involves scalars and vectors (0, or 1^{st} order tensors) except a few occasions such as inertia tensor in rotation[150])

3) Inner product between 4-vetcors is defined as:

$$\underset{\to}{A}\cdot\underset{\to}{B}\equiv A^0B^0-A^1B^1-A^2B^2-A^3B^3 \qquad (13\text{-}25)$$

Notes: the 0,1,2,3 are just indices of components. This inner product is a scalar meaning invariant upon transformation.

Proof: We have seen the length is invariant (a scalar) which is nothing but just: $|A|^2\equiv\underset{\to}{A}\cdot\underset{\to}{A}$, and A+B will be also a 4-vector from property 1:

$$(\underset{\to}{A}+\underset{\to}{B})\cdot(\underset{\to}{A}+\underset{\to}{B})=scalar$$

$$\underset{\to}{A}\cdot\underset{\to}{A}+\underset{\to}{B}\cdot\underset{\to}{B}+2\underset{\to}{A}\cdot\underset{\to}{B}=scalar$$

Because the lengths are scalars (a requirement in defining the transformation), then the inner product between any 4-vectors is a scalar too (meaning same in all frames).

13.6 Velocity 4-Vector and Momentum 4-Vector

We have defined what 4-vector is and used displacement 4-vector as example working out the LT transform, and all that in chapter 12 can follow from this. So the special relativity can be developed out of this 4-vector formalism. In this section, I will use the displacement vector as prefect to find out the formula for some more 4-vectors, especially

[150] For the interested students who wants to learn more on tensors and its analysis, I recommend a classical textbook: Borisenko and Tarapov (translated by Silverman) "Vector and Tensor Analysis with Applications" (Dover, 1955). We can elude tensors in SR (and many other courses, even QM) but have to resort to it in GR or field theory.

4-velocity and 4-momentum and what we had discussed in the first half of this chapter can be developed out of this.

In 3-D, the velocity is $\vec{v} = \frac{d\vec{r}}{dt}$, a naïve way to get velocity vector from displacement 4-vector would be $\underset{\rightarrow}{V} = \frac{d\underset{\rightarrow}{X}}{dt}$. But this won't work, the quantity defined like this is not a 4-vector, it is $(c, \vec{v})$ and it does not transform obeying LT. The problem is that dt is not a scalar anymore in SR, it changes from frame to frame. From property 1) above, a 4-vector multiplied (or divided by) a scalar will be another 4-vector, this is not satisfied by dividing dt. For a particle flying with velocity u across space-time, the time elapse dt is frame dependent, however we have seen that there is a time that is not changing, i.e. upon which all observers agree, it is a scalar. That time is proper time, measured in the rest frame of the particle. This is because since the $(\Delta s)^2 = (c\Delta t)^2 - (\Delta r)^2$ is a scalar, in the rest frame of particle where $\Delta r = 0$, $(\Delta s)^2 = c^2(\Delta\tau)^2$, thus the proper time $\Delta\tau$ is a scalar too. With this proper time interval, we can proceed constructing 4-vectors out of displacement 4-vector.

(1) 4-Velocity

$$\underset{\rightarrow}{V} \equiv \frac{d\underset{\rightarrow}{X}}{d\tau} \qquad (13\text{-}26)$$

Above is the *definition* of 4-velocity, to work out the detailed formula, especially we want to see the relation between it with regular velocities (dx/dt defined in each frame). We need relation between proper time and time, but this is just time dilation:

$d\tau = dt / \gamma_u$

In a frame that the particle is moving with velocity u, its time interval dt is related to the proper time by above relation.

$$\underset{\rightarrow}{V} \equiv \frac{d\underset{\rightarrow}{X}}{d\tau} = \gamma_u\left(\frac{cdt}{dt}, \frac{d\vec{r}}{dt}\right) = (\gamma_u c, \gamma_u \vec{u}) \qquad (13\text{-}27)$$

From this and transform property of 4-vector (LT), I hope you can work out the velocity transform relations that I had derived before in chapter 12. (It is straightforward and a bit tedious, so I left it as an exercise for you to finish)

Comment: In the real lab measurement what we measured is of course the regular 3-D velocity, and we can construct 4-velocity out of this by (13-27) which has the advantage that transforms obeying LT. In computation, using the 4-velocity or the velocity transform formula will be equivalent and about same amount of work.

The length of the 4-velocity:

$$|V|^2 = (\gamma_u c)^2 - \gamma_u^2 |\vec{u}|^2 = c^2 \qquad (13\text{-}28)$$

It can be proved by throwing the definition of gamma but most easily by the invariance of its value, and find it in the rest frame where u=0 and gamma=1.

The 4-velocity is not very useful in real practice, its purpose is an intermediate to find more 4-vectors out of it.

(2) 4-Momentum

Let's temporarily forget about the relativistic formula for energy and momentum and see how those arise from the construction of 4-momentum.

We have constructed 4-velocity above, and we know that rest mass is a scalar (it depends on internal energy but independent of motion), so we can construct a 4-momentum from 4-velocity:

$$\underset{\rightarrow}{P} \equiv m_0 \underset{\rightarrow}{V} = (\gamma_u m_0 c, \gamma_u m_0 \vec{u}) \qquad (13\text{-}29)$$

The spatial component is $\gamma_u m_0 \vec{u}$, and we can expand this as power series of u/c:

$$\gamma_u m_0 \vec{u} = (1 + \frac{1}{2}\beta^2 + ...) m_0 \vec{u}$$

We see that the 0th order term is just $m_0 u$, which is the momentum in Newtonian mechanics at low speed limit, higher orders would be relativity correction. Most important, since it is part of 4-vector, if this quantity is conserved in one frame, it will be conserved in other frames too. The $\gamma_u m_0 \vec{u}$ satisfies the correspondence and relativity principle and it is justified to define this term as relativistic momentum, which is exactly the same formula I derived before.

If I expand the time component (0th component):

$$\gamma_u m_0 c = (1 + \frac{1}{2}\frac{u^2}{c^2} + ...) m_0 c = m_0 c + \frac{1}{c}\frac{1}{2} m_0 u^2 + ...$$

The second term is obviously related to Newtonian kinetic energy, so this term is related to energy, a dimensional analysis tells us it is E/c. Since this term is also part of 4-vector and c is an universal constant, this means if this quantity is conserved in one frame, it is conserved in all frames too. Similar argument suggests it is justified to define this term as E/c, or $E=\gamma_u m_0 c^2$ which is our familiar relativistic energy formula. We have derived the formula for relativistic energy and momentum from 4-vector point of view.

So the energy and momentum are closely related in special relativity as time and space. The old argument in Newtonian mechanics centuries ago about whether the energy or the momentum is more fundamental now has a clear and more profound answer. They are equally important and are related to time and space, as I have mentioned at the very beginning of this course, that the energy conservation will be a result of time translational symmetry and momentum conservation is the result of space translational symmetry.

The 4-momentum vector for a particle is also called energy-momentum vector for obvious reason, but sometimes when I am sloppy, I just call it momentum

Because 4-Momentum is a 4-vector, it transforms obeying LT, and we have seen this explicitly in 13.4. The length of this vector is:

$|\underset{\to}{P}|^2=(\frac{E}{c})^2-|\vec{P}|^2$ and is a scalar. Its value is easily evaluated by choosing a frame in which the particle is at rest, so that u=0 (and

$\vec{P}=0, E=m_0c^2$ in such frame)and this will gives the value:

$$|\underset{\rightarrow}{P}|^2=(\frac{E}{c})^2-|\vec{P}|^2=m_0^2c^2 \qquad (13\text{-}30)$$

This is the energy momentum relation derived before.

I shall rework some of the examples in section 13.3 explicitly using 4-vectors, so that you may have a feel for this method. (I will not draw the figures and please refer to section 13.3 if necessary)

Example 1: Photon emission by a stationary atom/molecule in lab frame.

The initial momentum (4-vector): $\underset{\rightarrow}{P}_c$,

Final momentum: $\underset{\rightarrow}{P}_a$ for recoiled atom, $\underset{\rightarrow}{P}_b$ for photon (only consider 1-D, along x direction)

Energy and momentum conservation can be expressed with one equation:

$$\underset{\rightarrow}{P}_c=\underset{\rightarrow}{P}_a+\underset{\rightarrow}{P}_b$$

In order to apply for the invariant value, I direct product both side with P_c (equivalent to square both sides):

$$|\underset{\rightarrow}{P}_c|^2=|\underset{\rightarrow}{P}_a|^2+|\underset{\rightarrow}{P}_b|^2+2\underset{\rightarrow}{P}_a\cdot\underset{\rightarrow}{P}_b$$

All these values are scalars and I can choose the most convenient frames for their values, and clearly:

$$|\underset{\rightarrow}{P}_c|^2=m_c^2c^2;|\underset{\rightarrow}{P}_a|^2=m_a^2c^2;|\underset{\rightarrow}{P}_b|^2=0$$

To evaluate the cross terms, I shall just use lab frame, in which:

$$\underset{\rightarrow}{P}_a=(E_a/c,\vec{P}_a)$$
$$\underset{\rightarrow}{P}_b=(E_b/c,E_b/c,0,0)$$

Using momentum (3-D) conservation:

$$\vec{P}_a = -E_b / c \rightarrow \underset{\rightarrow}{P}_a = (E_a / c, -E_b / c, 0, 0)$$

$$\underset{\rightarrow}{P}_a \cdot \underset{\rightarrow}{P}_b = \frac{E_a E_b}{c^2} - (\frac{E_b}{c})(-\frac{E_b}{c}) = \frac{E_a E_b + E_b^2}{c^2}$$

Put all these back into squares of 4-momentum conservation:

$$m_a^2 c^2 + 2\frac{E_a E_b + E_b^2}{c^2} = m_c^2 c^2$$

There are still two unknowns E_a, E_b (m_a is related with m_c by $Q_0 = m_c c^2 - m_a c^2$), but we still have energy component conservation:

$$E_a = E_c - E_b = m_c c^2 - E_b$$

$$m_a^2 c^2 + 2\frac{(m_c c^2 - E_b)E_b + E_b^2}{c^2} = m_c^2 c^2 \rightarrow 2m_c c^2 E_b = m_c^2 c^4 - m_a^2 c^4 = m_c^2 c^4 - (m_c c^2 - Q_0)^2$$

$$E_b = Q_0 (1 - \frac{Q_0}{2M_0 c^2})$$

This is same as the result of example 2 in 13.3. (Above procedure may not be the quickest way to get answer, can you find a better alternative?) When I apply the scalar property of inner product of vectors, I chose P_c (4-vector), you may try other inner product, say direct product to both sides of 4-vector conservation equation with P_a and you will get same answers (try it yourself) using this slight different method.

Example 2 Total inelastic collision

Particle A and B collide to form one particle C, if we know the initial velocity of A, B and their rest masses m_A,m_B, find C's mass and velocity.

With the initial conditions, we can compute the initial momentum and energy from m, u, but I try to avoid as much as possible by using 4-vector's property(I cannot avoid it completely of course):

$$\underrightarrow{P}_A + \underrightarrow{P}_B = \underrightarrow{P}_C$$

$$|\underrightarrow{P}_A|^2 + |\underrightarrow{P}_B|^2 + 2\underrightarrow{P}_A \cdot \underrightarrow{P}_B = |\underrightarrow{P}_C|^2$$

$$|\underrightarrow{P}_A|^2 = m_A^2c^2; |\underrightarrow{P}_B|^2 = m_B^2c^2; |\underrightarrow{P}_C|^2 = m_C^2c^2$$

In the lab frame:

$$\underrightarrow{P}_A = (E_A / c, \vec{P}_A); \underrightarrow{P}_B = (E_B / c, \vec{P}_B)$$

$$\underrightarrow{P}_A \cdot \underrightarrow{P}_B = \frac{E_A E_B}{c^2} - \vec{P}_A \cdot \vec{P}_B$$

Put all these into equation:

$$m_A^2c^2 + m_B^2c^2 + 2\frac{E_A E_B}{c^2} - 2\vec{P}_A \cdot \vec{P}_B = m_C^2c^2$$

This is exactly same as the result in example 5 in section 13.3. With the mass of C computed, we can further find its velocity using conservation laws.

Example 3: This is example 6 in 13.3

$$p + p \rightarrow p + p + p + \overline{p}$$

The minimum energy of the moving proton hitting a target of rest proton and create 4 particles is what we need to compute.

The minimum energy is when the output's total energy is minimum. For the multi-particle system, I stated without proof that the

$E_{total}^2 - P_{total}^2 c^2 = E^2{}_{total(in\ CM)}$. Now I can prove this easily from 4-vectors:

The total energy-momentum vector is also a 4-vector:

$$\underset{\rightarrow}{P}_{total} = \sum_i \underset{\rightarrow}{P}_i = (E_{total} / c, \vec{P}_{total})$$

Its length is also a scalar invariant upon transformation between frame:

$$|\underset{\rightarrow}{P}_{total}| = \frac{(E_{total})^2}{c^2} - |\vec{P}_{total}|^2$$

Its value relates to the total energy in the C.M. frame of which is defined as total momentum (3-D) is zero:

$$|\underset{\rightarrow}{P}_{total}| = \frac{(E_{total})^2}{c^2} - |\vec{P}_{total}|^2 = (\frac{E_{total(CM)}}{c})^2$$

The minimum energy in CM frame is when all particles are at rest with total energy $4mc^2$. In other frames, $|\vec{P}_{total}|^2$ is some conserved value and this will give the minimum energy of E_{total} in that frame. The rest would be similar as before:

$$\underset{\rightarrow}{P}_A + \underset{\rightarrow}{P}_B = \underset{\rightarrow}{P}_C$$

$$|\underset{\rightarrow}{P}_A|^2 + |\underset{\rightarrow}{P}_B|^2 + 2\underset{\rightarrow}{P}_A \cdot \underset{\rightarrow}{P}_B = |\underset{\rightarrow}{P}_C|^2$$

$$|\underset{\rightarrow}{P}_A|^2 = |\underset{\rightarrow}{P}_B|^2 = m^2c^2, |\underset{\rightarrow}{P}_C|^2 = 16m^2c^2$$

$$\underset{\rightarrow}{P}_A = (E_A / c, P_A), \underset{\rightarrow}{P}_B = (mc, 0) \text{ in lab frame}$$

$$m^2c^2 + m^2c^2 + 2mE_A = 16m^2c^2 \rightarrow E_A = 7mc^2$$

Solving problems with conservation laws, you can either use the conservation laws separately as I did before or apply the 4-vector with the property of invariant inner products like what I just showed you here. The two methods are equivalent (as demonstrated by the examples I worked out) and which one you choose is somewhat personal taste. The 4-vector is generally more compact while the separate conservation laws are probably more familiar.

Chapter14 Acceleration and Force in Special Relativity

We have learned most of the basics of special relativity. This chapter is included mostly for the completion of story. Though in SR the force and acceleration lost their central positions in description of motion as in the Newtonian mechanics, we need to see why it is such.

The reason we hold dear on the old F=ma is because it is ***simple*** (and is correct under low speed limit). As I discussed at the very beginning of this course in chapter 1, the force and acceleration allow us to compute a complete trajectory of the particle, instead of counting the position and velocity of the particle at many different times. Knowing the form of the force and its distribution, we can solve the trajectory by solving 2^{nd} order differential equations, or 1^{st} order equations for velocity. All this made the designation of a physical quantity of interaction as force very useful. However, we will see that the simple relation F=ma is not correct at high speed (SR), force and acceleration when transformed between moving frames do not satisfy LT and are not 4-vectors, so as we discussed in the last chapter, F=ma cannot satisfy relativity principle. There is still a relation (called equation of motion as usual) between acceleration and force in SR, but the form is more complicated than its Newtonian counterpart. The calculation of trajectory is far more difficult in SR.

Knowing the complete trajectory (or history) of the particle would be nice, but fortunately never necessary for a understanding of physical systems. We have seen in this course that we can get the properties of system, velocity, momentum, energy, angular momentum without knowing the detailed trajectory. In microscopic world (quantum theory), knowing the trajectory is impossible because of the uncertainty relations. Modern physics is largely based upon measurement of momentum, energy and angular momentum and their conservation laws rather than trajectory. So it is a pity but not the end of the world if we cannot have a complete trajectory or history of the motion, we are just less god-like and his almighty maybe the only one knowing exactly where a particle came and where it will go☺.

Though the cherished simple equation of motion is no longer correct in SR, and force –acceleration becomes less important and knowing the complete trajectory is not essential in understanding the physical properties of a system, they do hold the central role in the first half of the course and they deserve a place of discussion here in SR.

The strategy in the following discussions will be similar as before: I shall discuss the acceleration and force without the help of 4-vector; then I shall invoke the 4-vector and show you how to construct 4-acceleraton and 4-force; their relation to the 3-D force and acceleration, and that the two formalisms are equivalent.

14.1 Acceleration

The acceleration is still defined as derivative of velocity over time, of course the time and velocity are measured in an inertial frame, i.e.

$$\vec{a}_{(S)} = \frac{d\vec{u}_{(S)}}{dt_{(S)}}$$

The subscript (S) is there just to show the dependence on frame explicitly, it is redundant as long as you do not forget such dependence.

(1) Transform formula of acceleration

Here we study the relation between the acceleration of same particle viewed in different frames, say S and S', S' is moving velocity v w.r.t. S as usual.

We have learned the relations between the velocity u, u' of the motion and time t, t', and starting from there we can find out relation between *a* and *a*'.

Let's first consider the x component of the acceleration vector (3-D), and x is the direction of motion between frames v:

$$a'_x \equiv \frac{du'_x}{dt'} = \frac{du'_x / dt}{dt' / dt}$$

$$dt' = \gamma(dt - \frac{v}{c^2}dx)$$

$$u'_x = \frac{u_x - v}{1 - vu_x / c^2}$$

$$\frac{du'_x}{dt} = \frac{(1 - vu_x / c^2)du_x / dt + \frac{v}{c^2}(u_x - v)du_x / dt}{(1 - vu_x / c^2)^2} = \frac{(1 - v^2 / c^2)a_x}{(1 - vu_x / c^2)^2} = \frac{a_x}{\gamma^2(1 - vu_x / c^2)^2}$$

$$dt' / dt = \gamma(1 - \frac{v}{c^2}\frac{dx}{dt}) = \gamma(1 - \frac{vu_x}{c^2})$$

$$a'_x \equiv \frac{du'_x}{dt'} = \frac{du'_x / dt}{dt' / dt} = \frac{a_x}{\gamma^3(1 - vu_x / c^2)^3} \qquad (14\text{-}1)$$

Same straightforward strategy and messy calculation will give:

$$\begin{aligned} a'_y &= \frac{a_y}{\gamma^2(1 - vu_x / c^2)^2} + \frac{1}{\gamma^2(1 - vu_x / c^2)^3}\frac{u_y v}{c^2}a_x \\ a'_z &= \frac{a_z}{\gamma^2(1 - vu_x / c^2)^2} + \frac{1}{\gamma^2(1 - vu_x / c^2)^3}\frac{u_z v}{c^2}a_x \end{aligned} \qquad (14\text{-}2)$$

The transformation is quite messy and not LT. The reverse transformation (expressing a in terms of a') is just change the sign of v. The relation is even more complicated by looking carefully to the coefficients, the coupling coefficient are not constant of motion. The u_x, u_y will change especially under acceleration, that means the coefficients are also changing.

(2) 4-Accelerator

Just like how we construct the 4-Velocity vector with space-time 4-vector divided by proper time, we can construct a 4-Accelerator similarly with 4-Velocity divided by proper time interval:

$$\underset{\rightarrow}{A}_u \equiv \frac{d\underset{\rightarrow}{V}}{d\tau} = \gamma_u \frac{d(\gamma_u c, \gamma_u \vec{u})}{dt} \qquad (14\text{-}3)$$

This is the 4-accelerator in one frame that the particle is moving with velocity u. Further expand the differentials:

$$\underset{\to}{A}_u = \gamma_u (c\frac{d\gamma_u}{dt}, \vec{u}\frac{d\gamma_u}{dt} + \gamma_u \frac{d\vec{u}}{dt}) = \gamma_u (c\frac{d\gamma_u}{dt}, \vec{u}\frac{d\gamma_u}{dt} + \gamma_u \vec{a})$$

$$\frac{d\gamma_u}{dt} = \frac{d(1-u^2/c^2)^{-1/2}}{dt} = (-\frac{1}{2})(1-\frac{u^2}{c^2})^{-\frac{3}{2}} \frac{-1}{c^2}\frac{d(\vec{u}\cdot\vec{u})}{dt} = \frac{\gamma_u^3}{c^2}\vec{u}\cdot\vec{a} \qquad (14\text{-}4)$$

Put these into the 4-Accelerator and we finally see the connection between this 4-accelerator with regular 3-D accelerator:

$$\underset{\to}{A}_u = \gamma_u (\frac{\gamma_u^3}{c}\vec{u}\cdot\vec{a}, \frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})\vec{u} + \gamma_u \vec{a}) \qquad (14\text{-}5)$$

Or write out each component explicitly:

$$A^0 = \frac{\gamma_u^4}{c}\vec{u}\cdot\vec{a}$$

$$A^1 = \frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})u_x + \gamma_u^2 a_x; A^2 = \frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})u_y + \gamma_u^2 a_y; A^3 = \frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})u_z + \gamma_u^2 a_z$$

And it is this 4-acceleration that will transform obeying LT, and we will see that the equation of motion can be expressed with this 4-acceleration. However its relation with regular acceleration is complicated, say suppose you know the Ai (i=0-3), solving for x(t) are a bunch of coupled differential equations.

The transform of regular acceleration (14-1,2) can also be derived from 4-acceleration and its transform (LT) between frames, and it is a straightforward and tedious computation and won't be presented here (you are encouraged to prove it yourself). So the two formalism on acceleration are equivalent, and we will see that the equation of motion when expressed with 4-acceleartion is much simpler.

Like all 4-vectors, its length is a scalar, the length for 4-accelerator expressed in terms of u, a is: (I skipped details and only copy the result)

$$|\underset{\rightarrow}{A}_u|^2 = (A^0)^2 - \sum_{i=1}^{3}(A^i)^2 = -\gamma_u^4 a^2 - \frac{\gamma_u^6}{c^2}(\vec{u}\cdot\vec{a})^2 \qquad (14\text{-}6)$$

This value is easiest to evaluate at what is called instantaneous rest frame of particle, i.e. a co-moving frame that travels with same velocity as particle at one moment, and since the velocity of particle under acceleration is changing, the frames will change accordingly from time to time (The lab observer is one frame S, and particle frame is S', while S' is changing from time to time). In such S' (instantaneous rest frame), the particle is stationary (u'=0, gamma (u')=1) and we see that:

$$|\underset{\rightarrow}{A}_u|^2 = -\alpha^2 \qquad (14\text{-}7)$$

α is the acceleration in the instantaneous frame, which is called proper acceleration.

We see that the transform relations between regular acceleration in different frames will be the simplest if one of the frame is the instantaneous frame S' (in which u'=0), also the relation between 4- and 3-acceleration is also simplest in such instantaneous frame (the relations involving instantaneous frame will be left for you to write out as a practice). This suggests the usefulness of using such frames in calculation involving acceleration in SR. However, keep in mind that the S' is instantaneous, meaning its velocity w.r.t. S (the lab frame) is changing continuously, i.e. terms involving u and gamma(u) are not constant. We shall see a simple example of solving the accelerated motion after we discuss the force in SR.

14.2 Force and 4-Force

(1) Definition of force and equation of motion in SR

First let's see the relation between force and energy and momentum. I had used force once before in the derivation of relativistic energy from work-energy theorem. There the force is defined as:

$$\vec{f} = d\vec{P} / dt \qquad (14\text{-}8)$$

This is the definition of force in SR (in a sense as discussed in footnote 137). We had seen that the energy coming out this satisfies the relativity (conserved in all frames) and correspondence principles, which justifies the (14-8) for force.

The work-energy theorem is:

$dK = \vec{f} \cdot d\vec{r}$, K is kinetic energy ($K=E-m_0c^2$)

$$\frac{dK}{dt} = \vec{f} \cdot \vec{u} \qquad (14\text{-}9)$$

Put in the total energy, and in the situation when ***the rest mass is a constant*** (no internal energy change of the particle):

$$\frac{dK}{dt}=\frac{d(E-m_0c^2)}{dt}=\frac{dE}{dt}=\vec{f}\cdot\vec{u} \qquad (14\text{-}10)$$

The (14-9) and (14-10) are clearly the analogy of power theorem in mechanics. These are relation between force and momentum-energy which is not completely new here, and what we learned in mechanics on impulse theorem and work-energy theorem can be applied in SR as well.

Now the question is what is the relation between force defined in (14-8) with acceleration? Do we still have F=ma?

Put the definition of momentum into (14-8):

$$\vec{f}=\frac{d(\gamma_u m_0\vec{u})}{dt}=\frac{d(\gamma_u m_0)}{dt}\vec{u}+\gamma_u m_0\frac{d\vec{u}}{dt}=\frac{d\gamma_u}{dt}m_0\vec{u}+\gamma_u m_0\vec{a}$$

Above I used assumption that the m_0, the rest mass of particle is a constant. We have seen in (14-4):

$$\frac{d\gamma_u}{dt}=\frac{\gamma_u^3}{c^2}\vec{u}\cdot\vec{a}$$

$$\vec{f}=\gamma_u m_0\vec{a}+m_0\frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})\vec{u} \qquad (14\text{-}11)$$

This is the relation between force and acceleration in SR, corresponding to the equation of motion F=ma in mechanics, and indeed we can see that at low speed limit, u<<c, the equation reduce to f=m_0a. However, generally the relation is more complicated in SR. Noticed that the acceleration *a* is not necessarily parallel with velocity u, in most cases, they are not along same direction. This means the

force and acceleration are not necessarily along same direction as shown in (14-11).

For example in 2-D for simplicity, if the particle with initial velocity (u_{x0},u_{y0}), and the force is along x direction only f=(f_x,0). The particle will have acceleration a_x along x direction and its u_x will increase. Consider the momentum along y direction which should be conserved for there is no force along this direction (14-8). $P_y = \gamma_u m_0 u_y$. As the u_x increases, the $u^2 = u_x^2 + u_y^2$ will increase and so will the $\gamma_u = 1/\sqrt{1-u^2/c^2}$. In order to keep the momentum along y to be conserved, the velocity along y, u_y has to decrease. So we see that though there is only force along x direction, the particle will have acceleration (deceleration in this case) along y too! This is quite different from that in Newtonian mechanics but in accordance with (14-11). Just rewrite the (14-11) to express explicitly from force to acceleration:

$$\vec{f} = \frac{d(\gamma_u m_0)}{dt}\vec{u} + \gamma_u m_0 \frac{d\vec{u}}{dt} = \frac{1}{c^2}\frac{dE}{dt}\vec{u} + \gamma_u m_0 \vec{a} = \gamma_u m_0 \vec{a} + \frac{1}{c^2}(\vec{f}\cdot\vec{u})\vec{u}$$

$$\vec{a} = \frac{1}{\gamma_u m_0}(\vec{f} - \frac{\vec{f}\cdot\vec{u}}{c^2}\vec{u}) \qquad (14\text{-}12)$$

In our simple example above, there will be non-zero y acceleration:

$$\vec{f} = (f_x, 0), \vec{u} = (u_x, u_y)$$

$$a_y = -\frac{1}{\gamma_u m_0}\frac{f_x u_x}{c^2}u_y$$

Only in some special cases, we have simple equation of motion in SR, for example if the acceleration is always perpendicular to the velocity (circular motion), the second term in (14-11) will disappear. Another case is when we choose instantaneous rest frame of moving particle, the equation of motion in such frame would be just:

$\vec{f}' = m_0\vec{\alpha}$ (14-13) in instantaneous rest frame

This makes perfect sense since in this inertial frame, the particle is at rest and the equation of motion would just be that in Newtonian mechanics. Unfortunately, transform the result back to lab frame won't be easy[151].

To find how the forces transform between frames, we could adopt the old strategy that:

$f' = dP'/dt'$ in S' and the transform of P and t are known (they obey LT), and we can find how the force is transformed, it won't surprise us that the force does not transform as LT (take a look at definition (14-8) and think why it is not LT yourself). I shall leave out the details of calculation here and give the transform relation after we construct 4-force.

(2) 4-Force

The force defined above is what we measured in lab but it does not transform obeying LT because it is momentum divided by dt, which is

[151] The situation is a bit similar to the free rotation of rigid body we discussed before. Choosing the principle axes as coordinate axes will simplify the equation of motion but the coordinate system is changing with time there. There is just no free lunch .

not a scalar upon transformation (same reason that the regular acceleration and velocity are not 4-vectors). Following the same procedure as constructing the 4-velcocity and 4-acceleration that have certain relations to the measurable quantities (the regular velocity and acceleration, such measurable physical quantity is also called observable) but transform as LT, we can construct 4-force that is relating to the observable force f and satisfies LT.

$$\underset{\rightarrow}{F} \equiv \frac{d\underset{\rightarrow}{P}}{d\tau} \qquad (14\text{-}14)$$

This is the 4-Force which is a 4-vector. Its relation to the observables can be directly worked out from formulas for 4-momentum and proper time:

$$\underset{\rightarrow}{P} = (E/c, \vec{P}), d\tau = dt/\gamma_u$$

$$\underset{\rightarrow}{F} \equiv \frac{d\underset{\rightarrow}{P}}{d\tau} = \gamma_u\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{P}}{dt}\right)$$

We can see from the above that the relation between the component of 4-vectors with physical observables: Its time component is related to the energy change over time; and its spatial components are related to regular force (so the 4-force is also called power-force vector) and writing out the component explicitly(for the case of constant m_0):

$$\begin{aligned} F^0 &= \frac{\gamma_u}{c}\frac{dE}{dt} = \frac{\gamma_u}{c}\vec{f}\cdot\vec{u} \\ F^{i=x,y,z} &= \gamma_u f_i \end{aligned} \qquad (14\text{-}15)$$

From this construction of 4-force and its relation with observables, we can find the transform formula among regular force (which I left out in previous section):

The two frames S, S' (relative v between them is along x direction) and forces measured by are f, f':

The relation between the 4-Force are already known (just LT):

$$F'^0 = \gamma(F^0 - \beta F^1)$$
$$F'^1 = \gamma(F^1 - \beta F^0)$$
$$F'^2 = F^2; \quad F'^3 = F^3$$

Throw in the relation of each component:

$$\frac{\gamma_{u'}}{c}\vec{f}'\cdot\vec{u}' = \gamma(\frac{\gamma_u}{c}\vec{f}\cdot\vec{u} - \beta\gamma_u f_x)$$
$$\gamma_{u'} f'_x = \gamma(\gamma_u f_x - \beta\frac{\gamma_u}{c}\vec{f}\cdot\vec{u})$$
$$\gamma_{u'} f'_y = \gamma_u f_y; \quad \gamma_{u'} f'_z = \gamma_u f_z$$

The relation between gammas is what we derived before:

$$\gamma_{u'} = \gamma\gamma_u(1 - \frac{u_x v}{c^2}) \qquad \text{(14-16)}[152]$$

The first equation gives us transformation between powers and the last 3 give us the transformation between regular force (***for the constant m₀***):

$$f'_x = \frac{f_x - \frac{v}{c^2}\vec{f}\cdot\vec{u}}{1 - u_x v / c^2}$$
$$f'_y = \frac{f_y}{\gamma(1 - u_x v / c^2)} \qquad \text{(14-17)}$$
$$f'_z = \frac{f_z}{\gamma(1 - u_x v / c^2)}$$

You see that the regular force do not transform as LT.

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[152] Which is proved in (13-1) for 1-D case; for general case here where u has components along x,y, z. The proof would be similar to that of (13-1) or using the transform property of 4-velocity, its temporal (0^{th}) component equation will lead directly to this relation, please try it yourself.

The equation of motion (relation between force and acceleration) can be written using 4-vectors in a very simple and familiar form:

$\underset{\rightarrow}{F} = m_0 \underset{\rightarrow}{A}$ (14-18)

The force and acceleration in the formula are 4-vectors, their relation to the observables are given by (14-15) and (14-5), throw these into (14-18), it is a practice for you to show that (14-18) indeed will give us relation between regular force and acceleration exactly as (14-11)

Generally the equation of motion as (14-18) or (14-11) are a bunch of coupled differential equations, knowing the force and initial conditions, solving the acceleration ,velocity and trajectory of the particle is not an easy task (most cases, you may not get an analytical answer), far more difficult than the Newtonian case. We shall take a look of a couple simple cases to have a taste of this where we may apply the instantaneous rest frame of the particle and simpler relations between force and acceleration in such frame.

Example1 Particle subject to a constant force

A particle is at rest initially, then a force field is turned on at t=0. The force will be constant and its direction is chosen as x-axis, i.e. $\vec{f} = (f_0, 0, 0)$ and the rest mass is m_0 for particle. After time t, what is the velocity and the distance travelled by the particle?

If it does not ask for acceleration, usually it is easier to work from momentum-force relation (14-8, impulse theorem):

Method 1:

$$\vec{f} = d\vec{P} / dt$$

P_y, P_z would be constant and since initially the particle is at rest these will be zero at all time, and so will be the u_y, u_z. The particle will only move along x direction.

$$dP_x / dt = f_0 \to P_x(t) = f_0 t$$

$$\frac{m_0 u_x}{\sqrt{1-u_x^2/c^2}} = f_0 t \to (m_0 u_x)^2 = (f_0 t)^2 (1-u_x^2/c^2) \to (m_0^2 + \frac{f_0^2 t^2}{c^2}) u_x = f_0^2 t^2$$

$$u_x = \frac{f_0 t}{m_0} \frac{1}{\sqrt{1+\frac{f_0^2 t^2}{m_0^2 c^2}}}$$

Once we know the velocity, we can find distance by integration over time, which is straightforward but may need integration table for the job.

Take a look at velocity over time, clearly at low velocity when time is small, the velocity reduces to Newtonian result, u=at, a=f/m. At longer time, the second term is not negligible, and at long enough time, the u will approach that of c. This makes perfect sense.

Method 2: I will also work this out using relation between force-acceleration.

Even when the force is constant, in lab frame the acceleration is not a constant (14-11 or 14-12):

$$\vec{a} = \frac{1}{\gamma_u m_0}(\vec{f} - \frac{\vec{f}\cdot\vec{u}}{c^2}\vec{u})$$

Initially u=0 and this will just give us a=f/m, the classical result. As velocity increases, the acceleration will decrease due to the increase of gamma and second term. Actually it is easy to see that as u approaches c, acceleration will be zero.

In our simple example, we can integrate above to find the $u_x(t)$:

$$a_x = \frac{du_x}{dt} = \sqrt{1-u_x^2/c^2}\,(\frac{f_0}{m_0} - \frac{f_0 u_x^2}{m_0 c^2})$$

This is a nonlinear 1st order ODE and certainly you can tackle it directly but here I can easily solve it by separation of variable:

$$a_x = \frac{f_0}{m_0}\sqrt{1-u_x^2/c^2}\,(1-u_x^2/c^2) = \frac{f_0}{m_0}(1-u_x^2/c^2)^{3/2}$$

$$\frac{du_x}{(1-u_x^2/c^2)^{3/2}} = \frac{f_0}{m_0}dt \rightarrow \int_0^{u_x} \frac{du_x}{(1-u_x^2/c^2)^{3/2}} = \frac{f_0}{m_0}t$$

The integration on the LHS can be found from integration table:

$\frac{cu_x}{(c^2-u_x^2)^{1/2}} = \frac{f_0}{m_0}t$ and u_x can be solved which will be same as I worked with momentum.

It is also interesting to work the second method from another point of view, using the instantaneous rest frame of particle. In this frame (I shall denote it S', it is moving with u_x relative to lab frame), u'=0 in such frame. The relation of transformation between force and acceleration in S and S' is simple:

$f'_x = f_x$ (easy to prove from 14-17)[153]

[153] You may try directly from (14-17) but it is more easy to work from the reverse transformation, i.e. express f_x in terms of f', the relation would be similar to (14-17) only change the sign involving v and switch label u to u'. Similar procedure applies to the transformation of acceleration too.

$a'_x \equiv \alpha = \gamma^3 a_x, \gamma = 1/\sqrt{1-v^2/c^2} = 1/\sqrt{1-u_x^2/c^2}$ (easy to prove from 14-1)

In the rest frame S', the relation between force and acceleration is just the classical mechanics one:

$\alpha = \frac{f'_x}{m_0} = \frac{f_0}{m_0}$ and in such instantaneous frame the constant force will give us constant acceleration. But we cannot do the integral in this instantaneous frame, because it is changing over time, we have to transform back to lab frame:

$\gamma^3 a_x = \alpha \to \frac{du_x}{(1-u_x^2/c^2)^{3/2}} = \frac{f_0}{m_0}dt$ this will give us exactly same integration as before so same answer. I hope I have shown you enough methods to treat this simple dynamical problem.

Example2: Particle with u_y subject to constant force (consider only in 2-D x-y).

Similar situation as above, but the particle has initial velocity along y. i.e. initial velocity is $(0,u_0)$, the force is along x direction $(f_0,0)$. What is the velocity of the particle at later time?

We have seen that it is easier to work with impulse theorem:

$$P_y = \gamma_u m_0 u_y = P_{y0} = const = \frac{m_0 u_0}{\sqrt{1-u_0^2/c^2}}, \gamma_u = \frac{1}{\sqrt{1-(u_x^2+u_y^2)/c^2}}$$

$$\frac{m_0 u_y}{\sqrt{1-(u_x^2+u_y^2)/c^2}} = P_{y0}$$

For the x component:

$$dP_x = f_0 dt \to P_x = f_0 t$$

$$\frac{m_0 u_x}{\sqrt{1-(u_x^2+u_y^2)/c^2}} = f_0 t$$

There are two unknowns with two equations, and luckily here the two equations can be squared, thus the two unknowns to be u_x^2 and u_y^2, and this becomes a couple of linear equation which can be solved with standard method. In the following calculation I shall choose the unit so that c=1; X= u_x^2; Y= u_y^2:

$$m_0^2 Y = P_{yo}^2(1-X-Y) \to P_{y0}^2 X + (m_0^2 + P_{y0}^2)Y = P_{y0}^2$$

$$m_0^2 X = f_0^2 t^2 (1-X-Y) \to (m_0^2 + f_0^2 t^2)X + f_0^2 t^2 Y = f_0^2 t^2$$

$$Y = \frac{P_{y0}^2}{m_0^2 + f_0^2 t^2 + P_{y0}^2} \to u_y^2 = \frac{P_{y0}^2}{m_0^2 c^2 + f_0^2 t^2 + P_{y0}^2} c^2$$

The last step is from dimensional analysis to put the c back into relation (of course if you never set c=1, you do not need worry about this step, however in complicated calculation, it is usually time saving to set c=1 at beginning and put it back through dimension analysis afterwards, like what I did here)

Similarly:

$$u_x^2 = \frac{f_0^2 t^2}{m_0^2 c^2 + f_0^2 t^2 + P_{y0}^2} c^2$$

From the results, it is easy to see that as time goes on, the x component of velocity will increase (but never exceed c) and the y component will decrease as we discussed earlier, there is deceleration along y though force only has x component, as you can see also from relation:

$$\vec{a} = \frac{1}{\gamma_u m_0}(\vec{f} - \frac{\vec{f} \cdot \vec{u}}{c^2}\vec{u}).$$

You may also try to solve the problem by using the force-acceleration relation above, and this will be left for you to explore. After finding the velocities, the trajectory can be worked out in a straightforward (but maybe nasty) integration, and I will not do that here.

Example 3 1-D harmonic oscillator

For a particle subject to a force f=-kx and the force can be written also in convention of $f = -m\omega^2 x$, the amplitude of the oscillation is A, what is the period of this oscillator?

Suppose m is initially at A, so u=0, x=A initially. The time for m travel from A to 0 is: $t_0 = \int_A^0 \frac{dx}{u}$, the period will be $4t_0$. The problem then will be find the relation between velocity and position u(x).

Starting from impulse or acceleration (which is derived from the impulse) would be equivalent here, so I shall use:

$$\vec{a} = \frac{1}{\gamma_u m_0}(\vec{f} - \frac{\vec{f} \cdot \vec{u}}{c^2}\vec{u})$$

$$a_x = \frac{du}{dt} = \frac{1}{\gamma_u m}(f - \frac{fu^2}{c^2}) = \frac{-\omega^2 x}{\gamma_u^3}$$

This result could also worked out from acceleration in instantaneous rest frame as in last example $\gamma^3 a_x = \alpha = -\omega^2 x$. We need to find u(x) so I rewrite the relation above as:

$$\frac{du}{dt}=\frac{du}{dx}\frac{dx}{dt}=u\frac{du}{dx}=\frac{-\omega^2 x}{\gamma_u^3}$$

$$\gamma_u^3 udu=-\omega^2 xdx$$

Doing integration on both side will give us u(x) and thus the period. The integral on the LHS is a bit nasty if doing it directly. Fortunately, we know (in the derivation of 14-11): $\frac{d\gamma_u}{dt}=\frac{\gamma_u^3}{c^2}\vec{u}\cdot\vec{a}\rightarrow d\gamma_u=\frac{\gamma_u^3}{c^2}udu$ in this case.

$$d\gamma_u=-\frac{\omega^2}{c^2}xdx\rightarrow\gamma_u=-\frac{1}{2}\frac{\omega^2}{c^2}x^2+const$$

The constant can be found from initial condition, x=A, u=0, gamma=1

$$const=1+\frac{1}{2}\frac{\omega^2}{c^2}A^2$$

$$\gamma_u=1+\frac{1}{2}\frac{\omega^2}{c^2}A^2-\frac{1}{2}\frac{\omega^2}{c^2}x^2=1+\frac{1}{2}\frac{\omega^2}{c^2}(A^2-x^2)$$

From this we can find out u directly from definition of gamma u:

$u=-\frac{c\sqrt{\gamma^2-1}}{\gamma}$ (taking minus sign because of our initial condition)

The period can be found out from integral:

$T=4t_0=4\int_0^A\frac{dx}{u}$ It is generally quite a nasty integration and I shall leave the formula as it is, and also from u(x), we can find x(t) through another nasty integration, so simple harmonic oscillation in classical mechanics is not so simple in SR.

Bibliography

Listed below are books that I read in the preparation of this lecture notes. Most books are read from beginning to end and some only a few chapters. The ones that I like most have already been listed in the reference list for the course.

Mechanics:

李复:力学教程 (上)

赵凯华,罗蔚茵 力学

郑永令 等 力学

Feynman, Richard et al. *Feynman Lectures on Physics* (Vol. I)

Forshaw, J.R and Smith A.G. *Dynamics and Relativity (The Manchester Physics Series)*

French, A. P. *Vibration and Waves*

Goldstein, Herbert et al. *Classical Mechanics (3rd edition)*

Greiner, W *Classical Mechanics, point particles and relativity; Classical Mechanics, System of particles and Hamiltonian Dynamics (2nd edition)*

Kleppner, D and Kolenkow, R *An Introduction to Mechanics*

Landau, L. D. and Lifshitz, E. M. *Mechanics (3rd edition)*

McCall, Martin *Classical Mechanics: From Newton to Einstein: An Modern Introduction*.

Morin, David *Introduction to Classical Mechanics with Problems and Solutions*

Serway and Jewett *Physics for Scientists and Engineers (6th edition)*

Taylor, John. R *Classical Mechanics*

Special Relativity:

刘佑昌 狭义相对论及其佯谬

Bohm, David *The Special Theory of Relativity*

Cheng, T. P. *Relativity, Gravitation and Cosmology*

Einstein, Albert *Relativity*

Einstein et al. *The Principle of Relativity* (this is a collection of English translated papers)

French, A. P. *Special Relativity*

Freund, Jürgen *Special Relativity for Beginners*

Moller, C. *The Theory of Relativity*

Pauli, Wolfgang *Theory of Relativity*

Rindler, W. *Relativity: Special, General and Cosmological*

Resnick, Robert *Introduction to Special Relativity*

Mathematics:

Arfken, G. *Mathematical Methods for Physicists (6th edition)*

Boas, M. *Mathematical Methods in Physical Sciences (2nd edition)*

Borisenkon and Tarapov *Vector and Tensor Analysis with Applications*

Edwards, C. H. and Penny, D. E. *Elementary Differential Equations (6th edition)*

Thomas's Calculus (11th edition)

Supplementary Materials on Math Needed in Mechanics

by Shuo Jiang

Preface

One difficulty (obstacle) for freshman physics is the math needed by the students. It requires Calculus (both single and multivariable versions) and some basics on solving ordinary differential equations, as well as Linear Algebra. The materials in the math are generally not required in high schools in this country, and will be taught simultaneous with physics in the first semester in college. While we are not going to encounter serious Linear Algebra in the first half of physics course, and I hope by the time I need LA at the second half of the semester, you will have learnt enough from the math course. However, the calculus and ordinary differential equations will be needed from the beginning of the physics course, especially calculus. I shall arrange the physical course that needed single variable calculus first, then multivariable (those partial derivatives etc.) calculus and then ordinary differential equation. I write this math supplementary accordingly in these three parts, hoping as a guidance or crutch that helps you understanding some basics and be able to use them in physics. This is not intending to replace the math textbook of course.

Supplementary I

Highlights of Differentiation and Integration

This is a short and crash course on part of calculus, dealing with differentiation (derivative of function) and integration (integral of function). Within this supplementary, I shall list some of the highlights of calculus (single-variable) and their applications in physics. There will be a few examples and not many rigorous proofs (you have to refer to math textbooks on calculus for the proof and more examples)[154].

Calculus is about the relation between functions. You have two functions which are related, each is a function of some variables explicitly or implicitly. Say function 1 is the distance traveled over time and function 2 is the velocity over time. So function 2 is the change rate of function 1 over the variable. i.e. function 2 express the change of function 1 as the variable changes. From function 1 you can calculate function 2, the function of rate of change, this is called differentiation. From function 2 (the rate of change), you can calculate the function 1, this is integration.

1. Differentiation of a function

This is also called derivative of a function. This will give you another

[154] A widely-used textbook (in US) on basic calculus for freshman is Thomas "Calculus". It has hundreds of examples and thousands of exercises, but it is almost 1600 pgs. That is why I decide to write this supplementary.

function which represents the rate of change of the initial function. To give a rigorous derivation on derivatives, you will need definitions regarding to functions, limits and continuity. All these would be too mathematical to present here for this supplementary, so I will dive into derivative directly.

The functions here we considered are what are called single variable functions, i.e. with only one input variable that changes. For the general case of multi-variable functions, I will leave that part to another supplementary on partial derivatives later.

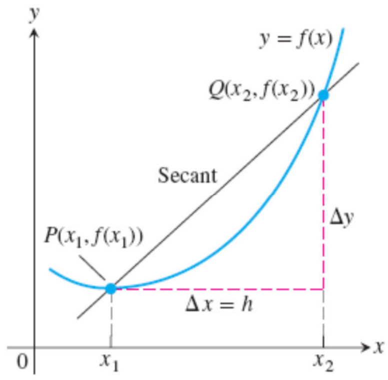
1-1. Geometric and physical interpretation of derivatives

Given a function of some single variable, say $y = f(x)$, where x is the variable (the input), and y is the output, f is a function represents their relations, i.e. knowing x you can calculate y. As the input x changes, the output y will also change. If the x is changed to some other value, say $x + \Delta x$, the y will change from $f(x)$ to $f(x+\Delta x)$. The rate of change is defined as:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (1)$$[155]

This is also called *average* rate of change, and is represented by the secant line in the graph below, the average rate of change is the slope of the secant line connection the two points P,Q.

[155] In the following discussion, sometimes I may mix f with y, they are the same anyway. y and f are just the symbols representing the same function. So in later notations, $y \equiv f$, $y' \equiv f'$, $dy/dx \equiv df/dx$



However, in many situations, the average rate of change may not be enough. We need the *instantaneous* rate of change. i.e. the average speed of a car between 2 second and 5 second may not provide enough information, we need the speed of the car right at the 2 second. This corresponds to making the Q approaching P in the above figure, and the Δx becomes smaller, approaching 0. Under this limit, we calculate the rate of change. That is the instantaneous rate of change of function f(x) at certain point, say x_1. This instantaneous rate of change is the physical interpretation of derivatives. The notation of derivatives of $y = f(x)$ has a few commonly used forms:

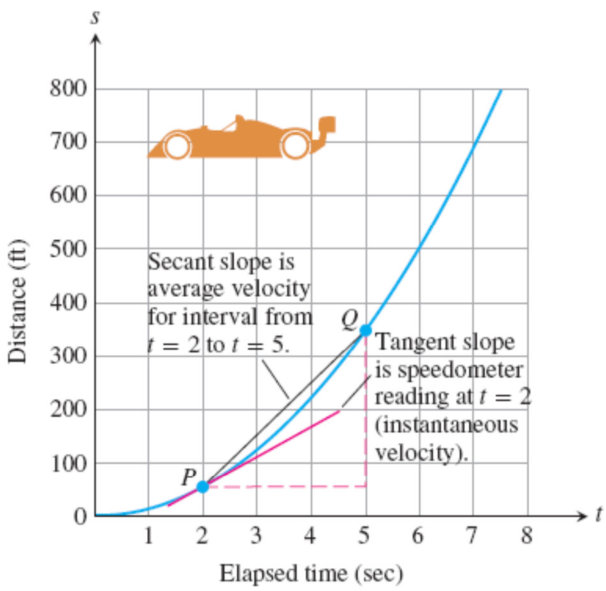
$$y' \equiv \frac{dy}{dx} \equiv \frac{d}{dx} f(x) \equiv Dy \qquad (2)$$

y' is the Newton notation; $\frac{dy}{dx}$ is the Leibniz notation; Dy is the operator notation which D represents the differentiation operator $\frac{d}{dx}$. The basic math definition of derivative is:

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (3)$$

The derivative is also a function of the same variables. For example, if f(t) is the distance, the $\frac{d}{dt}f(t)$ is the instantaneous velocity at time t, v(t). For the derivative over time, physicists also often use notation $\dot{f}$. If T(x) is a temperature distribution, then $\frac{d}{dx}T(x)$ is the temperature gradient g(x). If the Q(t) is the number of charge at one location, then $\dot{Q}=\frac{d}{dt}Q(t)$ is the current at that location, I(t).

The geometric interpretation is also clear from the figure above. As the Q approaches P, the secant line will become the *tangent* line at position P. So the derivative at x_1 is the slope of the tangent line of the original function passing x_1. (also refer to the figure below)



1-2. Derivatives of basic functions

Here I only listed derivatives of some basic functions without proof.

(1) Derivatives of Polynomial functions

$\frac{d}{dx}x^n = nx^{n-1}$, n is any integer (4)

$\frac{d}{dx}x^r = rx^{r-1}$, x>0, r is any real number (5)

(2) Derivatives of trigonometric functions

$\frac{d}{dx}\sin x = \cos x$ (6)

$\frac{d}{dx}\cos x = -\sin x$ (7)

(3) Derivatives of exponential functions

$\frac{d}{dx}e^x = e^x$ (8)

$\frac{d}{dx}\ln x = \frac{1}{x}$ (9)

$\frac{d}{dx}a^x = (\ln a)a^x$, a>0 (10)

The e^x is the natural exponential, (8) can be a definition of the natural exponential. From that, it can be represented in the polynomial forms as:

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \times 2} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (11) or

$e^x = \lim_{N \to \infty}(1 + \frac{x}{N})^N$ (12)[156]

When you put x=1 in the (11) or (12), you will get the numerical expression for e.

[156] These relations (including the proof of the relations (4), (6), (7)) are fundamental in derivatives. Please do check math book for their derivations! (for example, Thomas "Calculus", or the calculus textbook in use by you at present)

Of course you may check the math books for more formulas on derivatives of other functions. But generally given a function, you can calculate its derivative knowing only the derivatives of some basic functions (the relations (4), (6), (7) and (8) listed above). This is because we have some general rules of derivatives.

1-3. General rules for derivative

The f(x) and g(x) are some functions of variable x. f'(x) and g'(x) are their derivatives over x. Then:

Rules of linearity:

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx} \qquad (13)$$

$$\frac{d}{dx}(cf(x))=c\frac{df}{dx}\text{, c is constant} \qquad (14)$$

Production Rule:

$$\frac{d}{dx}(fg)=g\frac{df}{dx}+f\frac{dg}{dx} \qquad (15)$$

Quotient Rule:

$$\frac{d}{dx}(\frac{f}{g})=\frac{f'g-fg'}{g^2} \qquad (16)$$

For composite functions (implicit dependence), i.e. f(u) a function of variable u, but the variable is also a function of x, u(x), then what is the derivative of f with respect to x? We have the powerful (probably the most useful rule in derivatives) chain rule:

$$\frac{d}{dx}[f(u)]=\frac{df}{du}\frac{du}{dx} \qquad (17)$$

I did not give the proof of these rules from the definition of derivatives (relation (3)), please do it yourself or look for them in calculus textbooks. With the derivatives of basic functions and these general rules, the derivative of some general functions can be calculated.

Example 1: $f(x)=\tan x=\frac{\sin x}{\cos x}$, what is its derivative over x?

Ans.: $\frac{d}{dx}(\tan x)=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{1}{\cos^2 x}$

(using quotient rule)

Example 2: $f(x)=a^x$, a>0, its derivative (relation 10)

Ans.:
$$a=e^{\ln a}, a^x=e^{x\ln a}; let\ u=x\ln a$$
$$\frac{d}{dx}a^x=\frac{d}{dx}e^u=\frac{de^u}{du}\frac{du}{dx}=e^u\ln a=a^x\ln a$$

(using chain rule)

Example3: $f(x)=\tan(\frac{1}{x})$, what is its derivative?

Ans.: $let\ u=\frac{1}{x},\ \frac{d}{dx}\tan u=\frac{d\tan u}{du}\frac{du}{dx}=\frac{-1}{x^2\cos^2 u}=\frac{-1}{x^2\cos^2(1/x)}$

Useful techniques in calculating derivatives

Implicit derivative.

Sometimes the functional relation between the y and x are not explicitly given. For example, in the expression of an ellipse, the relation of y and x are given by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then what is the derivative of dy/dx?

This kind of problems can be solved by two methods.

Method 1: parametric functions.

The y and x are some functions of a parametric variable t. In the elliptical function above, we can introduce the parametric variable t, so that:

$$y = b\sin t$$
$$x = a\cos t$$

Then $\frac{dy}{dt} = b\cos t, \frac{dx}{dt} = -a\sin t$

To calculate the dy/dx, it is possible to prove from the definition of derivatives, that:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (18)$$ provided all derivatives exist.

Form 18, dy/dx for ellipse is:

$$\frac{dy}{dx} = -\frac{b\cos t}{a\sin t} = -\frac{b}{a}\cot(t) = -\frac{b^2 x}{a^2 y}$$

Method 2: Implicit derivative.

Take derivative of the equation over x. (the equations just express the equality of two functions, so you can take derivatives on both sides, since the functions on two sides always equal, the derivatives should be equal too) i.e.

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$$

$$\frac{dy}{dx}=y'=-\frac{b^2x}{a^2y}$$

This is same as the parametric method. Of course, if you want to get explicit expression of dy/dx as function of x, you will need explicit function form of y(x).

(1) Derivatives of inverse functions

This can be treated as a special case as implicit derivative. Sometimes the function form is the unfamiliar inverse function type, the example would be $y=\sin^{-1}x=\arcsin x$. For these kinds of functions, we can get its derivative using the implicit method.

Example: $y=\arctan x$, what is the derivative dy/dx?

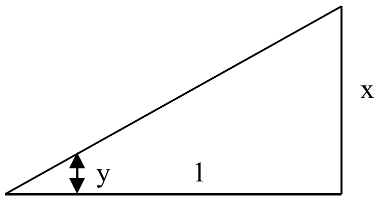
$$\tan y = x$$

$$\frac{d}{dx}(\tan y)=1$$

Ans.: $$\frac{d}{dx}(\tan y)=\frac{d\tan y}{dy}\frac{dy}{dx}=\frac{1}{\cos^2 y}\frac{dy}{dx}$$

$$\frac{dy}{dx}=\cos^2 y=\frac{1}{1+x^2}$$

The last relation is best easily seen from geometric graph below:



The above calculation can also be treated directly from the point of view of inverse function. i.e. y=f(x); x=f^{-1}(y).

$$\frac{dx}{dy}=\frac{d}{dy}(f^{-1}(y))=\frac{1}{dy/dx}=\frac{1}{df(x)/dx} \qquad (19)$$

In relation 18, 19, the derivatives are treated like conventional quotients, this is true in most cases, since the derivative from definition just the quotient of two small numbers, $\frac{\Delta y}{\Delta x}$ as denominator goes to 0. This is very useful and easy to remember, and that is why Leibniz notation is preferred over Newton's

(2) Logarithmic method

This is just another special case for implicit derivative. Sometimes (especially involve exponentials) it is more easy to take logarithm on both side and use implicit method to find out the derivative.

Example 1: Prove that $y=x^r$, x>0 and r is any real number (rational or irrational), the derivative is that in forms given by relation (5).

$$\ln y = r \ln x$$

$$\frac{d}{dx}\ln y = r\frac{d}{dx}\ln x$$

Ans.: $$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x}y = \frac{r}{x}x^r = rx^{r-1}$$

Example 2: $y = x^x$, x>0, what is dy/dx?

$$\ln y = x \ln x$$

Ans.: $$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x\ln x) = \ln x + x\frac{d\ln x}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = (\ln x + 1)y = (\ln x + 1)x^x$$

1-4. Higher order derivatives

From the original function, you can calculate its derivative now, which is also a function, representing the rate of change of the original one. Now we can also calculate the derivative of this function (the derivative of the derivative), which is the change rate of the rate of change. The process will go on forever, leads to higher orders derivatives of the original function. The most useful in physics is the second order derivative of the original function.

Given a function $y = f(x)$, its second order derivative is:

$$y'' \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) \equiv \frac{d^2y}{dx^2} \equiv D^2 y \qquad (20)$$

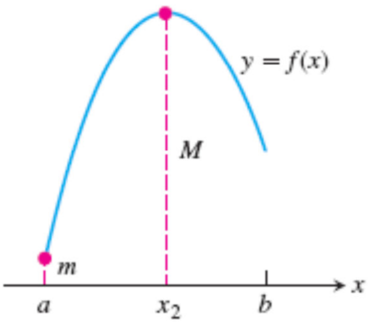
The above definition listed some mostly used notations, noted that the x^2 there has nothing to do with the usual square of x, just a bookkeeping. To calculate the second derivative from the original function is proceeding as the definition suggest: First calculate the first order derivative $f'(x)$ (a function of x), then calculate the derivative of the first derivative to get second order derivative $f''(x)$.

Examples in physics are relations between distance x(t), velocity v(t) and acceleration a(t):

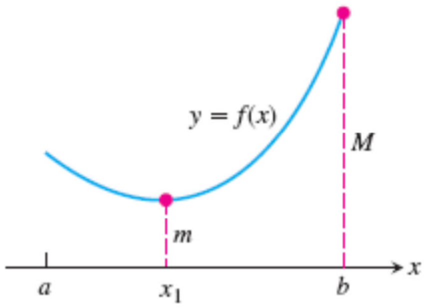
$$v(t) = \frac{dx(t)}{dt}$$

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}(\frac{dx(t)}{dt}) = \frac{d^2x}{dt^2}$$

Geometrically, the second order derivative is obviously the slope of the first derivative. But what is its relation to the original function? The second order derivative shows the curvature (or the 'bent') of the original function.

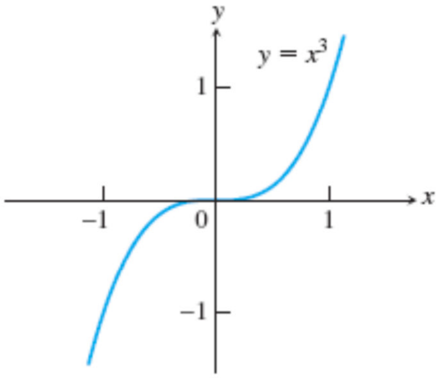


Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

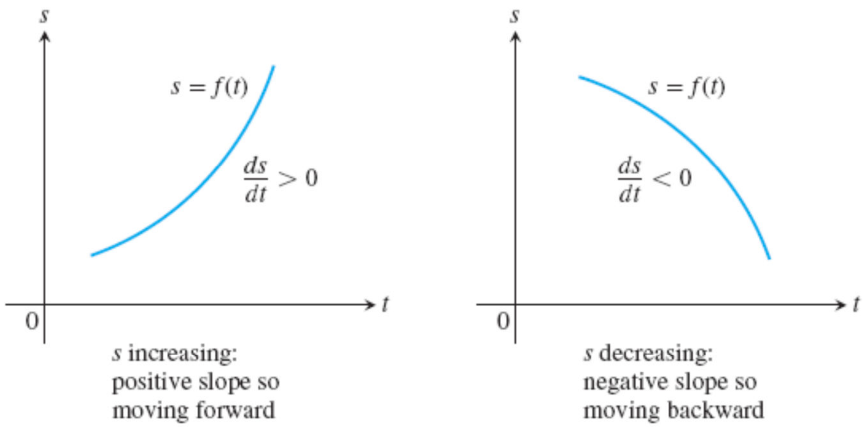
In the figure above, the left one with negative second order derivative, $d^2y/dx^2 < 0$, the curve is bending downward, is also usually called 'convex' curve. The curve representing the function at right with $d^2y/dx^2 > 0$, it is bending upward, also called 'concave' curve.



(a)

For the curve here, when x>0, $d^2y/dx^2 > 0$, its 'concave'; for x<0, $d^2y/dx^2 < 0$, it is 'convex'. At x=0, $d^2y/dx^2 = 0$, this point is called the *Inflection Point*.

It is also illustrating showing the curve with positive and negative first order derivative. The positive and negative derivative is related to the increasing or decreasing function over variables.



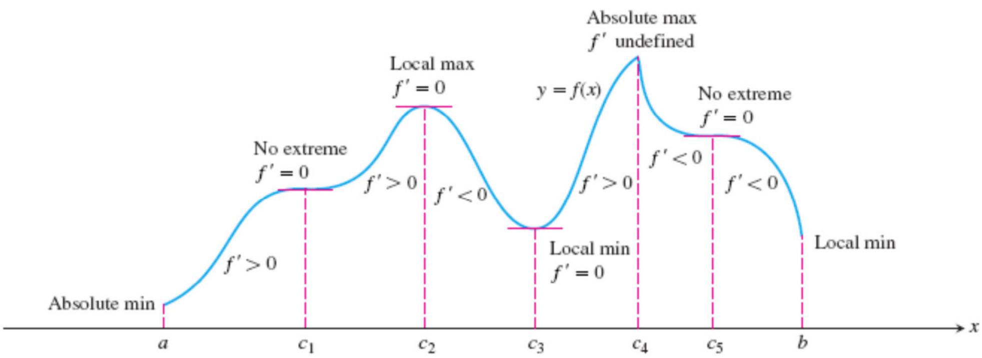
What happens if the $dy/dx = 0$? That is the local extreme points we shall discuss next.

1-5. Some applications of derivatrives

The most important application of derivatives is to evaluate the differentials of function, i.e. the samll change of the function as the variable has a samll change. That will be discussed in next section under differentials. Here only list out two other applications you will encounter quite often in physics.

(1) Finding the local extreme points (often related to finding the maximum and minimum values of the function)

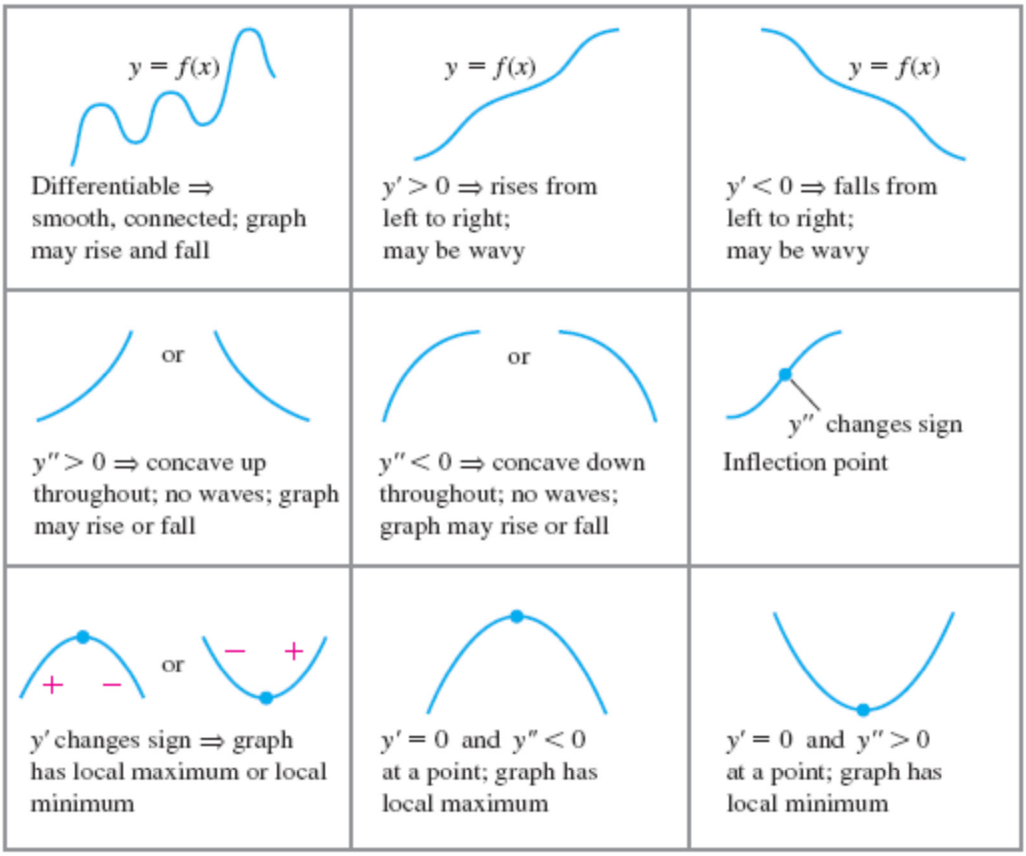
As stated in the last section, when $dy/dx = 0$, the points satisfy this relation are some special points on the curve of origianl function y=f(x). These points are called *critical points* and they can be a *local minimum* and *local maximum* or inflection points. These will be quite clear from the figure below.



In the figure, C2 and C3 are local maximum and minimum repectively. C1 and C5 are inflection points. The $dy/dx = 0$ at these points. To further distinguish which is local-max, or local-min, etc. You need to evaluate second order derivative at those critical points.

At the critical points ($dy/dx = 0$), if $d^2y/dx^2 > 0$ (a bending up concave curve), you will have a local-minimum. If $d^2y/dx^2 < 0$ (a bending down convex curve), you will have a local-maximum. If $d^2y/dx^2 = 0$ too, that is an inflection point.

To find out the absolute extremes, such as absolute maximum or minimum, you have to also check the points at discontinuity and points where derivatives are undefined (refer to the figure) and compare them with the value at local-extremes founded from derivatives. For local-extremes, maybe more easy to remember from the figure below:



So derivatives would be useful in problems of optimization process (such as which would be the path taking the least time…).

(2) L'Hopital's Rule[157]

This rule will allow us using derivatives to evaluate values of functions when direct calculation is impossible. Such as the sinc function at x=0, i.e.

$\frac{\sin x}{x} = ?$ at x=0.

Instead of writing down the rule myself, I just copy the rules from Thomas's Calculus.

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

[157] Pronounced as (lou pi ta)

1-6. Differentials and first order approximation (linear approximation) of a function change.

From the definition of derivatives, dy/dx, is just the ratio between two infinitesimal changes, dx is an infinitesimal change of the variable, and dy is the corresponding change of the function output. So sometimes the derivative is also called infinitesimal quotient (that is exactly how Leibniz treat it)

Differential is just the difference, the difference of what? The difference of functions $\Delta y = f(x+\Delta x) - f(x)$ as variable changes an amount of Δx. Of course if Δx approaches zero (infinitesimal small), the differential will be just nothing but rewriting derivative as:

$$dy = \frac{dy}{dx}dx = y'dx$$

In real applications the step took by the variable x may not be infinitesimal small as in derivative, but with limited-size steps of the general form Δx, then is it true the $\Delta y = f(x+\Delta x) - f(x) = f'(x)\Delta x$? Where f'(x) is the derivative over x at x.

It can be proved that the general form the change of functions due to Δx would be in forms of:

$$\Delta y = y'\Delta x + o(\Delta x) \qquad (21)$$

y' is just the first order derivative, $o(\Delta x)$ is a higher order term of Δx, such as $a(\Delta x)^2 + b(\Delta x)^3 + \ldots$, the more strict definition would be:

$$\lim_{\Delta x \to 0} \frac{o(\Delta x)}{\Delta x} = 0 .$$

The *first order approximation* of the differential:

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x \quad \text{small } \Delta x \qquad (22)$$

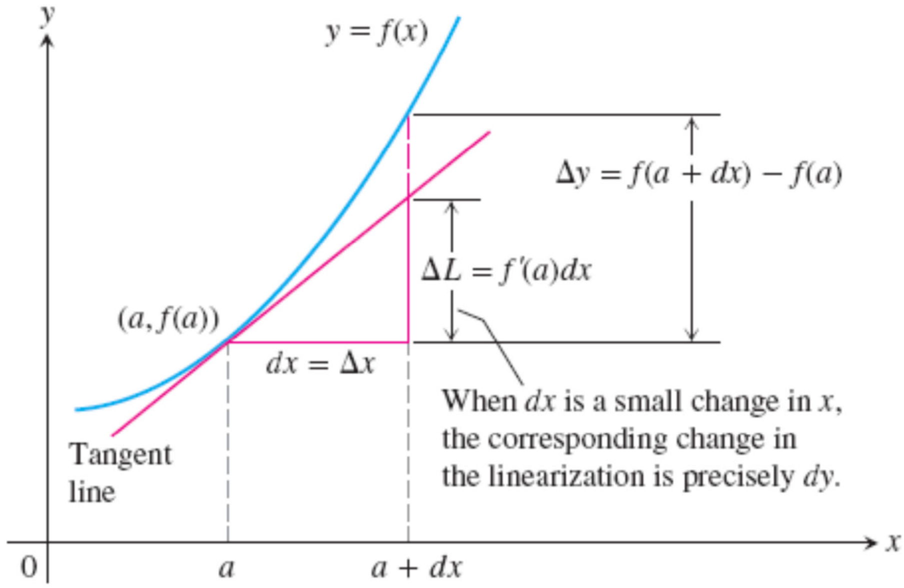
Or put it in another form which is just rearranging the order of (22):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) \qquad x \sim x_0 \qquad (23)$$

This expresses that we can evaluate the function in the proximity of certain initial point (x_0, $f(x_0)$), this is what I used to get motion of a little time later knowing the initial state (position and velocity). The relation 22 and 23 is equivalent to use a simple tangent line passing through the initial point to approximate the actual curve, which may be more complicated (see the figure below)

Such approximation would be valid if the step of $\Delta x \equiv x - x_0$ is sufficiently small, (if infinitesimal small, it would be strictly equal) so that higher order terms of Δx can safely neglected. This is called linear approximation, since the dependence on Δx is linear. This approximation is used widely in various branches of sciences. We always start from linear approximation, and make corrections if necessary regarding to the higher order terms.

The above ideas are clearly illustrated in the figure below.



Example: $y=(1+x)^N$. If the x changes from 0 to a small value x, i.e. $\Delta x = x-0 = x$, what is the value of y then?

$$y(0)=1$$

$$\Delta y = y(x)-1 = \frac{dy}{dx}\Big|_{x=0}\, x$$

$\frac{dy}{dx}\Big|_{x=0}$ means the derivative evaluated at point x=0.

$$\frac{dy}{dx}=\frac{d}{dx}(1+x)^N = N(1+x)^{N-1}$$

$$\frac{dy}{dx}\Big|_{x=0}=N$$

$$\Delta y = y(x)-1 = Nx$$

$$or\ y(x)=y(0)+Nx = 1+Nx$$

This gives a good formula to calculate y(x) when x is a small number.

So in the relations in special relativity, we come across the expression:

$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ quite frequently, and if v<<c, then the expression may

be simplified as: $\gamma = (1 - v^2 / c^2)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$

Other useful linear approximations we often use are (prove it yourself):

$$\begin{aligned}
&\sin x \approx x \quad (x \sim 0) \\
&\cos x \approx 1 \quad (x \sim 0) \\
&e^x \approx 1 + x \quad (x \sim 0) \\
&\ln(1+x) \approx x \quad (x \sim 0)
\end{aligned} \qquad (24)$$

These simplification would be very useful in the evaluation of functions, it replace the original complicated (sort of) function with simpler ones (the polynomial). You will get pretty good answers for values $\ln 1.01; e^{0.01}$ without using any electronic help.

I should also mention that in a lot of cases, we shall set x_0=0 (can always achieve this by shifting the origin of coordinate), then the formula 23 will become:

$f(x) = f(0) + f'(0)x$ when x~0. (25)

In cases that linear approximation is not good enough, most likely due to larger Δx, we can use quadratic approximation which involves 2nd order derivative:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \quad x \sim x_0 \qquad (26)$$

Or:

$$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 \quad \text{for } x \sim 0 \qquad (27)$$

You are encouraged to calculate the forms of relations in 24 above, when take quadratic term into consideration.

While the linear approximation is using a line to simulate the original curve, you see the quadratic approximation improved the simulation by using parabola. The reason of there is 1/2 in front of the quadratic term can be understood by: Suppose we approximate the original function by some quadratic forms, the most general one would be:

$f(x) = a + bx + cx^2$ (set x_0=0) We can find the expressions of the coefficients by:

$$a = f(0)$$
$$b = f'(0)$$
$$c = \frac{1}{2} f''(0)$$

So 1/2 simply comes from the derivative of the power series of x. (can you guess what will be the term if we include cubic terms, i.e. x^3, what is its coefficient in terms of derivatives? You will see I am approaching Taylor expansion)

1-7. Power series and Taylor Expansion

Power series is just an extension of polynomials, which are relatively speaking, simple functions, especially in differentiation and integration. Polynomials are:

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n \qquad (28)$$

If n goes to infinity, we have a power series.

$$P(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (29)$$

As the series go to infinity, there is a danger of divergence for certain x, i.e. the series may explode to infinity or underdetermined value if x gets too big. In such cases, the series will be useless, because we cannot handle it. However, it can be proved, the series will converge (loosely speaking, approaches to a fixed value as n approaches infinity) if $|x|<R$. R is a real positive number, which is called *radius of convergence* for the power series. Clearly the R will depend on the c's (the coefficients of the power series). We generally do not worry too much about the relation of R with c's in physics, because either the R may be quite obvious or the range of x that can change will be small enough to satisfy $|x|<R$. So we can only work with power series within the radius of convergence.

In physics, the power series is very useful in terms of Taylor expansion. That is to say we express the original function (may be complicated, with strange or unfamiliar form) into power series. i.e. the original function can be written as a power series:

$$f(x)=\sum_{n=1}^{\infty}c_n x^n \qquad (30)$$

This seems very much the same as (29). While (29) is a definition of power series, here the (30) states that an arbitrary function[158] can be expressed as power series. Of course, it requires that x should be within the convergence radius, i.e. $|x|<R$. This is an amazing fact that will

[158] Well, not too arbitrary, mathematician can think of functions that violate this, but all functions encountered in real world (physical problems) can be expressed in power series.

simplify the way we handle complicated functions because power series are relatively easy.

Then the question is: what are the expansion coefficients c? Knowing the original function (also called generating function of the Taylor series), the c's can be calculated straightforward. From (30), it is easy to see the following:

$$f(0) = c_0$$
$$f'(x) = c_1 + 2c_2x + 3c_3x^2 ... \to f'(0) = c_1$$
$$f''(x) = 2c_2 + 3\times 2c_3x + ... \to f''(0) = 2c_2$$
$$f'''(x) = 3\times 2c_3 + 4\times 3\times 2c_4x + ... \to f'''(0) = 3\times 2c_3$$
$$f^n(x) = n!c_n + (n+1)!x + ... \to f^n(0) = n!c_n$$

So the coefficients can be calculated by taking derivatives of original function and evaluated at x=0, and the (30) becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \qquad (31)$$

This is called Taylor expansion of function f(x), it is an expansion centered around x=0. There is another useful and more general way to make the expansion around a fixed $x=x_0$, then the formula will be:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x-x_0)^n \qquad (32)$$

The x would be centered at x_0, and the coefficients are also the derivatives evaluated at x_0.

Some comments on Taylor expansion:

(1) The range of x needs to be within the convergence radius of the power series.

(2) In order to make expansion, the f(x) has to be differentiable around x_0, you cannot expect the expansion will work if the function is discontinuous at x_0, or the derivatives do not exist. Even if the original function has discontinuity, but if our center of expansion is away from those 'bad points', locally (around x_0, in an interval that does not contains the bad points and within the convergence radius) Taylor expansion works fine.

(3) There is an issue of how accurate the expansion can replicate the original function, i.e. whether the power series on the right hand side of (31) or (32) will converge (by taking enough n) to the original function or not. If condition 1 and 2 are satisfied, it turned out for all functions in physics, the convergence can be achieved.

The most famous examples are (you should check below using (31)):

(a) Geometrics series:

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + ... \quad (|x| < 1) \qquad (33)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ...(-1)^n x^n ... \quad (|x|<1) \qquad (34)$$

(b) Exponentials and sinx, cosx:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} ... \quad (|x| < \infty) \qquad (35)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} .. \quad (|x| < \infty) \qquad (36)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} .. \quad (|x| < \infty) \qquad (37)$$

Noticed that the even function cosx only has the even function in power series, and odd function sinx only has odd components.

A very interesting and important fact is that we can now use power series to define something may appear strange at first look, such as e^{ix}, where x is real, i is the imaginary unit, i^2=-1. It seems meaningless from the definition of exponentials, which is multiplication of numbers, A^2=AA. What does this pure imaginary number do at the power of the e? From the power series (Taylor expansion), this expression makes sense:

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \ldots + \frac{(ix)^n}{n!} \ldots$$

$$= (1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots)$$

This may be impressive, but more striking fact lies by comparing it with (36) and (37), it won't require a genius to notice the relation between the above expansion of e^{ix} and those of sine and cosine. But it does take a genius to put *ix* into the exponential and find the connection. That is one of the great findings of Euler. So we proved the famous Euler formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (38)$$

Here just replace x which is a dummy variable anyway with the symbol usually used for angle. It is easy to see that $e^{i\pi} = -1$. It is the most beautiful relation in mathematics, relating the 4(probably the most) important numbers (e in calculus, 1 for real and i for imaginary number,

π in geometry) in one simple form. We shall see the extensive usage of Euler formula in oscillation and waves in physics.

Another important application of Taylor expansion in physics lies in the fact that we always try to neglect higher orders (large n's) terms of the power series in physics. This is justified in the region where x (or x-x_0) is small! In this case, the first order term (x) will give a pretty good approximation of the original function, we are talking about linear approximation of the last section. In case the first order is not good enough, we can include the second order (the x^2) term, and we have quadratic approximation. Seldom we go to higher orders (if necessary, we could), and this greatly simplify the calculation of complicated functions.

In relativity, we will often deal with terms like $\sqrt{1-\beta^2}$, and $\dfrac{1}{\sqrt{1-\beta^2}}$, where $\beta = \dfrac{v}{c}$. If v<<c, $\beta << 1$. Then the two expressions can be simplified. Do them yourself up to the second order. [159]

2. Integration of a Function

As stated as the beginning of this supplementary, the differentiation is starting from an original function, finding its instantaneous changing rate. i.e. Knowing a function F(x), we can calculate its derivative f(x), where

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[159] Answer: It is more easy to set $\beta^2 = x$, and obviously $x << 1$.

$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} + \dots = 1 - \frac{\beta^2}{2} - \frac{\beta^4}{8} \dots$ $\quad 1/\sqrt{1-x} = 1 + \frac{x}{2} + \frac{3}{8}x^2 = 1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4$

F'(x)=dF(x)/dx=f(x). Now we will ask the question reversely, if we know a function f(x) is the derivative of some original function, then can we find the original function F(x)? The physical example is that derivative is from the distance function to get velocity function; integration is from velocity function to get distance function.

This process (from derivative to original function) is called integration. You probably have heard of indefinite integrals and definite integrals. We shall discuss these below. They are different but closely related.

2-1. Antiderivative and Indefinite Integral

Antiderivative of a function f(x) is defined as:

If $\frac{dy}{dx} \equiv \frac{d}{dx}F(x) \equiv F'(x) = f(x)$ (39)

then y=F(x) is the antiderivative of the function f(x), i.e. F(x) is the original function in the derivative section. The definition of this antiderivative can also be looked as a differential equation (1st order, because only involves 1st order derivative), the problem is from f(x), finding the F(x). The above equation can also be rewritten in differential forms:

$dy \equiv dF(x) = f(x)dx$

Introducing the integral symbol, the equation can be written as:

$F(x) = \int f(x)dx$ (40)

(40) is just a bookkeeping, express the F(x) in terms of f(x) explicitly. It is

read as: the antiderivative of a function f(x) is the indefinite integral of the function f(x). The equation (40) is a bookkeeping of the definition because itself does not tell you how to calculate from f(x) to F(x).

Actually the antiderivative is not a single but a group of functions, because if F(x) is an antiderivative, then F(x)+c, where c is a arbitrary constant independent of variables would also be an antiderivative of f(x), because dc/dx=0, so that d(F(x)+c)/dx=d(F(x))/dx=f(x). So if we use indefinite integral to represent the whole group of antiderivative function, the (40) will become:

$$F(x) + c = \int f(x)dx \qquad (41)$$

This is the usual definition of indefinite integral.

Finding the antiderivative (or doing the indefinite integral) is much harder than finding the derivatives, though it is the reversed process of derivative. The basic rule from the definition is “guessing”. This is to say if we have a function f(x), we will guess what function form of F(x) would be so that its derivative will be f(x). This is much harder because we are short of the product rules and especially the chain rule in the integration, though the linearity still works. i.e. if F(x), G(x) are the antiderivatives of f(x), g(x), then the antiderivative for pf(x)+qg(x) would be pF(x)+qG(x). The product rules won’t work directly for integration[160]. The reason we can find the derivatives of many different functions lies in the chain rule and

[160] We will discuss a technique of integration by part which is the result of product rules from derivative.

product rule.

For simple function forms of f(x), its antiderivative may relatively easy to guess.

Examples1.
$$f(x)=0, F(x)=c$$
$$f(x)=c, F(x)=\int c\,dx = cx$$
$$f(x)=x^n, F(x)=\int x^n dx = \frac{1}{n}x^{n+1}$$
$$f(x)=e^{ax}, F(x)=\int e^{ax}dx = \frac{1}{a}e^{ax}$$
$$f(x)=\sin(kx), F(x)=-\frac{1}{k}\cos(kx)$$
$$f(x)=\frac{1}{x}, F(x)=\int \frac{1}{x}dx = \ln|x| \quad x\neq 0$$

All the F(x)'s above are subject to a shift of constant C, i.e. F(x)+C

The absolute symbol in the last relation may need a little comment. Comparing with equation (9) $d(\ln x)/dx = 1/x$, there is no absolute sign, because lnx already indicates x>0. For a function in forms of 1/x, x can be > 0 or < 0. So its antiderivative can be found by considering x>0 and <0 separately. For x>0, clearly F(x)=ln(x). For x<0, if you just write F(x)=ln(-x), and calculate d(ln(-x))/dx, you will get 1/x too. This is better illustrated using substitution: when x<0, introduce another variable u (u>0), and x=-u, and dx=-du. Then

$$\int \frac{1}{x}dx = \int -\frac{1}{u}(-du) = \int \frac{1}{u}du = \ln u = \ln(-x) = \ln|x|$$

For some other simple function forms which are variations of the basic functions, substitution method may give you quick results.

Example 2:

$$\int \sin x \cos x dx = \int \sin x d(\sin x) = \int u du = \frac{1}{2}u^2 = \frac{1}{2}\sin^2 x$$

$$or \int \sin x \cos x dx = \frac{1}{2}\int \sin 2x dx = \frac{1}{4}\int \sin 2x d(2x) = -\frac{1}{4}\cos(2x)$$

The two methods may seem giving two different antiderivative functions, but they are only different by a constant (check it yourself), so both are acceptable.

Example 3:

$$x > 0, \int \frac{dx}{x \ln x} = \int \frac{d(\ln x)}{\ln x} = \ln(|\ln x|)$$

There are more 'tricks' besides the simple substitution[161], and those won't be discussed here. Just remember the golden rule to get indefinite integral (equivalently antiderivative) is 'guessing', with the help of substitution. For complicated f(x), check the integral tables in handbook of mathematics for answer (at least you are allowed to do this in this course).

2-2. Definite Integral and Its Geometric Interpretation

The definite integral is to treat the problems like areas under a curve or volume enclosed by a surface, etc. For example, what is the area enclosed by a circle? The area enclosed by x axis and curves like $y = 2x^2 - 8$ etc. If we know the density distribution, then what is the mass of the object

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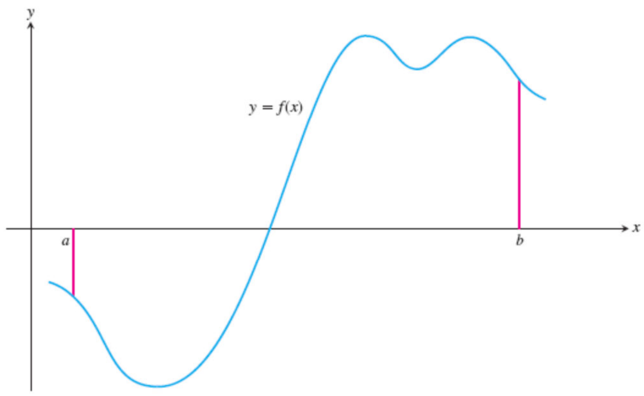
[161] Even the substitution may get uglier, try this indefinite integral: $\int \frac{d\theta}{\sin\theta} = ??$

with certain shape? In mechanics, we will need to calculate center of mass for an object, and moment of inertia around certain axis, these calculations all involves definite integral.

The geometric interpretation is the area ‘under’ the curve, or more precisely for single variable integration, the area enclosed by the curve and the x-axis, the sign is positive for area above the axis, negative for area below the axis (refer to figure below).

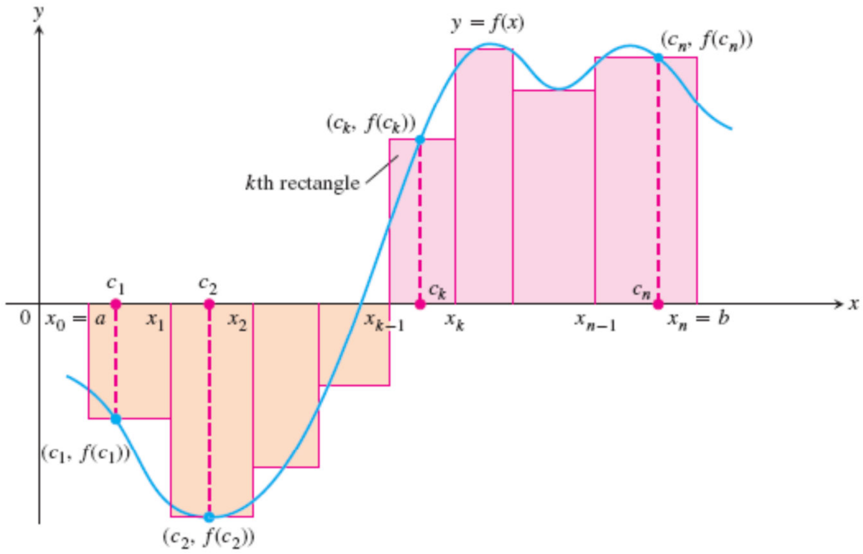


For example, we travel with certain velocity, within a time interval Δt_i around the time t_i, the velocity is $v(t_i)$, the distance traveled within that time interval would be $s(\Delta t_i) \approx v(t_i)\Delta t_i$, we can divide the whole time interval between [a, b] into many such small intervals, within each interval $v(t_i)$ is almost a constant. Then the total distance from form t=a to t=b would be a summation of all this distances:

$$Total\Delta S = \sum_{n=1}^{N} v(t_n)\Delta t_n \qquad (42)$$

If we make each Δt_i very small then the summation would approach the area under the curve of function v(t).

Relation (42) is also called Riemann Sum, illustrated in the figure below.



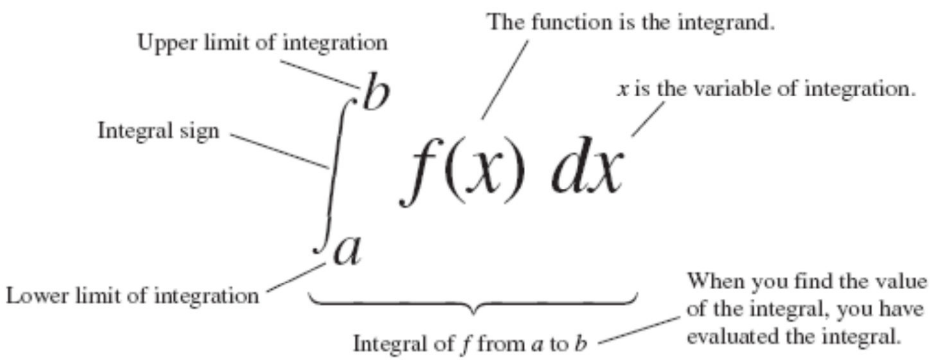
For a general function, we can divide the total closed interval [a,b] into smaller subintervals $\| P_i \| = \Delta x_i$, pick a $x_i = c_i$ within each interval, and use $f(c_i)\Delta x_i$ to approximate the area under the curve within that interval, then the total area between [a,b] would be approximated by Riemann sum:

$$F_{[a,b]} = \sum_{k=1}^{n} f(c_k)\Delta x_k \qquad (43)$$

Of course how good this approximation would depend on how you choose the subintervals and $x_i = c_i$. But as $\| P_i \| = \Delta x_i$ approaches 0, the choice would not matter, and the sum would approach to a fixed value which is the area under the curve. At this limit ($\| P_i \| \to 0$), the sum becomes what we called definite integral:

$$F_{[a,b]} = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)dx \qquad (44)$$

The meaning of the definite integral is summarized in the figure below:



There are some general properties of definite integral which are obvious from its definition and geometric interpretation:

TABLE 5.3 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then $\min f\cdot(b-a) \le \int_a^b f(x)\,dx \le \max f\cdot(b-a)$.	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

However, these properties and definition won't help us too much in the calculation of definite integrals, it is usually hard to get definite integral from the limiting case of the Riemann sum. And also you should start to wonder what the relation of this definite integral with the antiderivative? That relation is revealed by what is called fundamental theorem of calculus, it allows you to calculate the definite integral very easily from the antiderivative of the integrand function.

2-3 Fundamental Theorem of Calculus and Calculation of Definite Integral

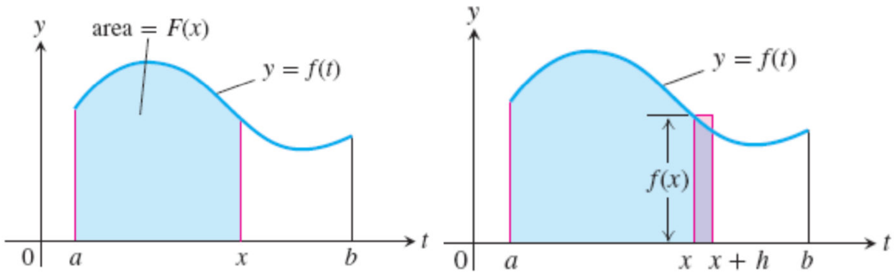
The fundamental theorem states that:

If f(x) is a continuous function for every point in interval [a,b], and F(x) is the antiderivative of f(x) in [a,b], then:

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (45)$$

Before prove it, the relation (45) makes perfect sense. Take the example of the total distance traveled between during a time period. We could calculate the distance by integrating the velocity function over time as in (42), or if we know the function of location with time (which is the antiderivative of velocity, or velocity is the derivative of it, same thing), then we can simply calculate the distance between the initial and final time by direct subtraction. The (45) simply states the two calculations would be same, certainly it better be.

I will give a prove of (45), please refer to the figure below.



We can define a function G(x) as the left figure suggests (we do not know the G(x) is the antiderivative yet, we shall prove this), G(x)=area under f(t) between [a,x], with the function f(t) known and initial point fixed, it is clearly vary as the end point x varies. So G(x) is defined as:

$$G(x)=\int_a^x f(t)dt$$ [162]

Clearly from this definition G(a)=0, and $G(b)=\int_a^b f(t)dt$

Let the x changes to x+h, then $G(x+h)=\int_a^{x+h} f(x)dx$ which is the area from a to x+h. The change of area is approximated by the rectangle in the figure on the right. In the limit of h approaches 0, we have:

$$G(x+h)-G(x)=f(x)h \quad h\to 0$$
$$\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}=f(x)$$

The above relation just states that f(x) is the derivative of the G(x) and G(x) is one of the antiderivatives of f(x), the general form of antiderivative would be $F(x)\equiv G(x)+C$, then:

[162] Here I changed the dummy variable from x to t in the integrand, since I used x at the upper bound of integral sign.

$$F(b)-F(a)=G(b)+C-[G(a)+C]=G(b)=\int_a^b f(x)dx \quad \text{Q.E.D.}$$

Actually from the above argument, we see that G'(x)=f(x), just write out explicitly, it becomes:

$$\frac{d}{dx}G(x)=\frac{d}{dx}\int_a^x f(t)dt=f(x) \qquad (46)$$

(46) and (45) are called part 1 and 2 of the fundamental theorem of calculus.

So the problem of finding any definite integral becomes a problem of 'guessing' the antiderivative[163]. However to evaluate the definite integral involves many tricks, such as how to choose the proper variable and interval (make clever 'cut' or 'slice'). These should be properly addressed in course of Calculus. I will only mention one trick which is called integration by part here.

From the product rule we know that $d(fg)=(f'g+g'f)dx$, f,g are functions of x. Take integration on both sides:

$$\int_a^b d(fg)=\int_a^b f'gdx+\int_a^b g'fdx \qquad (47)$$

$$\int_a^b f'gdx = fg|_a^b-\int_a^b fg'dx$$

$fg|_a^b$ is a short hand for $f(b)g(b)-f(a)g(a)$, it is not unusual that in many problems in physics, such value are 0 at the boundary, so this provides you a 'trick' to workout some integrals rather easily (especially

[163] This is how human calculate the integral. The computer of course takes the algorism similar to the Riemann sum.

those involves polynomials, since g' will reduce the order of polynomials. such as integrand xe^{ax}), and such trick will also be used often in derivation of physical relations (such as Euler-Lagrange equation in theoretical mechanics).

Finally I should mention that in this discussion of integration, I seem stressing the (45), that is to say whenever we need to find the definite integral, looking for the antiderivative of the integrand. However (46) is equally important, the $G(x)=\int_a^x f(t)dt$ gives you one method to construct functions, especially the 'unusual' ones which are called transcendental functions. Starting from f(t), you can construct functions that may not have any analytical forms (take the examples with $f(t)=e^{-t^2}, or\, f(t)=\sin(t^2)$, the error functions in statistics and Fresnel function in diffraction are defined this way). This is an important branch in math, since this supplementary is mainly for the preparation in physics, I will not elaborate this point further.

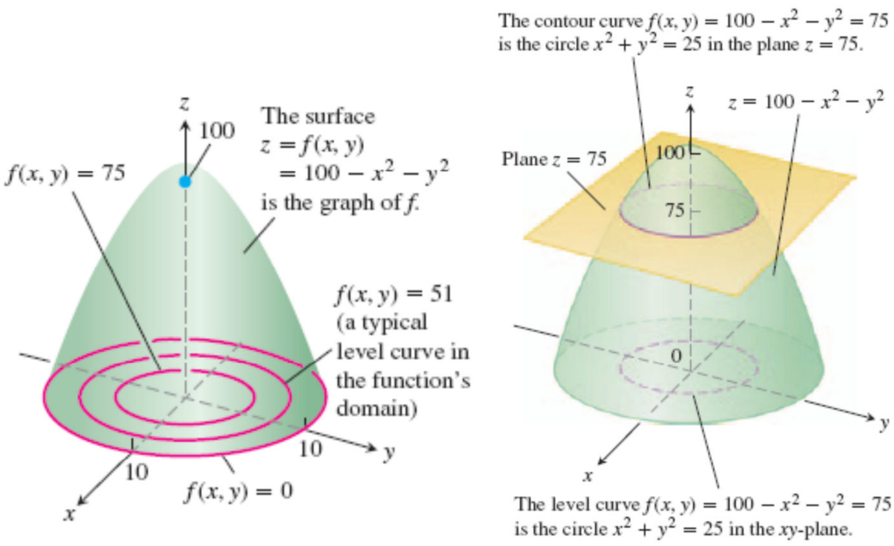
Supplementary II

Functions of Multi-Variables and Partial Derivative

In the supplementary 1, we discussed single variable calculus, i.e. given a function how we calculate its derivative (differentiation) or vice versa (integration). That is the foundation for what we are going to discuss here, the functions of multi-variables.

1. Functions of Multi-Variables

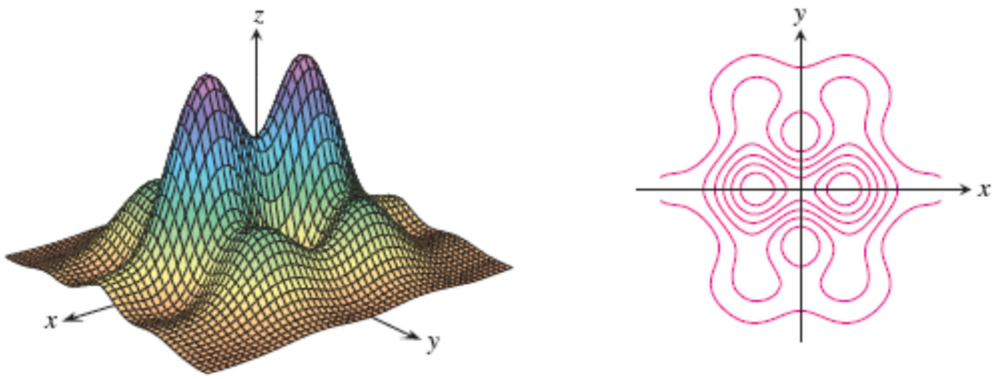
   In real applications, the function may depend on many variables, this is specified as $f(x,y)$[164]. The geometric meaning of this is a surface in 3-D. Let $z = f(x,y)$, the plot of z w.r.t (with respect to) x and y is a surface in the x-y-z coordinate.



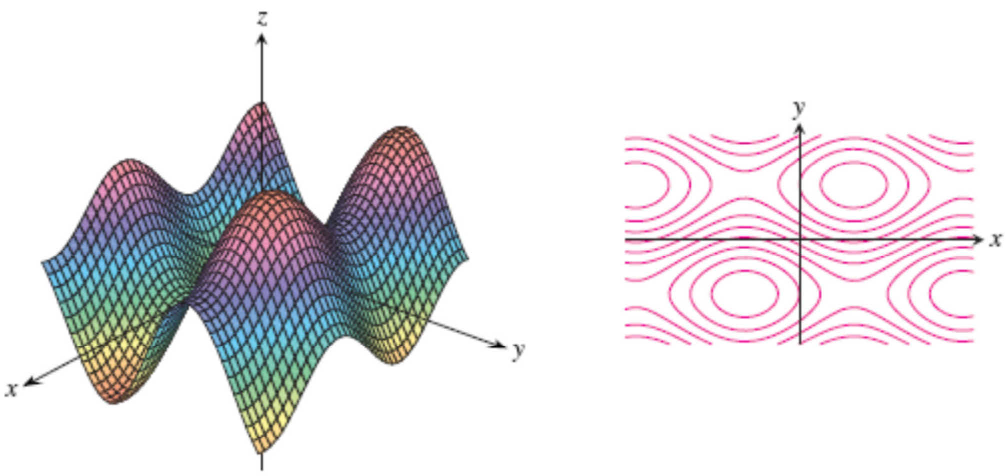
   The figure shows the example where $z = f(x,y) = 100 - x^2 - y^2$. It is a 3-D parabolic surface (the cut with x or y fixed is a parabola). The

[164] Here I shall only focus on two variables, the extension to more variables is straightforward.

circles in the figure are called level curves with $z = f(x, y) = c$, it is a curve on which the z=f(x,y) has the fixed value c, it is the cross section of plane z=c cuts through the surface, as shown in the figure to the right. Many times, instead of drawing a 3-D surface, a 2-D contour plot is used to show the f(x,y). The contour plot is just a group of level curves at different f(x,y)=c. The figure below shows more examples of 3-D surface and contour plot of some functions[165]:

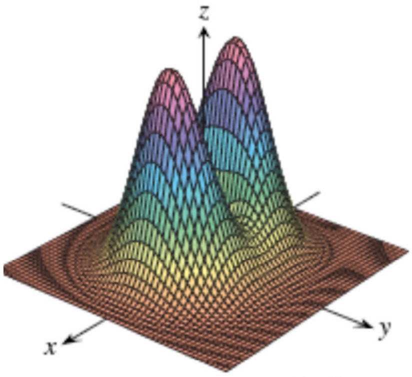


(a) $z = e^{-(x^2+y^2)/8}(\sin x^2 + \cos y^2)$

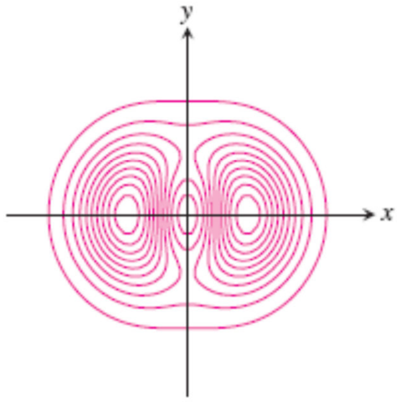


(b) $z = \sin x + 2 \sin y$

[165] Taken from Thomas 'Calculus' Chap.14, figure 14.10.



(c) $z = (4x^2 + y^2)e^{-x^2-y^2}$



2. Partial Derivatives

(1) 1st order Partial Derivatives

Now we asked the question, as the variable changes, how the function changes? This is a question related to the differentials of the function. For the single variable case, we see that:

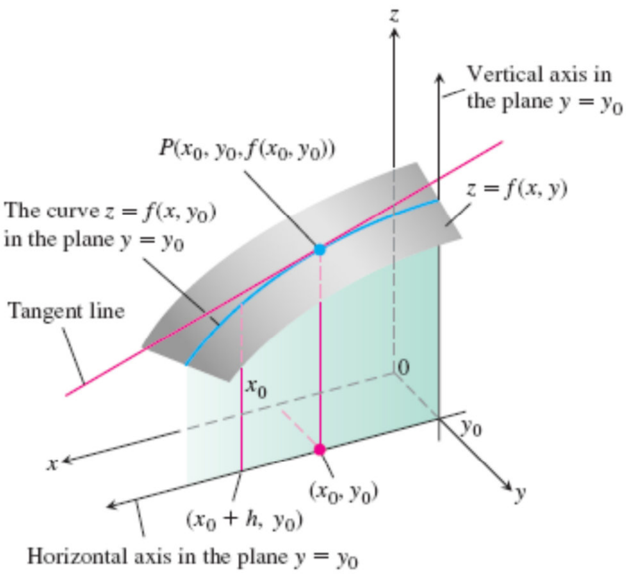
$$For\ y = f(x);\ dy = [\frac{d}{dx}f(x)]dx = f'(x)dx \quad (48)$$

Where $f'(x)$ is the derivative over x defined by relation (3), it is the rate of change of f over x; its geometric meaning is the slope of the tangent line of the curve y=f(x), passing through (x,f(x)).

Here for multi-variable function f(x,y), the change of f would be related to changes of both x and y. The rates of changes are defined through the partial derivatives:

$$\frac{\partial f}{\partial x}|_{x_0,y_0} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \qquad (49)$$

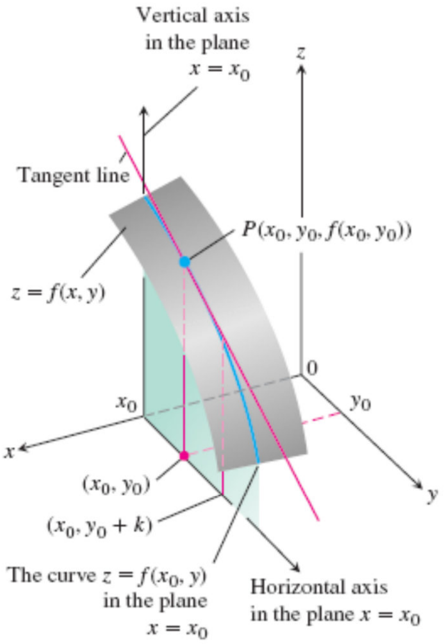
This is called partial derivative of f over x. Noticed that from its definition, it is a derivative at point (x_0,y_0), holding the y constant (y does not change) and varies the x. The geometric meaning of this is shown in the figure below. The $\frac{\partial f}{\partial x}|_{x_0,y_0}$ is the slope of a tangent line. This tangent line is tangent to a curve which is the cut of y=y_0 (a planes parallel to x-z) plane with the f(x,y) surface, and the tangent line passes through (x_0,y_0). (in the plane of y=y_0, the f(x,y) is reduced to f(x,y_0) and it is a single variable curve, and the partial derivative over x is just like an ordinary derivative)



Similarly the partial derivative over y can be defined as:

$$\frac{\partial f}{\partial y}|_{x_0,y_0} = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} \qquad (50)$$

The geometric meaning is given in the figure below:



So from the definition, the calculation of partial derivative is straightforward, just like the single variable derivative. When you calculate the $\frac{\partial f}{\partial x}$, hold y as a constant; and when calculating $\frac{\partial f}{\partial y}$, hold the x as constant, and all the techniques we talked about in derivatives can be applied here to get partial derivatives. There are some other often used notations (bookkeeping) for partial derivative:

$$\begin{aligned} \frac{\partial f}{\partial x} &\equiv \frac{\partial f}{\partial x}\Big|_{\text{constant y}} \equiv \left(\frac{\partial f}{\partial x}\right)_y \equiv f_x \\ \frac{\partial f}{\partial y} &\equiv \frac{\partial f}{\partial y}\Big|_{\text{constant x}} \equiv \left(\frac{\partial f}{\partial y}\right)_x \equiv f_y \end{aligned} \qquad (51)$$

The notation $\frac{\partial f}{\partial x}\Big|_{\text{constant y}}$ or $\left(\frac{\partial f}{\partial x}\right)_y$ are the best ones, it reminds you when taking partial derivative what is the constraint, i.e. here y=constant. We shall see later that if the constraint changes, the partial derivative can be quite different. But if you keep in mind of this constraint, all the notations are fine, it does take more time to type those better notations☹.

(2) 2^{nd} order Partial Derivatives

The 1^{st} order partial derivative of f(x,y) is generally a function of (x,y) too, so we can take the further partial derivative over the 1^{st} order partial derivative and this will give us 2^{nd} order partial derivatives.

The original function z=f(x,y), its partial derivative over x would be:

$\frac{\partial f}{\partial x} = f_x(x,y)$, when you take further partial derivative of $f_x(x,y)$, you would have:

$$\begin{aligned} \frac{\partial f_x}{\partial x} &\equiv \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) \equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{xx} \\ \frac{\partial f_x}{\partial y} &\equiv \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv f_{xy} \end{aligned} \qquad (52)$$

And similarly for $f_y(x,y)$:

$$\begin{aligned} \frac{\partial f_y}{\partial x} &\equiv \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) \equiv \frac{\partial^2 f}{\partial y \partial x} \equiv f_{yx} \\ \frac{\partial f_y}{\partial y} &\equiv \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) \equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{yy} \end{aligned} \qquad (53)$$

EXAMPLE 9 Finding Second-Order Partial Derivatives

If $f(x, y) = x\cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \cos y + ye^x$$

So

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = -x\sin y + e^x$$

So

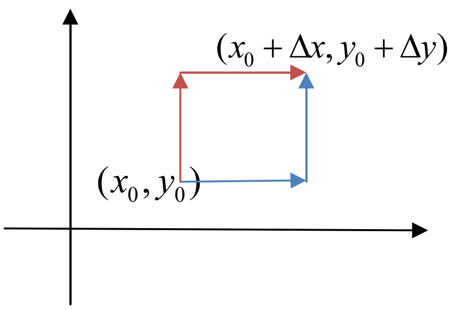
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y.$$

In the example, you will notice that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy} = f_{yx} \qquad (54)$$

This is no coincidence. The relation (54) holds for regular functions we shall encounter in physics.[166] The rigorous proof can be found in the math books, here I give you a reasoning why (54) is true:



Consider a function $f(x, y)$, as x, y changes the function will take value $f(x + \Delta x, y + \Delta y)$. The change of function f can be calculated

[166] It requires existence and continuity of the partial derivatives f_x, f_y, f_{xy}, f_{yx} in the open region where the partial derivatives are evaluated.

through two paths as shown in the figure, and two calculation results must be same, since the function is single valued at each point. If we take the blue path first: in the horizontal part, as $(x_0, y_0) \to (x_0 + \Delta x, y_0)$

$\Delta f_1 \approx \frac{\partial f}{\partial x}|_{x_0, y_0} \Delta x$, $\frac{\partial f}{\partial x}|_{x_0, y_0}$ is the partial derivative evaluated at (x_0, y_0)

In the vertical part of blue path, from $(x_0 + \Delta x, y_0) \to (x_0 + \Delta x, y_0 + \Delta y)$

$$\Delta f_2 \approx \frac{\partial f}{\partial y}|_{x_0+\Delta x, y_0} \Delta y \approx (\frac{\partial f}{\partial y}|_{x_0, y_0} + \frac{\partial^2 f}{\partial y \partial x}|_{x_0, y_0} \Delta x)\Delta y = \frac{\partial f}{\partial y}|_{x_0, y_0} \Delta y + \frac{\partial^2 f}{\partial y \partial x}|_{x_0, y_0} \Delta x \Delta y$$

The total change is of course $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \Delta f_1 + \Delta f_2$

The calculation along the red path would give us:

$\Delta f_1' \approx \frac{\partial f}{\partial y}|_{x_0, y_0} \Delta y$ and

$\Delta f_2' \approx \frac{\partial f}{\partial x}|_{x_0, y_0} \Delta x + \frac{\partial^2 f}{\partial x \partial y}|_{x_0, y_0} \Delta x \Delta y$ and

$\Delta f_1' + \Delta f_2' = \Delta f_1 + \Delta f_2$

This will show that $\frac{\partial^2 f}{\partial y \partial x}|_{x_0, y_0} = \frac{\partial^2 f}{\partial x \partial y}|_{x_0, y_0}$, since (x_0, y_0) is arbitrary, we 'proved' the relation (54).

3. Total Differentials and Linear Approximation

The question concerned here is the change of function due to the change of variables. As in the single variable case, we have relations like (21) and (22); for multi-variable case, the similar relations are:

$$\begin{aligned}\Delta f &\equiv f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \\ \Delta f &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y\end{aligned} \qquad (55)$$

The (55) requires 1st order partial derivatives exist in the region include (x_0, y_0) and they are continuous at (x_0, y_0). $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ are higher order terms of $\Delta x, \Delta y$, because as $\Delta x \to 0, \Delta y \to 0$, ε_1 and $\varepsilon_2 \to 0$. The functions f satisfies the (55) is called differentiable functions.

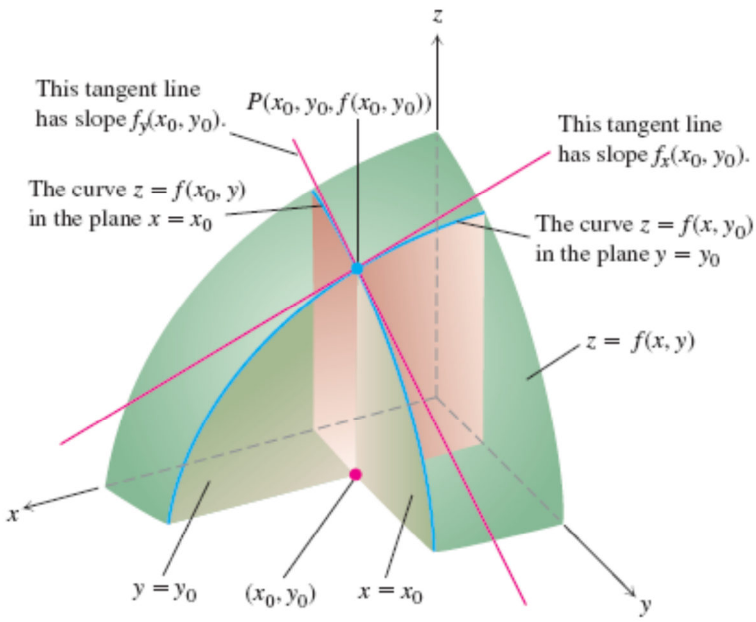
For the differentiable functions, as $\Delta x \to 0, \Delta y \to 0$, we have:

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy \qquad (56)$$

This is to say if we have an infinitesimal change of variables from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$, the change of function is given by (56). (56) is called ***total differential*** of the function. It is the most important relation in partial derivatives. If we hold y constant, i.e. dy=0, (56) will give us the rate of change of f due to x(with y fixed), which is the partial derivative f_x; and similarly for f_y.

Even if the change of variables are not infinitesimal as required for (56), as long as they are small enough so that the higher order terms in (55) can be discarded, we will have linear approximation for the change of function due to changes of variables:

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \quad \text{small } \Delta x, \Delta y \qquad (57)$$



In (57), we are using the tangent plane at (x_0, y_0) to approximate the original function surface. The tangent plane is formed by the two tangent lines at (x_0, y_0) that defined the partial derivatives. The exact change of function is of course along the surface, but for small $\Delta x, \Delta y$, the tangent plane is very close to the surface. All the discussions here are analogous to the single variable case.

4. Chain Rule

From the total differentials (56), we can have the powerful chain rule. Here we shall focus on the implicit dependence on variables.

(1) $f(x, y), x = x(t)$ and $y = y(t)$

The function only explicitly depends on x and y. However the x,y each is a function of some other variable t. Now the question is what is the derivative of the function over t. i.e. how the function changes as t varies.

The function changes because of x and y change, x and y change as t changes, so it is like a chain reaction. The change of function can be seen from the (56):

$$df = f_x dx + f_y dy$$

But now $dx = (\frac{dx}{dt})dt = x'dt, dy = (\frac{dy}{dt})dt = y'dt$, so:

$$df = (f_x x' + f_y y')dt$$

This is the differential of f with change of dt, then:

$$\frac{df}{dt} = f_x x' + f_y y' = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad (58)$$

This is the chain rule for this situation. Actually the product rule for derivative (15) can be derived from this:

$$Let\ F = f(x)g(x)$$

$$\frac{dF}{dx} = \frac{\partial F}{\partial f}\frac{df}{dx} + \frac{\partial F}{\partial g}\frac{dg}{dx} = g\frac{df}{dx} + f\frac{dg}{dx}$$

This is the product rule for ordinary derivatives.

Chain rule is the most important and useful rules in derivatives, like said in Tolkien's "Lord of Rings": One rule finds them all; One rule unites them all; One rule rules them all☺! I trust you can extend it to functions involves more variables like x,y,z…

There are cases in physics, a function not only explicitly depends on x,y, but also on t, and the x and y depend on t too. In such case, the derivative of the function over t is:

$$\frac{d}{dt}f(x,y,t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}\frac{dt}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t} \qquad (59)$$

(2) $f(x,y), x = x(s,t)$ and $y = y(s,t)$

In such case, the x and y are themselves function of multi-variables. An example would be the original function depends on x and y, now we switch to polar coordinate with (r,θ), then what is the derivative of function f over new variables?

Well, still starts from the total differential:

$$df = f_x dx + f_y dy$$

$$dx = \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt;\ dy = \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt$$

Then:

$$df = (\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s})ds + (\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t})dt \qquad (60)$$

This is the total differential expressed in the form that s,t are variables, and from this we can find partial derivatives of f over s (by let dt=0, holding t constant)or t:

$$\begin{aligned} \frac{\partial f}{\partial s}\Big|_{\text{constant t}} &\equiv \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t}\Big|_{\text{constant s}} &\equiv \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \end{aligned} \qquad (61)$$

This can be extended to more variables, such as the change from x-y-z to $r-\theta-\varphi$ spherical coordinate.

A puzzle for you: Let's say we have a function $f(x,y)$, now I make a substitution: $x = s, y = s + t$, replace the two variable with other two variables. Let's see what is $\partial f / \partial s$, we have:

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$ from the chain rule. Something appears strange here, since x=s, but clearly from the above: $\frac{\partial f}{\partial s} \neq \frac{\partial f}{\partial x}$, what is going on? [167]In this case, it would be clearer if we use the notation: $(\frac{\partial f}{\partial s})_t$ and $(\frac{\partial f}{\partial x})_y$ to avoid confusion. Such notation would become necessary when we deal with non-independent variable. A lot of these are in thermodynamics, such as internal energy $U(P,V,T)$, a function of pressure, volume and temperature, but the variables not independent, they may be related by some relations such as ideal gas equation. We will not meet many cases here in this course, so I will skip discussion on the partial derivatives for non-independent variables.

(3) Revisit of Implicit Derivative

In function of single variable, we discussed a technique of implicit derivative (section 1.4). Basically the y=f(x) is not given explicitly, the x,y relation are provided by an equation g(x,y)=0, and we can find the derivative dy/dx using implicit derivative method. Here we give a formula based on the partial derivatives.

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[167] Answer is that the constraint is different. One requires t constant, and since y=s+t and if s varies y has to change. The other requires y constant. The two really have different constraint. Or you may look it from geometrical point of view. The cutting plane under these two conditions are different, so the resulting curve and slope would be different too. Even though each curve stays on the surface specified by f, and passes through the same point (x_0,y_0) (or in the s-t, corresponding (s_0,t_0)), but still you have many curves can satisfy this, given different constraints.

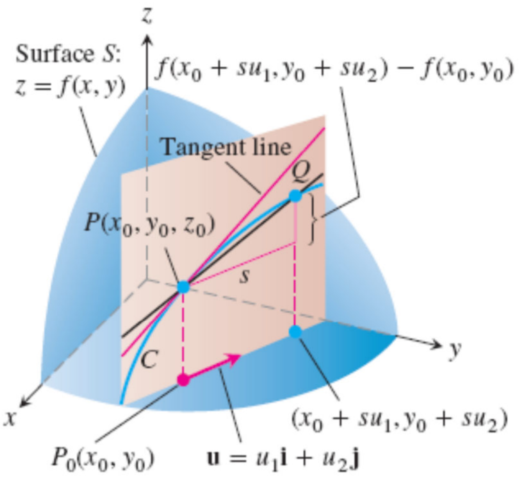
g(x,y)=0 means the as the x,y changes, they have to satisfy this constraint. So as x,y changes, the requirement g(x,y)=0 means g(x,y) does not change:

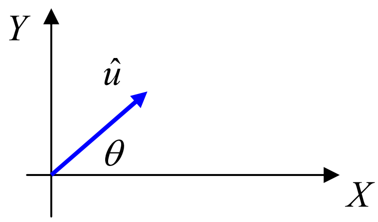
$$dg(x,y) = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$$
$$\frac{dy}{dx} = -\frac{\partial g / \partial x}{\partial g / \partial y} \qquad (62)$$

Please noticing the minus sign, it is not intuitive. You have to go through the derivation from total differentials to understand it. Please redo the example of ellipse in section 1.4 using (62).

5. Gradient Vector and Directional Derivative

The partial derivatives over x and y are rates of changes of a function along the two special direction, along the x-axis (or use vector symbol $\hat{i}$, a unit vector along +x direction) and y-axis $\hat{j}$. We can ask the question what are the changing rate along other directions? There are many ways to cut the surface of f(x,y) at (x_0, y_0), such as the figure below shows. The plane is not $x = x_0$ or $y = y_0$ as before, but a plane contain the point (x_0, y_0) and a direction vector $\hat{u} = u_1\hat{i} + u_2\hat{j} = \cos\theta\hat{i} + \sin\theta\hat{j}$. θ is the angle between the direction of u and x-axis, i.e. $\cos\theta = \hat{i}\cdot\hat{u}$. Now if we move along the $\hat{u}$ a small distance s, what is the change of function?





This is the meaning of directional derivative, the rate of change of function along any direction specified by $\hat{u}$. It is defined as:

$$(\frac{df}{ds})_{\hat{u},P_0} = \lim_{s\to 0}\frac{f(x_0+su_1,y_0+su_2)-f(x_0,y_0)}{s} \qquad (63)$$

The subscript of the derivative specifies the direction and point of interest, the u_1, u_2 are specified above, you certainly can use $\cos\theta, \sin\theta$ to replace them. The question now is how we calculate this directional derivative? Form the study of vectors, you learned that any vectors in x-y plane is a linear combination of base vectors $\hat{i}, \hat{j}$, can the directional derivative be expressed as linear combination of partial derivatives over x and y? The answer is positive. Below is the reasoning from a point of view of base transformation we discussed in chapter 3 of the notes, since we learned that there, I shall apply it here as a practice of transformation. There are other equivalent (even simpler) proofs in math books too.

Imagine we establish a transformed coordinate system. In the new coordinate system, the x'-axis would be along the direction of $\hat{u}$. This is just a rotation of the basis we discussed in chapter 3, $\hat{u}$ is the $\hat{i}'$ there. The directional derivative $(\frac{df}{ds})_{\hat{u},P_0}$ is now the partial derivative $\frac{\partial f}{\partial x'}$ in the transformed basis. The relations between the new and old coordinates are given in relation (3-20) and (3-21):

$$x' = x\cos\theta + y\sin\theta$$
$$y' = x(-\sin\theta) + y\cos\theta$$
$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

With these relations the directional derivative can be expressed with the original coordinates by chain rule:

$$(\frac{df}{ds})_{\hat{u},P_0} \equiv \frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \qquad (64)$$

This relation can also be expressed as:

$$(\frac{df}{ds})_{\hat{u},P_0} \equiv \frac{\partial f}{\partial x'} = (\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j})\cdot\hat{u} \qquad (65)$$

It is very useful to introduce a new vector, the gradient vector whose definition is clear from (65):

$$\nabla f \equiv \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \qquad (66) \qquad \text{in 2-D case}$$

∇f is called gradient vector (or simply gradient) of function f. It is a vector constructed from partial derivatives. ∇ is called differential operator. Its general definition in 3-D is:

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad (67)$$

This is a completely new math quantity. It is not a function or number. It is an operator because it is waiting you to act it on some functions or numbers.[168]

(1) Fundamental theorem of gradient

The gradient gives a way to evaluate the change of functions along any direction. Let's say along direction $\hat{u}$, we move an infinitesimal distance ds, this displacement will cause the value of function changes. The displacement along direction $\hat{u}$ can be expressed as a vector:

$d\vec{s} = ds\hat{u}$

With the gradient, the function change can be expressed as:

$df = \nabla f \cdot d\vec{s}$ (68)

(68) is called the fundamental theorem of gradient, it tells us that knowing the gradient (knowing the partial derivatives in certain basis, say x-y-z), the function change over a small distance along any directions can be evaluated. The (68) will give back the total differential relation (56) if we express the infinitesimal displacement vector in terms of dx and dy"

$$d\vec{s} = ds\hat{u} = (ds)\cos\theta\hat{i} + (ds)\sin\theta\hat{j} = dx\hat{i} + dy\hat{j}$$

$$df = \nabla f \cdot d\vec{s} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

So this fundamental theorem is a generalization of the total differential

[168] There are actually different ways how this operator acts on functions. If it acts on a scalar function, you will get gradient we discussed here. It can also acts as dot product with a vector function, which is called divergence of the vector function; it can also act as cross product with a vector function which is called curl of the vector function. Both divergence and curl have important roles in the field theory, you will definitely need them in the study of electro-magnetic field. But in this course, we do not need them yet. The detailed discussion on gradient, divergence and curl are covered in the subject of 'vector analysis'. There are many math text books on this. A short but very clear presentation is actually given in a physics book: D.Griffith, "Introduction to Electrodynamics", Chap.1.

we discussed before. In real applications we often need to evaluate changes of function over a small but not infinitesimal step, the (68) will be expressed in such case as:

$\Delta f \approx \nabla f \cdot \Delta\vec{s}$ |Δs| small (69)

(2) Geometric properties of the gradient

The fundamental theorem of gradient (68) and (69) also tells us the geometric property of the gradient. It is really easy to prove the following property: (a) The gradient is point toward the direction of fastest change (the steepest slope) of the function over same small change of variables. (b) The gradient always perpendicular to the level curve, i.e. the curve with f(x,y)=c.

For the property (a), if the displace step is same for all directions, i.e. ds or $|\Delta s|$ is same, then:

$df = \nabla f \cdot d\vec{s} = |\nabla f||ds|\cos\alpha$

Clearly at $\alpha = 0$, df has the largest positive value. This is when the direction of change overlaps with the direction of gradient. So the gradient points toward the direction of fastest increase of the function.

For property (b), if the displacement is along a level curve where f(x,y)=c. then clearly df=0. This means $\alpha = \frac{\pi}{2}$ and ∇f is perpendicular to the level curve. Here more comments may be needed for the perpendicular to the level curve f(x,y)=c. In this 2-D case, $d\vec{s}$ is the tangent to the level curve at certain point (x_0, y_0), perpendicular to the curve actually means perpendicular to the tangent line. For the 3-D case, $f(x,y,z)=c$ would be a surface instead of a curve, the gradient ∇f here would be perpendicular to the tangent plane of the level surface. So gradient is very useful to find the tangent line or plane of a curve (f(x,y)=c) or a surface (f(x,y,z)=c).

6. Taylor Expansion for Function of Two Variables

We have discussed linear approximation of functions from the total differential of the function, which is true for infinitesimal changes of variables. For limited steps of change, as long as the steps are small, we can use linear approximation. If we want to have higher accuracy, we have to include higher order of changes in variables, and Taylor expansion formula is just for this higher order approximation. Here I only give out the formula without proof, its form is quite similar to the single variable case in section 1-8 in supplementary 1.

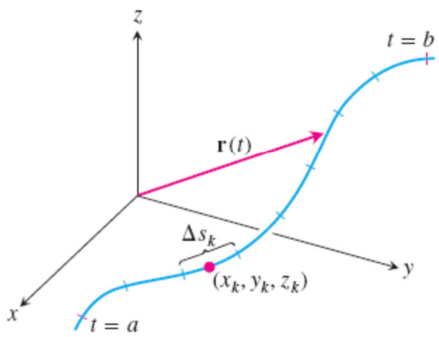
$$\begin{aligned}
f(x_0+\Delta x, y_0+\Delta y) &= f(x_0,y_0) + \frac{\partial f}{\partial x}|_{x_0,y_0}\Delta x + \frac{\partial f}{\partial y}|_{x_0,y_0}\Delta y + \\
&\left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}|_{x_0,y_0}(\Delta x)^2 + \frac{\partial^2 f}{\partial y^2}|_{x_0,y_0}(\Delta y)^2 + 2\frac{\partial^2 f}{\partial x\partial y}|_{x_0,y_0}\Delta x\Delta y\right] + \ldots
\end{aligned} \qquad (70)$$

7. Line Integral, Path Independence, Conservative Field and Green Theorem

This part of calculus is closely related to the conservative force and potential energy in physics. That is why I included this in this supplement.

I shall begin by introducing the general integral along a line (or curve), how to compute such integral. Then I will focus on a special type of line integral, the work by a force, and investigate under what condition the work done by a force is path independent, i.e. only depends on the initial and final position but not on the path connecting the two end points. Such force will be called conservative. Green theorem offers another way to evaluate the line integral and also gives the criteria that given a force, how do we know it is conservative or not.

7.1 Line Integral



The line integral is an integration, but in this integration, the x,y,z's are not free to vary, they have to stay on a curve in 2-D or 3-D space. For example, the curve is a wire, and its density may change along the wire, so to get the total mass of the wire, we may need to compute the integral:

$$M = \int_{C} \rho(x, y, z) ds$$

$\int_C$ is the symbol for line integral along a specific curve C, ds is the infinitesimal arc length of the curve. The question is how we compute it? The general method is using parametric definition of the curve. The curve is defined by parametric equations:

$x = x(t), y = y(t), z = z(t)$, where t is the parameter (it may corresponds to physical time; but more generally just a symbol for parameter) and x,y,z are functions of t.

We can visualize this by setting a position vector:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \equiv < x(t), y(t), z(t) > \qquad (71)$$

As t varies, the $\vec{r}(t)$ will scan the whole curve. If we know this parametric equation of the curve, the line integral can be calculated as following:

Take the example of only computing the arc length, we want to know how long the curve is:

$$L = \int_C ds$$

But the length ds can be viewed as travelling distance within time interval dt:

$ds = |v|\,dt$ (ds is defined as >0, so take magnitude of velocity, the speed in the computation)

The "velocity"[169] is simply:

$$\vec{v} = \frac{d\vec{r}}{dt} = < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > \qquad (72)$$

So the arc length expressed in term of the parameter t is:

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[169] Here the velocity with quotation mark is because this is analogous to physical velocity. If the parameter is time t, then it is velocity in common sense. If t is some general parameter, v is just the generalized velocity.

$$L = \int_C ds = \int_{t=a}^{t=b} [(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2]^{\frac{1}{2}} dt \qquad (73)$$

For other line integral of a function, such as mass or moment of inertia, center of mass etc, the integral will be:

$$\int_{t=a}^{t=b} f[x(t), y(t), z(t)][(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2]^{\frac{1}{2}} dt$$

Example: arc length of ellipse, for ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The parametric equation for such ellipse is:

$$\begin{matrix} x = a\cos t \\ y = b\sin t \end{matrix} \quad (0 \le t \le 2\pi)$$

The arc length will be:

$$L = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt$$

It is not an easy integral and please forgives my laziness and ask you checking the integral table for the result. It is easy to see the special case a=b (a circle) the line integral will give $2\pi a$ as expected.

7.2 Work as Line Integral

The work done by a force in physics is defined as:

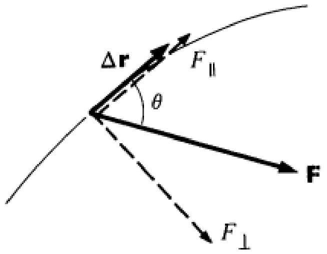
$$W = \int_C \vec{F} \cdot d\vec{r} \qquad (74)$$

$\vec{F}$ is the force vector, it may depend on position. This dependence of position and subsequent force distribution over space gives us a vector field:

$$\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k} \equiv < N, M, P > \qquad (75)$$

N,M,K are scalar functions of (x,y,z).

$d\vec{r}$ is the infinitesimal displacement vector *along the curve*:



It is a vector along the direction of tangent line at certain point on the curve, with magnitude of the arc length ds, i.e:

$$d\vec{r} = \hat{T}ds \qquad (76)$$

Then the work will be:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T}ds \qquad (77)$$

(77) shows that the work indeed is the line integral we defined in section 7.1, the line is the curve of the particle trajectory, and the integrand is a scalar function $\vec{F} \cdot \hat{T}$.

In some simple cases, where the form of tangent line is simple (such as for the straight line, circle etc), we can use (77) directly and apply what we learned in 7.1 to compute the line integral. However, the following equivalent formula which shows clearly the dependence on components and parameter may be more useful.

$$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz \qquad (78)$$

Combined with (75) will give us:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy + Pdz \qquad (79)$$

Or

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy \qquad (80) \qquad \text{In 2-D}$$

In 2-D, the curve C may be expressed as explicit functional form y=g(x), then the (80) can be reduced to regular integral over starting and ending point of x:

$$W = \int_{x_a}^{x_b} (M + N\frac{dy}{dx})dx \qquad (81)$$

More generally, the curve is expressed as a parametric form as in (71), and (79) or (80) will become:

$$W = \int_{t=a}^{t=b} (M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt})dt \qquad (82)$$

M,N,P are functions implicitly depend on the parameter t, and x,y,z are functions of t as in (71). The line integral would reduce to regular integral over parameter t.

It is also interesting to see that the above can be viewed or derived from the velocity point of view:

$$d\vec{r} = \vec{v}dt = (\frac{d\vec{r}}{dt})dt = (\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k})dt \qquad (83)$$

This will give (82) from definition of work in (74).

$$\vec{v} = \hat{T}\frac{ds}{dt}$$

$d\vec{r} = \vec{v}dt = \hat{T}ds$

This will lead to (77)

Now we know how to compute the work by a force along certain trajectory: (81) for 2-D case with explicit y=g(x); (82) for the more general case.

Noticed here, though in the above discussion, I assume the force is only depending on position as the fundamental force form does(formula (75)). In the calculation of line integral, this is not necessary. The force could also depend ***explicitly*** on time or velocity too. Parameterized integral (82) would work for this kind of force, such as choose the physical time t as parameter and line integral can be computed. However, for explicit time or velocity dependent force, the path integral will depend on the specific curve (it is not path-independent) and so cannot be conservative (I shall come back to this point after treatment for conservative force). That is why I stressed here that in the following sections where we are discussing conservative forces, the forces are only depending explicitly[170] on position.

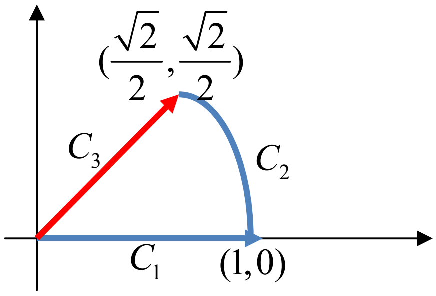
7.3 Path Independence and Conservative Force

Even for forces that only depend on position (no explicit dependence on time or velocity), the work computed from the above formula would

[170] Because of the motion, the positions will change over time, so the force can depend implicitly on time.

generally depend on the path (the curve or trajectory).

Let's first see a simple example:



From the origin (0,0) to the point $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$, we take two paths. The first one is C_1+C_2 (blue path), where C_2 is part of the circle with angle=$\frac{\pi}{4}$ and radius=1. The path 2 will be along C_3 (red path).

Let's first evaluate Force: $F_1 = x\hat{i} + y\hat{j}$ (can you draw the vector field of this force?)[171]:

Along C_1, whose parametric line form is $(x=t, y=0), t \in [0,1]$:

$$W_1 = \int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 t\,dt = \frac{1}{2}$$

Along C_2, whose parametric form would be $x=\cos t, y=\sin t$, $t \in [0,\frac{\pi}{4}]$

$$W_2 = \int_{C_2} \vec{F}_1\cdot d\vec{r} = \int_0^{\frac{\pi}{4}} \cos t(\frac{d\cos t}{dt}) + \sin t(\frac{d\sin t}{dt})dt = 0$$

$$W = W_1 + W_2 = \frac{1}{2}$$

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[171] It is a force field always point radially and its magnitude increase as radius increases.

For path 2, it is along C_3, the parametric equation is x=y=t, $t \in [0, \frac{\sqrt{2}}{2}]$

$$W_3 = \int_{C_3} \vec{F}_1 \cdot d\vec{r} = \int_0^{\frac{\sqrt{2}}{2}} (t+t)dt = \frac{1}{2}$$

So the work done by the force taking the two paths are same. I used the parametric method strictly (use formula (82)). For this simple computation, you may use (81) or even (77). The (77) is very useful to find out W_2, the work along the circular arc. Because the force given here is along radial direction, always perpendicular to the tangent of the arc, so W_2 is zero.

Now consider another force:

$$\vec{F}_2 = -y\hat{i} + x\hat{j}$$

Can you see what this force field look like[172]? Let's see the work done:

Along Path1, the C_1:

$$W_1 = \int_0^1 (0 + t0)dt = 0$$

This can also easily seen from the fact that the force is perpendicular with the path C_1 always.

Along Path C_2:

$$W_2 = \int_{C_2} \vec{F}_2 \cdot d\vec{r} = \int_0^{\frac{\pi}{4}} [(-\sin t)(\frac{d\cos t}{dt}) + \cos t(\frac{d\sin t}{dt})]dt = \frac{\pi}{4}$$

This could also be worked out in polar coordinate, where on the arc,

[172] It is always perpendicular with F_1, because the dot product between them is zero. So this force would point towards angular direction always.

$\vec{F}_2 = \hat{\theta}$, and $\hat{T} = \hat{\theta}$. So the line integral of work (77) would be just the arc length.

Along Path2, the C_3, the work by F_2: $W_3=0$.

You can work out the parametric formula, or directly see from the fact the force along the $\hat{\theta}$ would be perpendicular to the path. So the work done by F_2 is different for path 1 and path 2, it is depending on the path connecting the starting and ending points.

This simple example shows two kinds of force. The works done by one force are same for both paths; the work done by another force is dependent on the path. Of course in the above example, I only showed that F_1 is path independent for the two special paths of choice. The important fact is that the work done by this force is path independent for all paths. It only depends on the starting and ending points. This is the meaning of path independence of line integral by a vector field. I can use the results of the above example, and prove that for F_1 (the radial force) that the work will be path independent for any arbitrary path (can you do this?)[173], but I prefer to look at the broader picture first to see what kind of forces will have this property, and you will see that F_1 indeed falls into this category.

Definition of **Conservative Force**: *If the work done by a force is path independent for* ***any arbitrary*** *paths, only depends on the position of*

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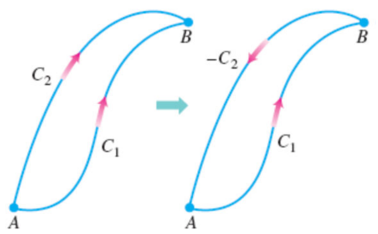
[173] You may refer to Example 4.8 in K&K or Feynman's book (Vol.1) section 13.2

initial and final points. Such force is conservative.

Corollary of this definition: *If the work done by a force along an* ***arbitrary*** *closed curve (a loop, of which the initial and final points overlap) is always zero, the force is conservative.*

It is easy to derive the corollary from the path independence and the theorem in integration: $\int_a^b fdt = -\int_b^a fdt$ and refer to the figure below.



7.4 Conservative Force=Gradient of Potential Function

Instead of giving a force and telling you whether it is conservative or not, I want to show you first that if the force equals to a gradient of a scalar function, i.e.

$\vec{F} = \nabla f$ (84)

In Cartesian, it is:

$$\vec{F}(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\hat{i} + \frac{\partial f(x,y,z)}{\partial y}\hat{j} + \frac{\partial f(x,y,z)}{\partial z}\hat{k} \equiv < f_x, f_{y,} f_z > \qquad (85)$$

Then the force is conservative. Please refer to section 5 if you forget about the gradient. From the fundamental theorem of gradient (68), i.e.: $df = \nabla f \cdot d\vec{r} = \nabla f \cdot \hat{T}ds$, we have the change of function f along direction $\hat{T}$ with step size ds is the dot product of its gradient with displacement vector. Along an arbitrary path, then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \int_C df = f(b) - f(a) \qquad (86)$$

a, b specify the starting and ending point of the path. (86) is called fundamental theorem of path integral of gradient. It is just the integral form of the fundamental theorem of gradient (68). It shows that the path integral of a gradient vector only depends on the difference of values of function f at the starting and ending point. The integral is thus path independent. So the force in forms of (84) is conservative, and the function f is called potential function. Notice here the potential function defined is a little different from the definition in physics, there is a minus sign difference, i.e.:

$$\begin{aligned} U_{physics} &\equiv -f \\ \vec{F} &= -\nabla U \end{aligned} \qquad (87)$$

The choice of minus sign is to make the summation of kinetic energy and potential energy K+U a constant, instead of K-f.

I have shown that the gradient of a potential function is a conservative force, i.e. gradient field is a sufficient condition for conservative force, but is it necessary? The answer is yes (I choose to skip the proof, it involves advanced calculus), a conservative force can always be expressed as gradient of potential function. There is a subtlety here, the force and the function have to be defined and differentiable everywhere in a domain, the domain has to be simply connected (I also choose not to discuss simply connected domain, because domains we deal with in this class will be simply connected).

Thus we conclude with the claim as the title: Conservative force=gradient of potential.

7.5 Characteristic of Conservative Force and from Force to Potential

Last section we see that if we know the potential, the conservative force can be computed by taking the gradient of the potential (math potential, for physics potential, use (87)). In this section, we study the reverse problem, i.e. provided a force, how do we know it is conservative? And if it is conservative, how can we find its potential.

Let's first take a look at 2-D case. Suppose the force is conservative, then:

$$\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \qquad (88)$$

We know form partial derivatives, there is a relation between the 2nd order partial derivatives, i.e. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. This provides the test of a force is conservative or not. For a conservative force, we must have:

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \qquad (89)$$

Or in shorthand: $M_y = N_x$.

So for the example given before, $F_1 = \langle x,y \rangle$ is conservative, since $M_y = N_x = 0$; $F_2 = \langle -y,x \rangle$ is not conservative. A sharp student may sense a loophole in the above argument: if the force is conservative, we have (89), and so if (89) is not satisfied, the force is not conservative. This is logically tight. But the reverse may not be true, i.e. if the force satisfies (89), is it necessarily conservative? This loophole will be closed when we talk about the Green theorem in the next section. For now we just accept that (89) is the sufficient and necessary condition for a conservative force.

In 3-D, a natural extension of (89) is obvious:

For a conservative force in 3-D: $\vec{F} = < N, M, P > = \nabla f$, we have:

$M_y = N_x,\ M_z = P_x,\ N_z = P_y$ (90)

(89) or (90) will be the test for a force is conservative or not.

Now let's tackle the problem of knowing the conservative force, how to compute the potential. This is best illustrated by an example:

The force is: $\vec{F} = < M, N > = < 4x^2 + 8xy, 3y^2 + 4x^2 >$.

First, test whether it is conservative, it is. (you do the math)

Then find out the potential function for this force. There are generally two methods.

Method 1, from definition of potential (86), i.e.:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \int_C df = f(b) - f(a)$$

Because the path in the equation is arbitrary, we shall choose the easiest path to do the integral. Here, let suppose we start from (0,0) and end with (x,y), the easiest path would be $(0,0) \to (x,0) \to (x, y)$ (please draw the path yourself)

From $(0,0) \to (x,0)$, y=0 and dy=0, so the line integral is just:

$$\int_0^x Mdx = \int_0^x (4x^2 + 8xy)dx = \int_0^x 4x^2 dx = \frac{4}{3}x^3$$

From $(x,0) \to (x,y)$, x is fixed, dx=0, line integral along this segment:

$$\int_0^y Ndy = \int_0^y (3y^2 + 4x^2)dy = y^3 + 4x^2 y$$

So the total line integral from (0,0) to (x,y) is:

$$f(x,y) - f(0,0) = \int_c Mdx + Ndy = \int_{C_1} Mdx + \int_{C_2} Ndy = \frac{4}{3}x^3 + 4x^2 y + y^3$$

The potential energy is defined within a constant f(0,0), usually we will set the f(0,0)=0 (or the potential energy at other reference point), because only the potential difference has physical significance.

Method 2: from the force is gradient of a potential:

$$M(x,y) = \frac{\partial f(x,y)}{\partial x}$$
$$N(x,y) = \frac{\partial f(x,y)}{\partial y}$$

From $4x^2 + 8xy = \frac{\partial f}{\partial x}$, we have (guess work):

$$f(x,y) = \frac{4}{3}x^3 + 4x^2 y + g(y) + C$$

g(y) is a function of y only, C is a constant. g(y) can be determined by the second relation:

$$N(x,y)=\frac{\partial f(x,y)}{\partial y}$$

$$3y^2+4x^2=\frac{\partial}{\partial y}(\frac{4}{3}x^3+4x^2y+g(y)+C)=4x^2+\frac{\partial g(y)}{\partial y}$$

$$\frac{\partial g(y)}{\partial y}=3y^2$$

$$g(y)=y^3$$

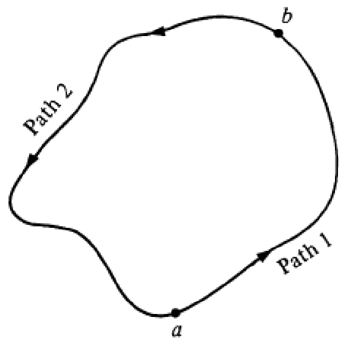
Then $f(x,y)=\frac{4}{3}x^3+4x^2y+y^3+C$, same as method 1, with f(0,0) replaced by C.

7.6 Green Theorem of line integral

I have shown that if the force is gradient of a potential function, the work will be path independent, and is determined by the value of potential at the starting and ending points. Could we prove the conservative force from point of view of its corollary? i.e. showing first that the work done by a force along any close curve (loop) is zero, then it is path independent. i.e. we want to prove that, for any loop c:

$$\oint_c \vec{F}\cdot d\vec{r}=0$$

$\oint_c$ is the symbol for line integral along a loop c, and positive direction of the path is chosen counterclockwise (right hand rule), as shown in the figure below.



Green Theorem states that if the vector field $\vec{F}$ (any vector filed, conservative or not)is defined and differentiable (its partial derivatives are also defined) everywhere in the domain enclosed by the loop (note this constraint is wider than the line integral which only requires vector field is defined and differentiable along the curve, thus this put the limit on the applicability of Green Theorem, or you play the trick by cutting holes to exclude the singularity points in the domain, then the loop will be a little complicated), then the loop line integral equals to an area integral (double integral over the enclosed area by the loop):

In 2-D: $\vec{F} = M(x, y)\hat{i} + N(x, y)\hat{j}$

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \equiv \iint_R (N_x - M_y) dxdy \qquad (91)$$

R is the area enclose by the loop c.

The proof of this Green Theorem is given in K&K's section 5.6. Actually (91) is only one type of Green Theorem in 2-D, it is called tangential form of Green theorem, because it deals with the line integral of the dot product with the tangent vector of the curve[174]. It offers another way to evaluate the line integral of closed loop, sometimes the computation on the right hand side of eqn.(91) may be easier.

The $N_x - M_y$ on the R.H.S of 91 also defines a physical quantity of the vector field in 2-D, it is called curl of the field:

$$curl(\vec{F}) \equiv N_x - M_y \qquad (92)$$

And (91) is often written in terms of curl as:

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y)dxdy \equiv \iint_R curl(\vec{F})dA \qquad (93)$$

Now let's look for the special case of conservative force. From (93), we see that if $N_x = M_y$, then $\oint_c \vec{F} \cdot d\vec{r}$ =0 for any loops in the domain. This is the condition we are looking for as stated at the beginning of this section. So that for a vector field in 2-D, if $N_x = M_y$ or equivalently curl(F)=0, then the force is conservative, this closed the loophole in the argument of last section.

Gradient theorem allows us to show that the work will be path independent if the force is gradient of potential function. Now Green theorem shows that if the curl of the force is zero, work done along a loop will be zero. The condition for the curl(F)=0 is exactly $N_x = M_y$, which is relation (89) or (90) in the previous section!

---

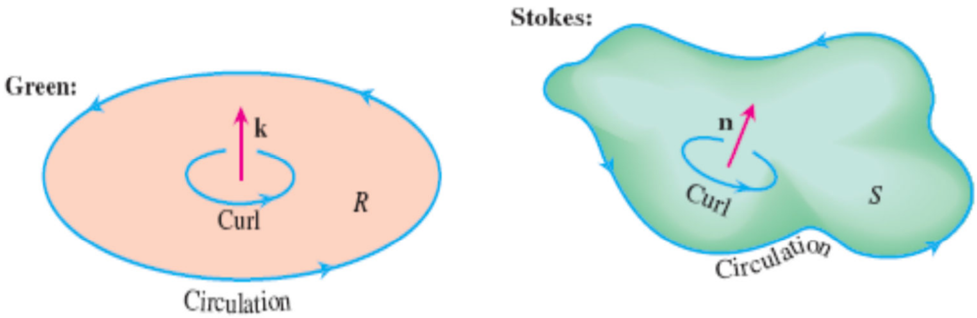
[174] There is another Green theorem (Normal form) which deals with the line integral of the dot product of a vector field with the normal direction of the curve. Such line integral is related to the flux in physical problem. We do not discuss this here, you will certainly learn it in the study of electro-magnetic field

In 3-D case, the Green theorem in 2-D is extended to:

$$\vec{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k} \equiv < M, N.P >$$

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_S curl(\vec{F}) \cdot \hat{n} dA = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dA \qquad (94)$$

This is called Stokes theorem. S is *any* surface enclosed by the loop c in 3-D, $\hat{n}$ is the unit normal vector of an area element, its positive direction is defined using right hand rule. (the definition positive direction of $\hat{n}$ is related to how you travel along the c)



The curl of the vector field in 3-D is defined as:

$$curl(\vec{F}) \equiv \nabla \times \vec{F} \qquad (95)$$

∇ is the differential operator given in (67) $\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ in Cartesian. Using the rule of cross product, we have:

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = (P_y - N_z)\hat{i} + (M_z - P_x)\hat{j} + (N_x - M_y)\hat{k} \qquad (96)$$

So the 2-D curl is just a special case of this definition (P is always zero in 2-D, and M,N independent of z). From the Stokes theorem, if $\nabla \times \vec{F}$ =0, then the loop integral will be zero and the force is conservative. The condition for $\nabla \times \vec{F} = 0$ is exactly the condition (90) from the gradient point of view.

Now we can make a summary for conservative force, the following statements are equivalent for the conservative force (Don't forget that the premise is that the force only explicitly depends on position):

1) The work done is path independent
2) The work done through a loop is zero
3) The force is a gradient of a potential function
4) The curl of the force is zero.

1 And 2 are definitions and corollary of conservative force, they are equivalent. The previous sections show that 3 will lead to 1 and 4 will lead to 2. So naturally you will conclude that 3 and 4 will be equivalent too. Indeed, we can show that:

$\nabla \times \nabla f = 0$ (97)

It is a famous equation in vector analysis. Please prove this yourself; it is just two lines using (90) and (96).

Comments on Time Dependent Force

(1) Time Dependent Force is not Conservative

If the force has explicit dependence on time:

$\vec{F}(x,y,z,t) = \langle M(x,y,z,t), N(x,y,z,t), P(x,y,z,t) \rangle$

Then the work done by this force can be computed using (82), doing the line integral with time as parameter. The problem here is this line integral will depend on the path of choice.

The reasoning (not a rigorous math proof) is following: Consider a closed loop line integral:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C [M(x,y,z,t)\frac{dx}{dt} + N(x,y,z,t)\frac{dy}{dt} + P(x,y,z,t)\frac{dz}{dt}]dt \quad (98)$$

The integral can be evaluated knowing the path and its dependence on time (the trajectory). However in this case, the starting point and ending point even for a spatially closed loop is not same, the time is different. The particle comes back to the original position but at later time. So the result of the above integral will generally be:

$$\oint_C \vec{F} \cdot d\vec{r} = f_C(t_2) - f_C(t_1)$$

f_C means that the function form may depend on the path C too. There is no guarantee that this value will be zero for *arbitrary* closed loop as required by the path independence. Because you may choose arbitrary loop, so the time delay (t_2-t_1) could be any value, If $f_C(t_2) - f_C(t_1)$ is always 0 for any time delay, this is only possible that the f_C must be a constant, that means it is not dependent on time at all! That is only possible if the force does not explicitly depend on time. So for the time dependent force, the work done for a closed loop will not be zero for arbitrary loops, this is same as saying the work will be **path dependent**. The force is not conservative.

(2) Dose Time Dependent Associated to a Potential

Previously we proved that for conservative force (which is only depending explicitly on position), it is a gradient of potential function $U_c(x,y,z)$,the subscript c here stands for conservative.

$\vec{F}_c = -\nabla U_c = \nabla f$ (relation 87)

Now for the force that explicitly depends on time, it is possible to define a potential function like the above, as long as the spatial dependence of F satisfies similar relations as in (89) or (90), or equivalently treating time as an independent variable, if $\nabla \times \vec{F} = 0$, then the force could also be written as:

$$\vec{F}(x,y,z,t) = -\nabla U(x,y,z,t) = -[\frac{\partial U(x,y,z,t)}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}] \quad (99)$$

This is valid math manipulation, so even for time-dependent force, we can define a potential. However I have to stress that the potential defined this way, their difference does not equals to the path integral in (98). The path integral would be path dependent and no potential function can be derived there. The difference in math is a bit subtle, the (x,y,z,t) in the path integral are not really independent variables, the x,y,z depends on time there. While the (99) treat the x,y,z,t as completely independent variables.

The important fact that we can relate the potential of conservative force with path integral is because we shall have mechanical energy conservation. So for the conservative force, we have mechanical energy conservation from work-energy theorem. For time dependent force and potential defined above, we do not have mechanical energy conservation! The common derivation to show that the mechanical energy (T+U) is not conserved, i.e. it changes with time is from Lagrangian formalism[175]of mechanics. That is why the time dependent potential defined in (99) is best termed non-conservative potential to distinguish from the conservative potential..

Basically:

$$T = \frac{1}{2} m v^2$$

$$\frac{dT}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{F} \cdot \vec{v} = -\left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial Z}\frac{dz}{dt}\right)$$

$$\frac{dU(x,y,z,t)}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} + \frac{\partial U}{\partial t}$$

So: $\frac{d(T+U)}{dt} = \frac{\partial U}{\partial t}$ is not zero in general. The mechanical energy is not conserved, and the mechanical energy is converted into other energy, lost to the outside world. Here please remember our discussion in chapter1, the reason that we have time dependent force is that our system is not closed, so the energy will flow in and out of our mechanical system[176].

As a final example, I will show you the form of gradient vector in polar coordinate. This example will use almost all the technique we learned so

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[175] See for example, Landau and Lifshitz “Mechanics”, chapter 2.

[176] Please read Taylor’s “classical mechanics”, section 4.5 for more discussion and example.

far. I will only work the two dimensional case from Cartesian to Polar. The definition of a gradient vector in Cartesian is (66), i.e.:

$$\nabla f \equiv \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Then what this vector's expression in polar coordinate? i.e. everything expressed in terms of $r, \theta, \hat{r}, \hat{\theta}$.

We know the relations between the base vectors and coordinates:

$$r = \sqrt{x^2 + y^2}; \tan\theta = \frac{y}{x}$$
$$x = r\cos\theta; y = r\sin\theta$$
$$\hat{i} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$
$$\hat{j} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$$

Then $\hat{i}, \hat{j}$ can be expressed in terms of $\hat{r}, \hat{\theta}$, what about the $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$?

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} \end{aligned} \quad \text{(chain rule)}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \sin\theta$$

To calculate $\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}$, I will use implicit method and total differential:

$$d(\tan\theta) = \frac{\partial}{\partial x}(\frac{y}{x})dx + \frac{\partial}{\partial y}(\frac{y}{x})dy$$

$$\frac{d\theta}{\cos^2\theta} = -\frac{y}{x^2}dx + \frac{1}{x}dy$$

$$\frac{\partial\theta}{\partial x} = -\cos^2\theta\frac{y}{x^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{x} = \frac{\cos\theta}{r}$$

Now we throw all above and relation of $\hat{i},\hat{j}$ with $\hat{r},\hat{\theta}$ into $\nabla f \equiv \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$:

The procedure is too long to type, final result is:

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} \qquad (100)$$

The differential operator in polar coordinate is in form:

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} \qquad (101)$$

This is a pretty long calculation, but I do not use any trick and stick to what we learned so far. It is the 'stupidest' method but the 'bread-butter' one, which I can derive without referring to books after many years. There are simpler derivations based on the theory of coordinate transformation, but won't be covered here[177]. You may work out the formula of the gradient and curl in Cylindrical or Spherical coordinate, in 3-D, similarly like what I did here (it is just more time consuming). The results of these in non-Cartesian are generally listed in the standard textbook, so no need to memorize them, except the forms in Cartesian,

[177] You may refer to Greiner's book Chap. 10, 11 for some details.

you have to know them by heart.

Supplementary III

Ordinary Differential Equation (ODE)

In this supplementary, I shall give a brief introduction on subject of solving ordinary differential equations[178]. Solving ODE is not an easy task and not all types of ODE have analytical solutions (In fact majority of them will not have). I shall focus on some special forms of ODE which have analytical solutions and outline how we get them. The ODE will be either 1st order (means only contains first order derivative) or 2nd order (the highest order of derivative will be 2), these are the common cases you will meet in physics.

The general strategy solving differential equations always involves process of "guess" work, i.e. guess what the solution should look alike with some undetermined parameters and solve for the parameters, thus reducing the differential equations to algebraic equations.

1. First Order ODE

1.1 Easy Ones with Methods of Separation of Variables

(1) Indefinite Integral, General and Specific Solution Initial Value

The simplest 1st order ODE that I can think of is:

$$\frac{dy}{dx} = f(x) \qquad (1)$$

[178] As usual, you may refer to a math textbook for details. Such as 'Elementary Differential Equations' 6th edition by C. Edwards and D. Penney.

Its general solution will be just the indefinite integral (antiderivative):

$$y = \int f(x)dx + C \qquad (2)$$

This is the general solution for y, i.e. all solutions of (1) will satisfy this form. C is the integration constant. To get a specific solution, we need to fix the C. This is achieved by knowing the initial value, and get the specific solution is also called initial value problems (IVP) or boundary value problems. The conventional initial value will be in form of: $x = 0, y(0) = y_0$ and C can be determined from (2) by inserting the initial value.

Example: velocity of a car changes with time v=2t, find its position over time, with t=0, x=2.

$$\frac{dx}{dt} = 2t \to x = t^2 + C \overset{t=0,x=2}{\to} C = 2;\ x = t^2 + 2$$

Another important model (both in physics and other science) is exponential decay or growth:

$$dx / dt = kx$$

$$\frac{dx}{x} = kdt \to \ln x = kt + C \to x = Ae^{kt}$$

(k>0 for population growth, money interest etc. k<0 for nuclear decay, temperature loss etc.)

(2) Separation of Variables

The next simplest form would be:

$\frac{dy}{dx} = f(x, y)$ with f(x,y) in some specific form:

$$f(x,y) = g(x)h'(y) = \frac{g(x)}{h(y)}$$

Then we can separate variables:

$$h(y)dy = g(x)dx \qquad (3)$$

Integral on both side:

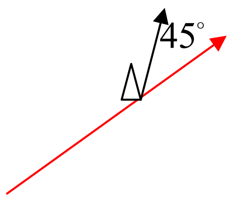
$$H(y) = G(x) + C \qquad (4)$$

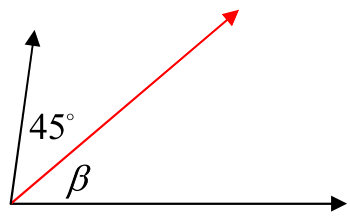
C need to be determined from initial conditions for specific solution. Generally what you get is an equation between y and x from (4), and explicit function y=F(x) may not be possible.

You have /would have seen quite a few examples in the physics notes (orbit function of planetary motion etc.). Simple as this separation of variables is, it is the 'bread-butter' in solving differential equations for this course.

Sometimes, it may require tricks of *substitution* to get the ODE in forms of (3) as the example below illustrate.

Example:





Suppose there is a lighthouse shines a beam of light (the read beam) on a little boat. The boat will always try to travel along a direction that has 45° with respect to the light beam, and the light beam at the same time follow the boat, then what is the trajectory of the boat, i.e. its y(x).

If the boat at (x,y), the light beam will along direction:

$$\tan\beta = \frac{y}{x}$$

The direction of boat's velocity is:

$$\tan\alpha = \tan(45+\beta) = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\tan\beta + \tan 45}{1-\tan\beta\tan 45} = \frac{y/x+1}{1-y/x}$$

This is the differential equation need to be solved to get y(x), and it may appear that we cannot use separation of variables. The trick is use substitution, by introduce new variable:

$$z = \frac{y}{x}, y = zx, y' = z + xz' \quad (y' \equiv dy/dx)$$

$$xz' + z = \frac{1+z}{1-z} \rightarrow xz' = \frac{1+z^2}{1-z}$$ now we can separate variables:

$$\frac{1-z}{1+z^2}dz = \frac{dx}{x}$$

The left hand is a nasty integral, but I can check the table:

$$\tan^{-1} z - \frac{1}{2}\ln(1+z^2) = \ln x + C$$

$$\tan^{-1} z = \ln x\sqrt{1+z^2} + C = \ln\sqrt{x^2+y^2} + C$$

In polar coordinate above equation is:

$$\theta = \ln r + C \rightarrow r = Ae^{\theta}$$

1.2 Linear 1st Order ODE and Integration Factor

The linear equation is in the form of:

$$a(x)y'+b(x)y+c(x)=0 \qquad (5)$$

This is linear because if y_1,y_2 are solutions, their linear combination uy_1+vy_2 is a solution for the equation with $c_{1+2}(x)=uc_1+vc_2$ (c=0 is a special case which is called homogeneous). In engineering, the c(x) is called input, the y(x) solution is called response. Linear here means the response has a linear relation with the input. Equation (5) is often reorganized into standard form of linear ODE:

$$y'+P(x)y=Q(x) \qquad (6)$$

The integration factor is a function of x, say $\rho(x)$, and it is multiplied to both ends of the equation (6), and make the left hand side a total differential (this is the guess work that such integration factor exists), i.e.:

$$\rho(x)y'+\rho(x)P(x)y=\rho(x)Q(x)$$

$$\rho(x)y'+\rho(x)P(x)y=d(\rho(x)y)/dx \qquad (7)$$

$$d(\rho(x)y)/dx=\rho(x)Q(x)$$

$$\rho(x)y=\int\rho(x)Q(x)dx+C$$

$$y=\rho(x)^{-1}(\int\rho(x)Q(x)dx+C) \qquad (8)$$

To find out the integration factor $\rho(x)$, we can use the requirement of (7):

$$d(\rho(x)y)/dx=\rho' y+\rho y'=\rho y'+\rho Py$$

$$\rho'=\rho P(x)$$

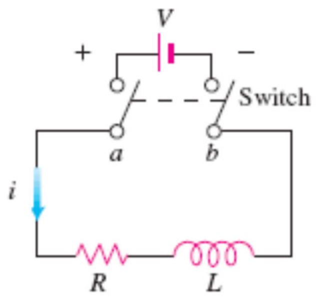
$$\frac{d\rho}{\rho}=P(x)dx$$

$$\ln \rho = \int P dx$$
$$\rho = e^{\int P dx} \qquad (9)$$

So the integration factor can be computed (at least in principle) from (9), and then put it back to (8) will give the general solution for y:

$$y = e^{-\int P dx}\left(\int e^{\int P dx} Q(x) dx + C\right) \qquad (10)$$

Example 1: RL circuit



The voltage drop across resistor R is: iR (current x resistance)

The voltage drop across inductor: $L\dfrac{di}{dt}$

$L\dfrac{di}{dt} + iR = V$, out it in standard form:

$i' + Pi = Q,\ P = \dfrac{R}{L}, Q = \dfrac{V}{L}$ (actually this problem can be solved by separation of variables, but I shall use integration factor)

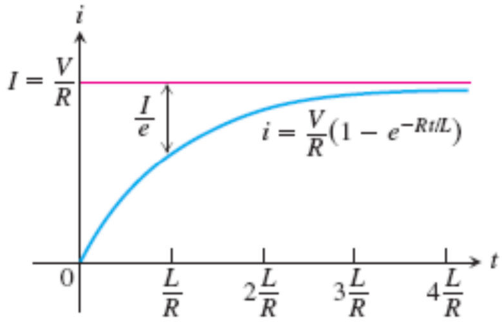
$$\rho = e^{\int P dt} = e^{Pt}$$

$$\int \rho Q dt = Q \int e^{Pt} dt = \frac{Q}{P} e^{Pt} = \frac{V}{R} e^{Pt}$$

$$i = e^{-Pt}\left(\frac{V}{R} e^{Pt} + C\right) = \frac{V}{R} + C e^{-\frac{R}{L}t}$$

Let the initial condition is: $t = 0, i = 0$, thus $C = -\dfrac{V}{R}$

$$i(t)=\frac{V}{R}(1-e^{-\frac{R}{L}t})$$



The current will reach to its steady value V/R over time, how fast it is going to reach depends on L/R. This is the effect that the inductor 'fights' against current change.

Example 2: Falling object subject to air resistance:

air resistance: $F=-kv$, gravity: mg

$$m\frac{dv}{dt}=mg-kv$$

$$v'+\frac{k}{m}v=g$$

Similar to the above and at t=0, v=0:

$$v=\frac{mg}{k}(1-e^{-\frac{k}{m}t})$$

You may try to find the position change over time from there.

What happened if the air resistance is in forms of $F=-kv^2$? The equation will not be linear, and the integration factor method will fail. However, you may still apply the separation of variables here. The details will be left for you to explore.

2. 2nd Order Linear ODE

The linear 2nd order ODE is generally in forms of:

$A(x)y''+B(x)y'+C(x)y+D(x)=0$

If the D(x)=0, it is called homogeneous equation. This homogeneous linear equation (2[nd] order) has very important features (theorems)

(1) Superposition principle of solution for homogeneous equation

If y_1, y_2 are solutions for the homogeneous equation, then their linear combination is a solution too: i.e. $ay_1 + by_2$ is also a solution.

(2) Completeness of solution:

If y_1, y_2 are solutions and they are independent (means $ay_1 + by_2$=0 only when a=0,b=0; another way saying it is y_1, y_2 are not a constant multiplier of each other), then the general solution for the homogeneous equation(2[nd] order here) will be in form of:

$$y = c_1y_1 + c_2y_2 \qquad (12)$$

c_1, c_2 are arbitrary constant for the general solution, and can be determined to give specific solution provided with initial values, such as $y(0), y'(0)$.

The proofs of the theorems will not be our focus here (the 1[st] one is easy to prove; the 2[nd] one is a bit headache but not so hard[179]). We shall focus on how to find the independent solutions y_1, y_2, given the homogeneous equation first; then see how to handle the inhomogeneous ones (i.e. D(x) is not zero). Unfortunately, with coefficients varies with x, as the equation above (A, B,C are functions of x), there is no general methods to get

[179] See for example: chap.2 of the book in footnote 25.

solutions. The one we can get simple answers are the special class of equations called constant coefficients where A,B,C are constants. That is what we shall study below.

2.1 Constant Coefficients 2nd Order Homogeneous Linear ODE

The long title tells that we are dealing with a very special class of ODE. The standard form is:

$$ay''+by'+cy=0 \qquad (13)$$

Let's take a guess of what the form of function y(x) may look like. The functions after taking derivative will generally changes to another function except for the exponential ones $e^{\lambda x}$ (λ a constant which can be either real or complex). So let's take the $e^{\lambda x}$ for y_1, y_2 (the two independent solutions in (12)) as a trial. Put $e^{\lambda x}$ in (13), we get:

$$e^{\lambda x}(a\lambda^2+b\lambda+c)=0$$

$$a\lambda^2+b\lambda+c=0 \qquad (14)$$

This (14) is called characteristic equation of the ODE (13). There could be 3 cases for its roots:

(1) $b^2-4ac>0$

There will be two real roots for (14):

$$\lambda_1=\frac{-b+\sqrt{b^2-4ac}}{2a}, \lambda_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$y_1=e^{\lambda_1 x}, y_2=e^{\lambda_2 x}$ are two independent solutions and the general solution to equation (13) will be:

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x} \qquad (15)$$

The constants c_1, c_2 shall be determined provides with initial conditions (such as y(0) and y'(0)).

(2) $b^2 - 4ac < 0$

There will still be two roots but are complex:

$$\lambda_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a} = \alpha + i\beta, \lambda_2 = \frac{-b - \sqrt{4ac - b^2}}{2a} = \alpha - i\beta$$

$\lambda_1 = \lambda_2^*$ complex conjugate with each other

The two solutions $y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$ will still be independent, so the general solution can be written as:

$$y = \tilde{c}_1 e^{\lambda_1 x} + \tilde{c}_2 e^{\lambda_2 x} \qquad (16)$$

$\tilde{c}_1, \tilde{c}_2$ are two *complex* constants. The form (16) is ok for the solutions that include complex functions. But in some applications, such as the oscillations in mechanics, the solutions should be real functions, this is also able to be satisfied by (16), it just put a requirement on $\tilde{c}_1, \tilde{c}_2$. Since $\lambda_1 = \lambda_2^*$, if $\tilde{c}_1 = \tilde{c}_2^*$ are complex conjugate with each other, then the (16) will be a real function.

$$\tilde{c}_1 = u + iv, \tilde{c}_2 = u - iv$$

$$e^{\lambda_1 x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x),\ e^{\lambda_2 x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

Euler formula (38) is used here.

Put these into (16):

$$y = e^{\alpha x}(2u\cos\beta x - 2v\sin\beta x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \qquad (17)$$

Note c_1, c_2 are real constants, they are not $\tilde{c}_1, \tilde{c}_2$ but are related to their real and imaginary parts.

Of course (17) can also be proved by other methods, because any combination of $ay_1 + by_2$ is also a solution to (13), so we can set:

$y_1' = \dfrac{y_1 + y_2}{2} = e^{ax}\cos\beta x$; $y_2' = \dfrac{y_1 - y_2}{2i} = e^{\alpha x}\sin\beta x$ as two independent real solutions to (13), and the general solution can be written as:

$y = c_1 y_1' + c_2 y_2'$ which is (17) we derived above.

There is also another very useful variation of (17), and we shall derive it from (16), this also offers a good practice on Euler expression for complex numbers and introduces the important "vector" representation of complex number[180].

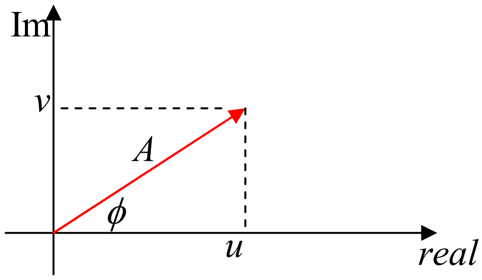
$\tilde{c}_1 = u + iv, \tilde{c}_2 = u - iv$, let's express it into Euler form:

$$\tilde{c}_1 = Ae^{i\phi} \qquad (18)$$

$$\begin{aligned} &A = \sqrt{u^2 + v^2},\ \tan\phi = \frac{v}{u}; \\ &u = A\cos\phi, v = A\sin\phi \end{aligned} \qquad (19)$$

(19) expressed the module A and phase angle ϕ. The meaning of (18) and (19) is extremely clear from the geometric representation of complex numbers. They are quite analogous to Cartesian and Polar representation of vectors:

[180] The reason I put quotation mark around vector is because generally the physical vectors we encounter in mechanics live in real space. Here for complex numbers, we have to go to complex space. You can think this as extrapolation of vector to complex space.



The above figure use a "vector" like arrow represents the complex number $\tilde{c}_1 = u + iv$, with length and phase angle given by (19). This "vector" representation is usually called phasor, and they do add up as vectors do (as complex numbers do). This phasor will be very useful for computations involving complex numbers, when we discuss general wave theory in later courses.

With the $\tilde{c}_1 = Ae^{i\phi}$, $\tilde{c}_2 = \tilde{c}_1^{\,*} = Ae^{-i\phi}$, equation (16) can be written as:

$$y = Ae^{\alpha x}[e^{i(\beta x+\phi)} + e^{-i(\beta x+\phi)}] = 2Ae^{\alpha x}\cos(\beta x+\phi) = e^{\alpha x}c\cos(\beta x+\phi) \quad (20)$$

(20) and (17) are equivalent expressions for the general solutions, though on the surface they look slightly different. This difference can be set aside by the trigonometric relation:

$$\begin{aligned} &a\cos\theta + b\sin\theta = c\cos(\theta-\varphi) \\ &c = \sqrt{a^2+b^2}, \quad \tan\varphi = \frac{b}{a} \end{aligned} \qquad (21)^{181}$$

Apply this to (17):

$$c_1\cos\beta x + c_2\sin\beta x = \sqrt{c_1^2+c_2^2}\cos(\beta x-\phi') = 2\sqrt{u^2+v^2}\cos(\beta x-\phi') = 2A\cos(\beta x-\phi')$$

$$\phi' = \tan^{-1}\frac{c_2}{c_1} = \tan^{-1}(-\frac{v}{u}) = -\tan^{-1}\frac{v}{u} = -\phi$$

[181] Please prove this useful relation yourself. Hint: cosine expansion will work; or think about dot product of vectors and its geometric meaning; or try to apply Euler formula…. Many ways to prove this

So indeed (17) and (20) are equivalent. In (17) the constants c_1, c_2, and in (20) constants A, ϕ can be determined with the initial conditions.

(3) $b^2 - 4ac = 0$

Here we have degenerate roots:

$$\lambda_1 = \lambda_2 = \frac{-b}{2a}$$

There is only one independent $y_1 = e^{\lambda x}$ which is an incomplete base for the general solution of the 2nd order ODE. We have to find another independent solution. This is done by variation technique:

Suppose another solution in forms of: $y_2 = u(x)e^{\lambda x}, u(x)$ is a polynomial of x, and we need to find this $u(x)$ and it is better be not a number:

$$y_2' = u'e^{\lambda x} + \lambda u e^{\lambda x}$$

$$y_2'' = u''e^{\lambda x} + \lambda u'e^{\lambda x} + \lambda u'e^{\lambda x} + \lambda^2 u e^{\lambda x} = u''e^{\lambda x} + 2\lambda u'e^{\lambda x} + \lambda^2 u e^{\lambda x}$$

$$ay_2'' + by_2' + cy_2 = 0 \to ue^{\lambda x}(a\lambda^2 + b\lambda + c) + e^{\lambda x}(au'' + 2a\lambda u' + bu') = 0$$

Because $a\lambda^2 + b\lambda + c = 0, \lambda = -\dfrac{b}{2a}$, we have from above:

$$u'' = 0 \to u(x) = px + q$$

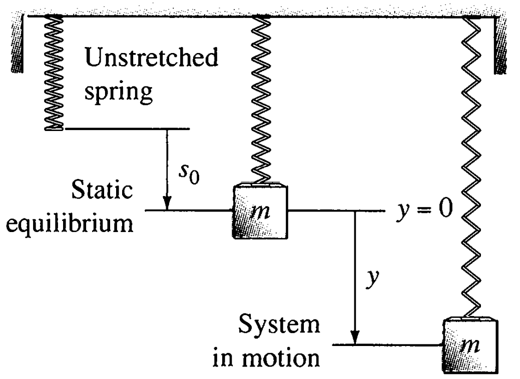
The simplest form of $u(x)$ (which is not a number) is just $u(x) = x$ (here we only want one solution that is independent of y_1, so we choose the simplest form satisfying this).

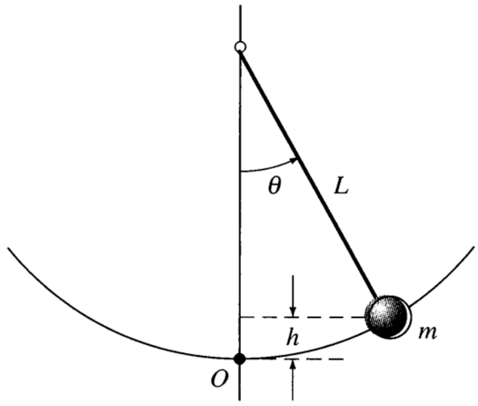
$y_2 = xe^{\lambda x}$ will satisfy the ODE and is independent of y_1, the general solution for the ODE with double degenerate roots:

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x} = (c_1 + c_2 x)e^{\lambda x} \qquad (22)$$

Noticed that to get the $u'' = 0 \to u(x) = px + q$, we need to use the fact of double degeneracy, i.e. $\lambda_1 = \lambda_2 = \frac{-b}{2a}$, so (22) only applies to this condition (this is why in case 1 and 2 where $b^2 - 4ac \neq 0$, solutions cannot be expressed with single exponential)

The examples will be the harmonic oscillators discussed in chapter 10 of the physics notes:





(1) Free Oscillators without damping

We have seen many such oscillators in physics, be it the mass-spring on the left, or the simple pendulum or the right, or the physical pendulum we discussed in chapter 7 on rotation. You will see more such models in other physical applications, such as RCL circuit, molecular vibration etc.

The reason of such wide applicability of Harmonic Oscillator model comes from the fact mentioned awhile ago: The potential energy curve around the equilibrium point (i.e. U is minimum, the force are restoring force pointing towards minimum) can be approximated by a harmonic potential, i.e. $U \propto x^2$, where x is the displacement from the equilibrium. The restoring force will be thus in forms of $F \propto -\frac{dU}{dx} \propto -x$. The equation of motion can be written as:

$m\ddot{x} = -kx$ (23)[182]

Equation (23) can be applied to all the harmonic oscillators, be it mass-spring or pendulum. For mass-spring m is mass, x is displacement from equilibrium, -kx is Hook's law; For pendulum, m is related to moment of inertia, x is angular displacement, -kx is related to torque. All these different harmonic models can be expressed to the form in (23). Here free means no other driving force besides the restoring force; no damping is neglecting the effect of friction etc.

(23) is usually written in another form:

$\ddot{x} + \omega_0^2 x = 0$ (24)

$\omega_0 = \sqrt{k/m}$ (25)

Solving (24) with the standard method of 2nd ODE:

x_1, x_2 will be in forms of $e^{\lambda t}$, and

$\lambda^2 + \omega_0^2 = 0 \rightarrow \lambda_1 = i\omega_0; \lambda_2 = -i\omega_0$

From the discussion in case (2) above, the general solution of (24) is:

$x(t) = \tilde{c}_1 e^{i\omega_0 t} + \tilde{c}_2 e^{-i\omega_0 t}$

The displacement function is obviously a real function, so:

$x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t = A\cos(\omega_0 t + \phi)$ (26)

This is the general solutions for the free oscillator with no damping. The

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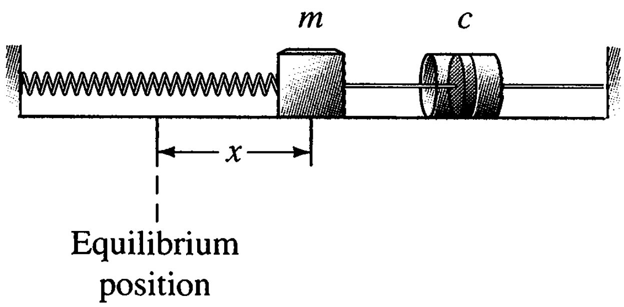
[182] Here we study the change over time. The independent variable is time t (i.e. x=t), and dependent variable is x (i.e. y=x). The temporal derivative is usually expressed as $\dot{x}, \ddot{x}$.

choice of cosine instead of sine and $+\phi$ instead of $-\phi'$ is a convention rather than necessity.

(2) Free oscillators with damping



Suppose we introduce the damping as shown explicitly in the figure. The damping force will be always against the direction of motion, i.e. against the velocity, and magnitude is proportional to speed:

$f_d = -bv$

The equation of motion will be:

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x}$$

Usually write it as:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \qquad (27)$$

$$\gamma = b/m; \omega_0 = \sqrt{k/m}$$

Characteristic equation is:

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0 \qquad (28)$$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

Case A: $\frac{\gamma^2}{4} - \omega_0^2 > 0$, strong damping

λ_1, λ_2 are real numbers, general solution:

$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

Note λ_1, λ_2 are negative numbers, so the x(t) will decay to almost 0 at long run[183].

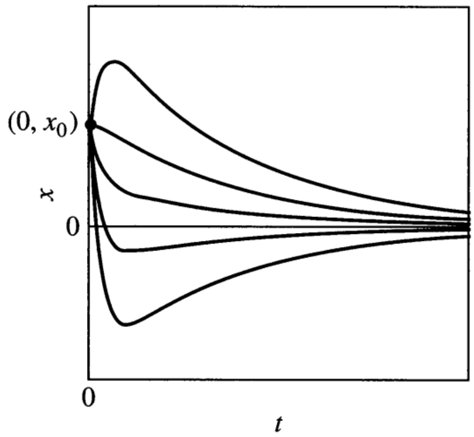


FIGURE 2.4.7. Overdamped motion: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ with $r_1 < 0$ and $r_2 < 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

Case B: $\frac{\gamma^2}{4} - \omega_0^2 = 0$, critical damping

$\lambda_{1,2} = -\frac{\gamma}{2}$

The general solution would be:

$x(t) = (c_1 + c_2 t) e^{-\frac{\gamma}{2} t}$

It approaches to zero at longer t. It actually approaches zero faster than case A. This is called critical damping, $\frac{\gamma^2}{4} - \omega_0^2 = 0$ is the critical damping condition. This has wide applications in situations where oscillation is not wanted, such as stabilization system in automobile;

[183] Figure is taken from 'Elementary Differential Equations' 6th edition by C. Edwards and D. Penney.

oscillation isolators for optical tables etc.

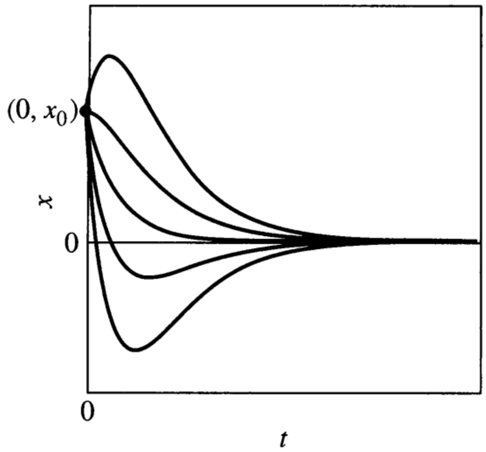


FIGURE 2.4.8. Critically damped motion: $x(t) = (c_1 + c_2 t)e^{-pt}$ with $p > 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

Case C： $\dfrac{\gamma^2}{4} - \omega_0^2 < 0$ weak damping:

Introduce:

$$\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \qquad (29)$$

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm i\omega'$$

General solution is:

$$x(t) = e^{-\frac{\gamma}{2}t}(c_1 \cos\omega' t + c_2 \sin\omega' t) = e^{-\frac{\gamma}{2}t} A\cos(\omega' t + \phi) \qquad (30)$$

When the damp is very weak, $\omega_0 >> \gamma$, the solution is like a oscillator with a slow decaying amplitude, as figure below shows.

This concludes our strict solutions to free oscillators.

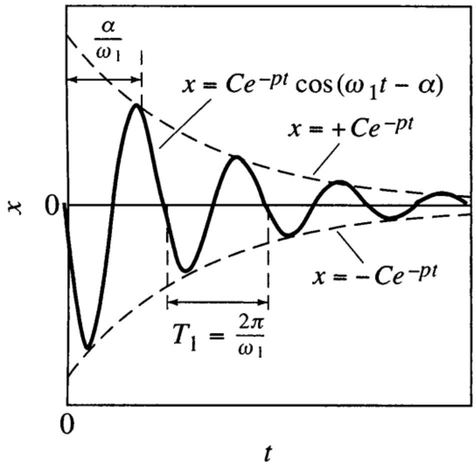


FIGURE 2.4.9. Underdamped oscillations:
$x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$.

2.2 Constant Coefficients 2[nd] Order **Inhomogeneous** Linear ODE

Here $D(x) \neq 0$ anymore, and the ODE is usually written as:

$ay'' + by' + cy = f(x)$ (31)

$f(x)$ is called driving force or input of the system, the solution y is called the response or the output of the system.

The fundamental theorem for this kind of inhomogeneous equation is the following, the general solution of (31) can be expressed as:

$y = y_p + y_c$ (32)

y_p is one (anyone) *particular* solution to (31) and need to be independent to y_c; y_c is called *complementary* solution, which is the general solution to the associated homogeneous equation, i.e. y_c is the general solution of $ay'' + by' + cy = 0$, same a, b, c in (31).

This theorem is not hard to prove. First if y_p is a solution to (31),

$y_p + y_c$ is also a solution. It is just like adding a 0 to the RHS of (31) and applies the linearity of the ODE. Next we need to prove all solutions to (31) can be expressed as (32). Suppose besides y_p, we have another solution to (31) which is y'_p. Then using linearity, $y'_p - y_p$ would be a solution to the complementary homogeneous equation. But the general solution of this is expressed as y_c, i.e. $y'_p - y_p = y_c$. Q.E.D.

The focus of the inhomogeneous ODE will be on how we get the particular solution. The solution of it will heavily depend on the detailed function form of $f(x)$. There are only a few simple forms of $f(x)$ that we can get simple answers for particular solutions.

(1) $f(x)$ is a polynomial of x

The general method will be undetermined coefficients, illustrated as example here:

$$y'' + 4y = 4x^3$$

The 'guess' is that the particular solution would be also a polynomial with same order, i.e.:

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

Throw them in the equation:

$$y'' + 4y = 4Ax^3 + 4Bx^2 + (6A + 4C)x + (2B + D) = 4x^3$$

Equate the coefficients of each order of x, clearly:

$A = 1; B, D = 0; C = -\frac{3}{2}$

$y_p = x^3 - \frac{3}{2}x$

The complementary solution is: $y_c = A\cos(2t + \phi)$, y_p is independent to y_c. So $y = x^3 - \frac{3}{2}x + A\cos(2t + \phi)$.

(2) $f(x) = e^{\alpha t}$ (this also includes $\cos\alpha t, \sin\alpha t$ cases)

$ay'' + by' + cy = e^{\alpha x}$ (33)

In this case, you probably can guess the particular solution to (33) is in form of: (this is also the method of undetermined coefficients)

$y_p = Ae^{\alpha x}$ (34)

Substitute this into (33) to determine the coefficient:

$Ae^{\alpha x}(a\alpha^2 + b\alpha + c) = e^{\alpha x}$

$A = \frac{1}{a\alpha^2 + b\alpha + c}$

$y_p = \frac{e^{\alpha x}}{a\alpha^2 + b\alpha + c}$ (35)

There is a subtlety here, in order to get (35) $a\alpha^2 + b\alpha + c \neq 0$, i.e. α is not a root for the characteristic equation. If the α is a root, that is if α equals to $\lambda_1 or \lambda_2$, the particular solution y_p will be same as one of the complementary solution y_c. We need to investigate this.

(a) $a\alpha^2 + b\alpha + c \neq 0$, i.e. $\alpha \neq \lambda_{1,2}$, (35) applies

(b) $\alpha = \lambda_1 \neq \lambda_2$, i.e. α equals to one of the roots and the roots are not degenerate.

$y_c = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

The particular solution need to be independent with y_c, so the guess of $y_p = Ae^{\alpha x}$ won't work now. Instead, let's guess:

$y_p = Axe^{\alpha x}$, $a\alpha^2 + b\alpha + c = 0$

$y_p' = Ae^{\alpha x} + \alpha Axe^{\alpha x}$

$y_p'' = \alpha^2 Axe^{\alpha x} + 2\alpha Ae^{\alpha x}$

$ay'' + by' + cy = Axe^{\alpha x}(a\alpha^2 + b\alpha + c) + Ae^{\alpha x}(2a\alpha + b) = e^{\alpha x}$

$$A = \frac{1}{2a\alpha + b}$$

$$y_p = \frac{1}{2a\alpha + b} xe^{\alpha x} \qquad (36)$$

(c) $\alpha = \lambda_1 = \lambda_2, \alpha$ equals to double degenerate roots

$y_c = (c_1 + c_2 x)e^{\lambda x}, y_p = Axe^{\alpha x}$ will be same as one of the y_2 in y_c, so it is not a valid independent solution anymore. We need to take:

$y_p = Ax^2 e^{\alpha x}$, $a\alpha^2 + b\alpha + c = 0$, $\alpha = -\dfrac{b}{2a}$

Similar procedure will give us:

$$A = \frac{1}{2a}$$

$$y_p = \frac{1}{2a} x^2 e^{\alpha x} \qquad (37)$$

Let's look at an example: Driven Oscillator

The oscillator with natural frequency and damping coefficient is driven by an external force $F = F_0 \cos(\omega t)$

The 2nd ODE is:

$m\ddot{x} = -kx - b\dot{x} + F_0 \cos(\omega t)$

Express it in the standard form:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t) \qquad (38)$$

The complementary solution of the associated homogeneous equation had been worked out before: for weak damping, $\gamma << \omega_0$

$$x_c = e^{-\frac{\gamma}{2}t}(c_1 \cos\omega' t + c_2 \sin\omega' t) = e^{-\frac{\gamma}{2}t} A\cos(\omega' t + \phi), \quad \omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

The question is the particular solution. The force is in forms of cosine function not the exponential we treated above. One method is to use Euler formula $\cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$, where $\alpha = i\omega$ and $\alpha = -i\omega$. We can use the results above and linearity of the equation to solve for the particular solution, i.e. $x_p = x_{1p} + x_{2p}$, x_{1p} corresponds to $\alpha = i\omega$ and x_{2p} corresponds to $\alpha = -i\omega$. This method is left for you to carry out. Here I shall use another method.

Let me introduce a supplementary function y, which satisfies:

$\ddot{y} + \gamma \dot{y} + \omega_0^2 y = \frac{F_0}{m} \sin(\omega t)$, then

$$i\ddot{y} + \gamma i\dot{y} + \omega_0^2 iy = i\frac{F_0}{m} \sin(\omega t)$$

Introduce complex function:

$$\tilde{z} = x + iy \qquad (39)$$

$$\ddot{\tilde{z}} + \gamma \dot{\tilde{z}} + \omega_0^2 \tilde{z} = \frac{F_0}{m} e^{i\omega t} \qquad (40)$$

Solving $\tilde{z}_p$, and its ***real*** part will be the particular solution of x_p.

$$\tilde{z}_p = \frac{F_0}{m}\frac{e^{i\omega t}}{(i\omega)^2 + i\gamma\omega + \omega_0^2} = \frac{F_0}{m}\frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

The denominator is a complex and I will write it as phasor:

$$\omega_0^2 - \omega^2 + i\gamma\omega = Ae^{i\phi'}$$

$$A = \sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}, \quad \phi' = \tan^{-1}(\frac{\gamma\omega}{\omega_0^2 - \omega^2}) \qquad (41)$$

$$\tilde{z}_p = \frac{F_0}{m}\frac{e^{i(\omega t - \phi')}}{A} = \frac{F_0}{m}\frac{e^{i(\omega t - \phi')}}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

The real part is:

$$x_p = \frac{F_0}{m}\frac{\cos(\omega t - \phi')}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \qquad (42)$$

Sometimes it is also expressed as (depends on our convention):

$$x_p = \frac{F_0}{m}\frac{\cos(\omega t + \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

$$\phi = -\phi' = \tan^{-1}(\frac{\gamma\omega}{\omega^2 - \omega_0^2}) \qquad (43)$$

The complete solution of (38) will be:

$$x(t) = x_p + x_c$$

$x_c = e^{-\frac{\gamma}{2}t}(c_1\cos\omega' t + c_2\sin\omega' t) = e^{-\frac{\gamma}{2}t}A\cos(\omega' t + \phi)$ will die out due to the damping after awhile. So it is called ***transient*** solution. The particular solution on the other hand is due to the driven force and it will last as long as there is driven force. So x_p is called ***steady state*** solution.

All above the technique is called undetermined coefficients, because we guess the answer for the particular solution and throw it back in the equation to find out the coefficient. There is another 'trick' called variation of parameters which is a standard tool to solve for the particular solution. I will introduce it below but this is not required for the course since we are not going to use it for the problems of physics in this class.

(3) Variation of Parameters

$ay''+by'+cy=f(x)$

The complementary solution is: $y_c = c_1y_1 + c_2y_2$, the guess for the particular solution would be:

$y_p = v_1(x)y_1 + v_2(x)y_2$ (44)

$v_1(x), v_2(x)$ are *functions* need to be determined. The only way to determine them is throwing (44) back to the ODE:

Notices: $ay''_{1,2} + by'_{1,2} + cy_{1,2} = 0$ because they are solutions to the homogeneous equation. Differentiate (44):

$y'_p = v_1y'_1 + v_2y'_2 + (v'_1y_1 + v'_2y_2)$

If we keep differentiate this to 2nd order, we will get 2nd order terms in v''_1, v''_2, these will not help us because we get another 2nd order ODE for the parameters, unless the things in the parenthesis go to zero, i.e.:

$v'_1y_1 + v'_2y_2 = 0$ (45)

This is the extra condition that we want to have parameters to satisfy. Then, the substitution into ODE will get another equation:

$v'_1y'_1 + v'_2y'_2 = f(x)/a$ (46)

The two equation (45), (46) are linear equations for v_1', v_2', and they may be solved to get v_1', v_2', then through integration, v_1, v_2 can be determined.

Example: $y'' + y = \tan x$

This is not solvable for the undetermined coefficients method, we will try the variation of parameters:

$y_c = c_1 \cos x + c_2 \sin x$, $y_1 = \cos x; y_2 = \sin x$ (from $y'' + y = 0$)

$y_p = v_1(x)\cos x + v_2(x)\sin x$

(45) will lead to:

$v_1' \cos x + v_2' \sin x = 0$

(46) will lead to:

$v_1'(-\sin x) + v_2' \cos x = \tan x$

Solve the pair linear equations (using matrix method, Gauss elimination or use the formula with inverse matrix):

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}$$

$$v_1' = \frac{-\sin^2 x}{\cos x}$$

$$v_2' = \sin x$$

$$v_1(x) = \int \frac{-\sin^2 x}{\cos x} dx \overset{\substack{check \\ table}}{=} -\ln|\sec x + \tan x| + \sin x$$

$$v_2(x) = -\cos x$$

(We do not worry about integration factors above, because we only need to find one particular solution) The y_p is determined and the general solutions can be written out.

All these methods could only handle a small class of function forms of $f(x)$. The most general treatment for arbitrary function $f(x)$ would be the task of Fourier Transform and Laplace Transform, which will not be covered here. Also recall that we only treat constant coefficients a, b, c here, for the more general case a, b, c may be functions of x. In such case, we shall either need power series technique or have no analytical solutions. The power series technique will not be covered here, if interested you can find it in the textbook on ODE. Finally if the a, b, c are functions of y, the ODE may not be linear anymore, there is generally no analytical solution for y(x).

2.3 Coupled Differential Equations[184]

2.3-1 1st Order Coupled Equations

There are many applications where not one dependent variable but a few dependent variables that their changes are coupled. In this kind of problems, usually the independent variable is taken as time t, the dependent variables are x,y (or other symbols), both are functions of time. Let's look at an armament race scenario: x is the military budget of Russia and y is that of US. The change of budget (a government decision) will not only depend on how much money already invested (a kind of wasted in arm race case; but this model is also applied to education fund,

[184] This is extra meat I throw in, not required for this course.

company competition etc) by our side, but also depend on how much invested by the enemy.

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \qquad (47)$$

In the armament race scenario, a, d are negative number (we already spend too much money), b, c are positive (in response to the enemy's increase of budget, here x=money already invested − normal budget). For other models,0 they can be any numbers.

The more general form of linearly coupled systems (1st order) will be:

$$\begin{aligned} \dot{x} &= ax + by + r_1(t) \\ \dot{y} &= cx + dy + r_2(t) \end{aligned} \qquad (48)$$

I shall only treat the (47) case, which is called homogenous coupled equations.

(1) Direct substitution

Example:

$$\begin{aligned} \dot{x}_1 &= -2x_1 + 2x_2 \\ \dot{x}_2 &= 2x_1 - 5x_2 \end{aligned}$$

Then by elimination, first:

$x_2 = \dfrac{\dot{x}_1 + 2x_1}{2}$ from the first equation, substitute this into second:

$$\frac{\ddot{x}_1 + 2\dot{x}_1}{2} = 2x_1 - 5\frac{\dot{x}_1 + 2x_1}{2}$$

$\ddot{x}_1 + 7\dot{x}_1 + 6x_1 = 0$ this is a 2nd order homo. ODE:

$$x_1 = c_1 e^{-t} + c_2 e^{-6t}$$

$$x_2 = \frac{\dot{x}_1 + 2x_1}{2} = \frac{1}{2}c_1 e^{-t} - 2c_2 e^{-6t}$$

From initial values $x_1(0), x_2(0)$, c_1, c_2 can be determined.

(2) Matrix Method

This is more powerful and elegant than the simple substitution, and also it has wider application, such as in solving the inhomogeneous coupled equations, and it is another example of eigenvalue and eigenvector of matrix, so I will show you the details below.

For the homogeneous couple equations:

Introduce column vector representation of the dependent variables:

$\vec{x} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \dot{\vec{x}} \equiv \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$, this is just the usual bookkeeping for vector in column matrix. The coupled ODE will be:

$$\dot{\vec{x}} = A\vec{x} \qquad (49)$$

A is the matrix of coefficients:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(49) is just a shorthand for the equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (50)$$

The solution for the coupled equation will be reduced to solving the eigenvalues and eigenvectors of matrix A:

Assume the appropriate solutions are in forms of:

$x_1 = a_1 e^{\lambda t}, x_2 = a_2 e^{\lambda t}$, a_1, a_2 are coefficients need to be determined. In matrix, it is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t}$$

Put above into (50):

$$\lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad (51)$$

This equation is exactly the eigenvalue-eigenvector equation for matrix A.

Its eigenvalues λ_1, λ_2, and their associated eigenvectors $\vec{\alpha}_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\vec{\alpha}_2 = \begin{bmatrix} a_1' \\ a_2' \end{bmatrix}$ can also be determined (at least within a constant factor, if we require they are normalized, this ambiguity is removed).

Then the general solution of (50) would be in form of:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \vec{\alpha}_1 e^{\lambda_1 t} + c_2 \vec{\alpha}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} e^{\lambda_2 t} \qquad (52)$$

Example 1: same problem above but worked with matrix method

$$\dot{x}_1 = -2x_1 + 2x_2$$
$$\dot{x}_2 = 2x_1 - 5x_2$$
$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

$A\vec{\alpha} = \lambda \vec{\alpha}$ is the eigenvalue-eigenvector equation

$(A-\lambda I)\vec{\alpha}=0 \qquad (53)$[185]

The eigenvalues are computed from

$|A-\lambda I|=0$

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix}=0 \rightarrow \lambda^2+7\lambda+6=0$$

This is exactly the characteristic equation for the 2nd ODE in substitution method before.

$\lambda_1=-1, \lambda_2=-6$

For $\lambda_1=-1$, eigenvector can be solved from (53):

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=0 \rightarrow \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=0$$

Let $a_1=2, a_2=1$, so $\vec{\alpha}_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is the eigenvector. (I do not normalize it to $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, because it is not necessary here, the constant here will be absorbed in the general solution constant c_1)

For $\lambda_2=-6$:

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=0 \rightarrow \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=0$$

$a_1=1, a_2=-2$, $\vec{\alpha}_2=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-t}+c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-6t}$$

This is exactly same as the substitution method. (except that the constant

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[185] Note A-λ is meaningless, since A is a matrix, and λ is a number. ***I*** is the identity matrix, it comes from the fact $\lambda\vec{\alpha}=\lambda I\vec{\alpha}$.

c_1 here is half of the constant there)

Example 2: In case of 'good' degeneracy in eigenvalues:

$$\dot{x}_1 = -2x_1 + x_2 + x_3$$
$$\dot{x}_2 = x_1 - 2x_2 + x_3$$
$$\dot{x}_3 = x_1 + x_2 - 2x_3$$

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = \lambda(\lambda+3)^2 = 0$$

$\lambda_1 = 0$, $\lambda_{2,3} = -3$ double degeneracy

For $\lambda_1 = 0$, through Gauss elimination:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1.5 & 1.5 \\ 0 & 1.5 & -1.5 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1.5 & 1.5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

The nontrivial one (the *a*'s are not all 0) is let free variable[186] $a_3 = 1$, then

$a_2 = a_1 = 1$

So $\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda_{2,3} = -3$:

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[186] This means this variable has a free choice of values.

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=0\rightarrow\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=0$$

There are two free variables a_2, a_3. Let's first choose $a_2 = 1, a_3 = 0$, then $a_1 = -1$; Next set $a_2 = 0, a_3 = 1$ (guarantee independence from the first choice), then $a_1 = -1$. So $\vec{\alpha}_2 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$; $\vec{\alpha}_3 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$. These two are independent vectors associated with $\lambda_{2,3} = -3$[187].

$\vec{x} = c_1\vec{\alpha}_1 e^{0t} + c_2\vec{\alpha}_2 e^{-3t} + c_3\vec{\alpha}_3 e^{-3t}$

This degenerate case is what's called 'good' degeneracy, because we still can find enough independent eigenvectors.

A hindsight: The 2nd order ODE we discussed before can be treated as 2 coupled 1st order equations. It is like the reversed process of substitution method in solving the two-coupled 1st order equation will result in one 2nd order ODE. For a 2nd order ODE (homogeneous for simplicity):

$a\ddot{x} + b\dot{x} + c = 0$ Previously we solve it with characteristic equations, now play a 'trick', by introducing a second dependent variable y:

$\dot{x} = y$

Now we see that the 2nd ODE is equivalent to couple 1st order ODEs:

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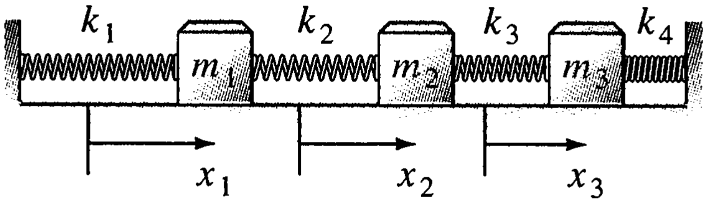
[187] They will automatically independent of the eigenvector associated with λ_1 from the theory of linear algebra. Actually it is easy for you to see that they are orthogonal to $\vec{\alpha}_1$, because the matrix here is symmetric (symmetric means with respect to the diagonal, or $A=A^T$.

$$\dot{y} = -\frac{b}{a}y - \frac{c}{a}x$$
$$\dot{x} = y$$

Solving this coupled equations will give answers to the 2nd order ODE, so Linear Algebra find its application in ODE.

2.3-2 2nd Order Couple Equations and Normal Modes

Besides coupled 1st order ODE, we also encounter many coupled 2nd order ODE, such as coupled oscillators in mechanics (though this part is not required for the course)[188], as the figure below shows:



In this section, I shall only introduce the simplest case in coupled 2nd order ODE, i.e. homogeneous (no external driving force) and no damping (no velocity dependent force). The inclusion of those factors will complicate problems and may render them insolvable analytically.

The equations for the above figure are:

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) = -(k_1 + k_2)x_1 + k_2x_2$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3(x_2 - x_3) = k_2x_1 - (k_2 + k_3)x_2 + k_3x_3$$
$$m_3\ddot{x}_3 = -k_3(x_3 - x_2) - k_4x_3 = k_3x_2 - (k_3 + k_4)x_3$$

Expressed above with matrix:

$$M\ddot{\vec{x}} = K\vec{x} \qquad (54)$$

188 If you are interested, please read chapter 5 in 'vibration and waves' by A.P. French.

$$\vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, M=\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, K=\begin{pmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{pmatrix}$$

Usually (54) is rewritten in the standard form:

$$\ddot{\vec{x}} = A\vec{x},\ \ A = M^{-1}K \qquad (55)$$

The method of solving this is undetermined coefficients we should be familiar with by now:

Guess the solution in form of:

$\vec{x} = \vec{\alpha} e^{\lambda t},\ \ \vec{\alpha} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ the coefficients need to be determined along with the λ. Put this trial solution into (55):

$$A\vec{\alpha} = \lambda^2 \vec{\alpha} \qquad (56)$$

This is just eigenvalue problem all over again, we can compute the eigenvalue and get the value of $\lambda = \pm\sqrt{\lambda_{1,2,\ldots}}$ and associated eigenvectors of λ_i. The general solutions would be:

$$\vec{x} = c_1\vec{\alpha}_1 e^{\sqrt{\lambda_1}t} + c_1'\vec{\alpha}_1 e^{-\sqrt{\lambda_1}t} + c_2\vec{\alpha}_2 e^{\sqrt{\lambda_2}t} + c_2'\vec{\alpha}_2 e^{-\sqrt{\lambda_2}t} + \ldots \qquad (56)$$

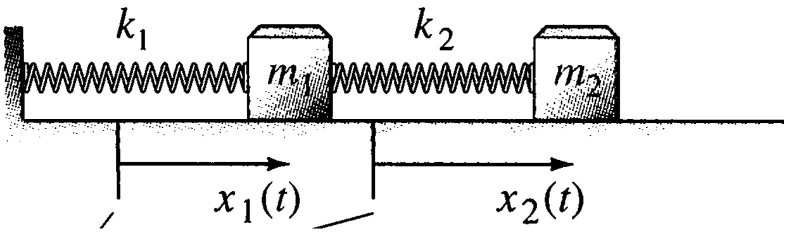
For the oscillation case, the eigenvalues will be generally a negative number:

$$\lambda_i = -\omega_i^2$$

$$c_1\vec{\alpha}_1 e^{\sqrt{\lambda_1}t} + c_1'\vec{\alpha}_1 e^{-\sqrt{\lambda_1}t} = \vec{\alpha}_1(\tilde{c}_1 e^{i\omega_1 t} + \tilde{c}_1' e^{-i\omega_1 t}) \xrightarrow{\substack{require \\ real}} \vec{\alpha}_1(a_1\cos\omega_1 t + b_1 \sin\omega_1 t)$$

$$\vec{x} = (a_1\cos\omega_1 t + b_1\sin\omega_1 t)\vec{\alpha}_1 + (a_2\cos\omega_2 t + b_2\sin\omega_2 t)\vec{\alpha}_2 + \ldots \qquad (57)$$

Let's workout an example:



Mass $m_1 = 2kg, m_2 = 1kg$, $k_1 = 100, k_2 = 50$ ISU. What is the motion of the system?

The equation of motion is:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solve for eigenvalues for A:

$$| A - \lambda I | = \begin{vmatrix} -75-\lambda & 25 \\ 50 & -50-\lambda \end{vmatrix} = 0 \rightarrow \lambda_1 = -25, \lambda_2 = -100$$

Or $-\omega^2 = \lambda \rightarrow \omega_1 = 5, \omega_2 = 10$

For $\lambda_1 = -25$:

$$\begin{pmatrix} -50 & 25 \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{\alpha}_1$$

For $\lambda_2 = -100$:

$$\begin{pmatrix} 25 & 25 \\ 50 & 50 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{\alpha}_2$$

The general solution would be:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a_1 \cos 5t + b_1 \sin 5t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (a_2 \cos 10t + b_2 \sin 10t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The constants can be determined from initial conditions.

There seem having two harmonic motions with frequency 5 and 10 (determined by the eigenvalues of A). This is no coincidence; these two are the frequencies of the **Normal Mode**, which I shall prove. The goal is to decouple the coupled equation, i.e. we need to find a combination of x_1, x_2, for example:

$$y_1 = ax_1 + bx_2; y_2 = cx_1 + dx_2 \qquad (58)$$

so that the differential equation of $\ddot{y}_1$ will only involve y_1; $\ddot{y}_2$'s will only involves y_2. Then the coupled 2nd order ODE will be reduced two decoupled ODE and can be solved individually. $y_1 = ax_1 + bx_2; y_2 = cx_1 + dx_2$ satisfy this is called normal mode of the system.

You may try to start form (58) and see what condition a, b, c, d would decouple the equations. But linear algebra already tells us the answer (i.e. what a, b, c, d should be). The goal is to make matrix A diagonal, it is not under x_1, x_2 bases. But under the basis of eigenvectors of the matrix, it will be a diagonal form: $A_{eigenbasis} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. The transform matrix is the matrix formed by eigenvectors[189]:

$$S = \begin{bmatrix} \vec{\alpha}_1 & \vec{\alpha}_2 \end{bmatrix} \qquad (59)$$

This means the first column of matrix S is the eigenvector associated with 1st eigenvalue, etc.

$$\vec{y} = S^{-1}\vec{x} \text{ or } \vec{x} = S\vec{y} \qquad (60)$$

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[189] This will require the matrix A is a matrix with independent columns (A in this problem certainly satisfies this, it actually comes from a symmetric matrix K) which almost all physical matrix satisfy..

This will give the combination of x_1, x_2 that decouple the equations (make A diagonal). Proof:

$\ddot{\vec{y}} = S^{-1}\ddot{\vec{x}} = S^{-1}A\vec{x} = S^{-1}AS\,\vec{y}$

$S^{-1}AS = \Lambda$ a *diagonal* matrix [190]

Then $\ddot{\vec{y}} = \Lambda\vec{y} \to \ddot{y}_1 = \lambda_1 y_1;\ \ddot{y}_2 = \lambda_2 y_2 \ldots$

Let' use the above example to illustrate this:

We have solved the eigenvectors:

$$S = [\vec{\alpha}_1 \quad \vec{\alpha}_2] = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad S^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

In this special example S, S^{-1} happens to have same form, but this is not general at all.

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = S^{-1}\vec{x} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The common factor 1/3 does not play any roles and can be neglected:

$$y_1 = x_1 + x_2$$
$$y_2 = 2x_1 - x_2$$

Now you differentiate these y's, it is straightforward to show that:

$$\ddot{y}_1 = -25y_1$$
$$\ddot{y}_2 = -100y_2$$

The above combination is also very clear from the general solution we solved for x_1, x_2, such combination will only leave one frequency component.

Though I only discussed a special class of systems (homogeneous, no

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[190] I used results from linear algebra here, where the matrix A can be written as: $A = S\Lambda S^{-1}$

damping), these systems are important in many applications. The above analysis is the basics for analyzing complicated coupled oscillations, such as molecular vibration. For instance, the benzene molecule (C6H6) has 12 atoms, and will have 3N-6=36-6=30 degree of freedom in vibration (the 6 degree are 3 translational motions of C.M., 3 rotational motions of the whole molecule, and what is left are vibrations). These corresponds to 30 decoupled normal modes of vibration, each is a combination of certain motions of carbon and hydrogen atoms. The procedure will be same as I illustrated above, just the dimension of matrix gets bigger.