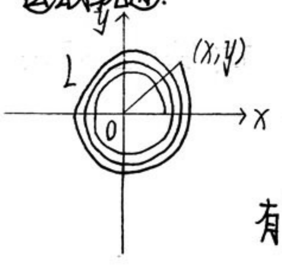
1.18 已知：光盘内径 $R_1=2.2cm=0.022m$，外径 $R_2=5.6cm=0.056m$，径向音轨密度 $N=650$ 条/mm $=650000$ 条/m，激光速度 $v=1.3m/s$.

求解：总时间 t_0 及 $r=5cm$ 时的角速度 ω 和角加速度 α.

解：(1) 设所用时间为 t 时，所走路程为 s，转过总弧度 θ，与圆心相距为 ρ.

以原点 O 作光盘圆心，以光盘为参考系，建立如图所式平面直角坐标系，曲线 L 表示激光的运动轨迹.



则曲线 L 的参数方程为：

$$\begin{cases} x=(R_1+\frac{\theta}{2\pi N})\cos\theta \\ y=(R_1+\frac{\theta}{2\pi N})\sin\theta \end{cases}$$

有

$$\begin{cases} \frac{dx}{d\theta}=[\frac{\cos\theta}{2\pi N}-(R_1+\frac{\theta}{2\pi N})\sin\theta] \\ \frac{dy}{d\theta}=[\frac{\sin\theta}{2\pi N}+(R_1+\frac{\theta}{2\pi N})\cos\theta] \end{cases}$$

则 $S=\int_0^\theta\sqrt{(dx)^2+(dy)^2}=\int_0^\theta\frac{1}{2\pi N}\sqrt{(\theta+2\pi NR_1)^2+1}\,d\theta$

$=\frac{1}{4\pi N}\left[(\theta+2\pi NR_1)\sqrt{(\theta+2\pi NR_1)^2+1}+\ln\left((\theta+2\pi NR_1)+\sqrt{(\theta+2\pi NR_1)^2+1}\right)\right]\Big|_0^\theta$

$=\frac{1}{4\pi N}\left[(\theta+2\pi NR_1)\sqrt{(\theta+2\pi NR_1)^2+1}+\ln(\theta+2\pi NR_1+\sqrt{(\theta+2\pi NR_1)^2+1})-2\pi NR_1\sqrt{(2\pi NR_1)^2+1}-\ln(2\pi NR_1+\sqrt{(2\pi NR_1)^2+1})\right]$

$=vt$

又 $\rho=R_1+\frac{\theta}{2\pi N}$，$\theta+2\pi NR_1=2\pi N\rho$

$\therefore S=\frac{1}{4\pi N}\left[2\pi N\rho\sqrt{(2\pi N\rho)^2+1}+\ln(2\pi N\rho+\sqrt{(2\pi N\rho)^2+1})-2\pi NR_1\sqrt{(2\pi NR_1)^2+1}-\ln(2\pi NR_1+\sqrt{(2\pi NR_1)^2+1})\right]=vt$

$\therefore t_0=\frac{1}{4\pi Nv}\left[2\pi NR_2\sqrt{(2\pi NR_2)^2+1}+\ln(2\pi NR_2+\sqrt{(2\pi NR_2)^2+1})-2\pi NR_1\sqrt{(2\pi NR_1)^2+1}-\ln(2\pi NR_1+\sqrt{(2\pi NR_1)^2+1})\right]$

$=\frac{1}{4\pi\times650000\times1.3}\left[2\pi\times650000\times0.056\times\sqrt{(2\pi\times650000\times0.056)^2+1}+\ln(2\pi\times650000\times0.056+\sqrt{(2\pi\times650000\times0.056)^2+1})\right.$
$\left.-2\pi\times650000\times0.022\times\sqrt{(2\pi\times650000\times0.022)^2+1}-\ln(2\pi\times650000\times0.022+\sqrt{(2\pi\times650000\times0.022)^2+1})\right]\text{s}$

$=4165.75\text{s}$

$=69.43\text{min}$

这应该是准确解

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(2) 由(1)，$S=vt$，两边同时对 t 求导得

$\frac{ds}{dt}=\frac{ds}{d\theta}\cdot\frac{d\theta}{dt}=\frac{ds}{d\theta}\cdot\omega=v=S'\cdot\omega=v$

$\therefore\omega=\frac{v}{S'}$

再对 t 求导则

$\frac{d\omega}{dt}=\alpha=-\frac{v}{(S')^2}\cdot\frac{ds'}{dt}=-\frac{vS''}{(S')^2}\cdot\frac{d\theta}{dt}=-\frac{vS''}{(S')^2}\cdot\frac{v}{S'}=-\frac{v^2S''}{(S')^3}$

$S'=\frac{1}{2\pi N}\sqrt{(\theta+2\pi NR_1)^2+1}=\sqrt{r^2+\frac{1}{4\pi^2N^2}}$

$S''=\frac{1}{2\pi N}\cdot\frac{\theta+2\pi NR_1}{\sqrt{(\theta+2\pi NR_1)^2+1}}=\frac{1}{2\pi N}\cdot\frac{r}{\sqrt{r^2+\frac{1}{4\pi^2N^2}}}$

考虑到 N 很大，可近似为

$\omega\approx\frac{v}{r}$

$\alpha\approx-\frac{v^2}{2\pi Nr^3}$

$\therefore\omega=\frac{v}{\sqrt{r^2+\frac{1}{4\pi^2N^2}}}=\frac{1.3}{\sqrt{0.05^2+\frac{1}{4\pi^2\times650000^2}}}\text{rad/s}=26\text{ rad/s}$ 就是"所谓标准答案"．

$\alpha=-\frac{1}{2\pi N}\cdot\frac{v^2r}{(r^2+\frac{1}{4\pi^2N^2})^2}=-\frac{1.3^2\times0.05}{2\pi\times650000\times(0.05^2+\frac{1}{4\pi^2\times650000^2})^2}\text{rad/s}^2=-3.31\times10^{-3}\text{rad/s}^2$