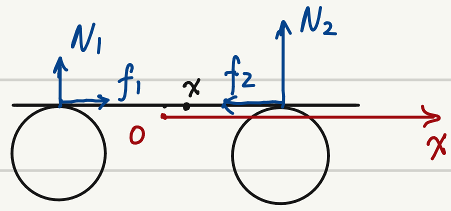
8-6

证明：(1).



由受力平衡与力矩平衡：

$$\begin{cases} N_1 + N_2 = mg \\ N_1(x+l) = N_2(l-x) \end{cases} \Rightarrow \begin{cases} N_1 = \frac{l-x}{2l}mg \\ N_2 = \frac{l+x}{2l}mg \end{cases}$$

水平方向上，$m\ddot{x} = -f_2 + f_1 = -\mu\frac{x}{l}mg$

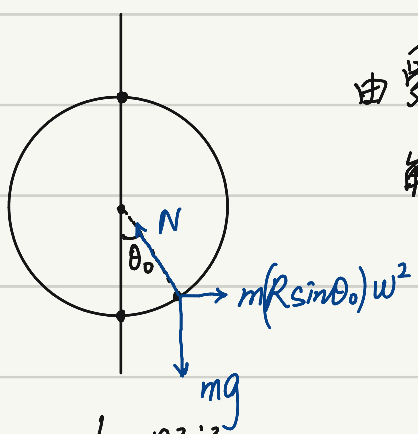
即 $\ddot{x} + \frac{\mu g}{l}x = 0$，作简谐运动，$\omega = \sqrt{\frac{\mu g}{l}}$

解：(2)、旋转相反时，$\ddot{x} - \frac{\mu g}{l}x = 0$，

杆的运动会因偏离于质心的微小扰动后会向一侧一直运动，直至脱落

8-8.

(1).



由受力平衡：$mR\sin\theta_0 w^2 = mg\tan\theta_0$

解得 $\theta_0 = 0, \pi$，或 $\cos^{-1}\frac{g}{Rw^2}$

(2). $E_k = \frac{1}{2}mR^2\dot{\theta}^2$

$$E_p = -\frac{1}{2}mR^2\sin^2(\theta_0+\Delta\theta)w^2 + mgR[1-\cos(\theta_0+\Delta\theta)]$$

$$= E_{p0} - \frac{1}{2}mR^2w^2\cos2\theta_0(\Delta\theta)^2 + \frac{1}{2}mgR\cos\theta_0(\Delta\theta)^2$$

[注意 Taylor 展开!!]

$$= E_{p0} + \frac{1}{2}mR(g\cos\theta_0 - Rw^2\cos2\theta_0)(\Delta\theta)^2$$

$$E = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\left(\frac{g}{R}\cos\theta_0 - w^2\cos2\theta_0\right)(\Delta\theta)^2$$

$\theta_0 = 0$ 时：若 $Rw^2 < g$，则 $w_0 = \sqrt{\frac{g}{R} - w^2}$

若 $Rw^2 \geq g$，则为不稳定平衡

$\theta_0 = \pi$ 时：不稳定平衡

$\theta_0 = \cos^{-1}\frac{g}{Rw^2}$ 时：$w_0 = w\sin\theta_0 = w\sqrt{1-\frac{g^2}{R^2w^4}}$

8-9.

(1)、$mg\cos\theta + k(x_0 - l_0) = m(x_0\sin\theta)w^2\sin\theta$

得 $x_0 = \frac{kl_0 - mg\cos\theta}{k - mw^2\sin^2\theta}$

(2)、设新平衡位置为 x_0'，偏移平衡位置位移为 x.

则有 $m\ddot{x} = -kx + (m(2w)^2\sin^2\theta)x$

$\Rightarrow w_0 = \sqrt{\frac{k}{m} - 4w^2\sin^2\theta}$

$$t = \frac{\pi}{w_0} = \frac{\pi}{\sqrt{\frac{k}{m} - 4w^2\sin^2\theta}}$$

8-10

解：(1)、$m_2 g = m_1 \frac{v_0^2}{l_0} \Rightarrow l_0 = \frac{m_1 v_0^2}{m_2 g}$

(2)、记 $h = l_0 v_0 = \frac{m_1 v_0^3}{m_2 g}$

则有：$-m_2 g = (m_1+m_2)\ddot{r} - m_1 \frac{h^2}{r^3}$

取 $r = l_0 + \Delta r$ 得：

$$-m_2 g = (m_1+m_2)\ddot{r} - m_2 g\left(1 - \frac{3\Delta r}{l_0}\right)$$

即 $(m_1+m_2)\ddot{r} + \frac{3m_2^2 g^2}{m_1 v_0^2}\Delta r = 0$

角频率 $\omega_0 = \sqrt{\frac{3m_2^2 g^2}{m_1(m_1+m_2)v_0^2}}$

8-13

解：(1)、$J_1 = \frac{1}{3}ml^2 + \frac{1}{2}m_0R^2 + m_0 l^2$

$J_1\ddot{\theta} \approx -(\frac{1}{2}mgl + m_0 l)\theta \Rightarrow \ddot{\theta} + \frac{(\frac{1}{2}m+m_0)gl}{(\frac{1}{3}m+m_0)l^2 + \frac{1}{2}m_0R^2}\theta = 0$

$\Rightarrow T_1 = 2\pi\sqrt{\frac{(2m+6m_0)l^2+3m_0R^2}{(3m+6m_0)gl}}$，$l_1' = \frac{(2m+6m_0)l^2+3m_0}{(3m+6m_0)l}$

(2)、圆盘可自由旋转时，其自转与系统无关，

$J_2 = \frac{1}{3}ml^2 + m_0 l^2$，其他不变。

故有：

$$T_2 = 2\pi\sqrt{\frac{(2m+6m_0)l}{(3m+6m_0)g}}, \quad l_2' = \frac{2m+6m_0}{3m+6m_0}l$$

8-16.

解：$J = \frac{1}{3}ml^2 + \frac{1}{3}(2m)(2l)^2 = 3ml^2$

$E = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}k(l\theta)^2 + \frac{1}{2}(2m)gl\theta^2$

$\Rightarrow \omega = \sqrt{\frac{kl+2mg}{3ml}}$，$T = 2\pi\sqrt{\frac{3ml}{kl+2mg}}$

8-17.

解：(1)、设管截面积为 S.

$(\rho_1 SL)gL = \int_0^L \rho_0 Sxg\,dx = \frac{1}{2}\rho_0 SgL^2$

$\Rightarrow \rho_1 = \frac{1}{2}\rho_0$

(2)、设偏离平衡位置 x.

$(\rho_1 SL)\ddot{x} = -(\rho_0 Sx)g \Rightarrow \ddot{x} + \frac{2g}{L}x = 0$

故做简谐运动，振幅 $\frac{1}{2}L$，周期 $T = 2\pi\sqrt{\frac{L}{2g}}$

(3)、取 $\rho_1' = \frac{4}{3}\rho_1 = \frac{2}{3}\rho_0$，平衡位置距水面 $\frac{2}{3}L$（即振幅），$\omega = \sqrt{\frac{3g}{2L}}$

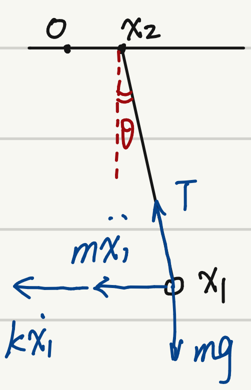
简谐运动段：$t_1 = \frac{1}{\omega}\left(\frac{\pi}{2} + \sin^{-1}\left(\frac{1}{3}/\frac{2}{3}\right)\right) = \frac{2}{3}\pi\sqrt{\frac{2L}{3g}}$

匀减速运动段：$a = \frac{\rho_0 SL - \rho_1' SL}{\rho_1' SL}g = \frac{1}{2}g \Rightarrow t_2 = \frac{\frac{2}{3}\omega L\cos\frac{\pi}{6}}{a} = \sqrt{\frac{2L}{g}}$

$\Rightarrow t = (\sqrt{3} + \frac{2}{3}\pi)\sqrt{\frac{2L}{3g}}$

8-27

解：(1)、空气阻力满足：$f=-kv$.



在悬点系中，加一惯性力.

$$ml^2\ddot{\theta}=-(m\ddot{x}_2+k\dot{x}_1)l-mgl\theta$$

代入 $\theta\approx\frac{x_1-x_2}{l}$ 得：

$$\ddot{x}_1+\frac{k}{m}\dot{x}_1+\frac{g}{l}x_1=\frac{g}{l}x_2$$

取 $\delta=\frac{2k}{m}$，即有 $\ddot{x}_1+2\delta\dot{x}_1+\omega_0^2x_1=\omega_0^2x_2$

(2)、由条件知：$e^{-50\delta T}=\frac{1}{e}\Rightarrow\delta=\frac{1}{50T}=\frac{\omega_0}{100\pi}$，其中 $\omega_0=\sqrt{\frac{g}{l}}$

代入 $\tilde{x}_1=\tilde{A}_1e^{i\omega t}$，得：

$$-\omega^2\tilde{A}_1+2i\delta\omega\tilde{A}_1+\omega_0^2\tilde{A}_1=\omega_0^2A$$

$$\Rightarrow\tilde{A}_1=\frac{\omega_0^2A}{(\omega_0^2-\omega^2)+2i\delta\omega}$$

共振时，$\omega=\omega_0$，于是 $|\tilde{A}|=\frac{\omega_0}{2\delta}A=50\pi A=0.157\,m$

(3)、取 $|\tilde{A}_1|=\frac{\omega_0}{4\delta}A=25\pi A$ 得：$\omega=0.994\omega_0$ 或 $1.005\omega_0$

代入 $\omega_0=\sqrt{\frac{g}{l}}$，得 $\omega=3.113\ rad\cdot s^{-1}$ 或 $3.148\ rad\cdot s^{-1}$

8-32.

解：(1)、由图可知：固有频率 $\omega_0\approx40\ s^{-1}$

锐度 $S=\frac{\omega_0}{BW}=\frac{40}{\frac{5}{2}}=16$

进而 $Q=S=16$

(2)、$Q=\frac{\omega_0}{2\delta}\Rightarrow\delta=\frac{\omega_0}{2Q}=\frac{5}{4}s^{-1}$

能量衰减 $\propto e^{-2\delta t}$

故 $t=\frac{5}{2\delta}=2s$，周期数 $N=\frac{\omega_0T}{2\pi}\approx12.73$，即经过13周

8-33.

解：(1)、质点1：$m\ddot{x}_1=-kx_1+k(x_2-x_1)+b(\dot{x}_2-\dot{x}_1)$

质点2：$m\ddot{x}_2=-kx_2-k(x_2-x_1)-b(\dot{x}_2-\dot{x}_1)$

$$\xRightarrow{\text{整理}}\begin{cases}m\ddot{x}_1=-2kx_1+kx_2+b(\dot{x}_2-\dot{x}_1) & ①\\ m\ddot{x}_2=-2kx_2+kx_1-b(\dot{x}_2-\dot{x}_1) & ②\end{cases}$$

(2)、$y_1=x_1+x_2$，$y_2=x_1-x_2$

①+② $\Rightarrow m\ddot{y}_1+ky_1=0$

①−② $\Rightarrow m\ddot{y}_2+2b\dot{y}_2+3ky_2=0$

两式相互独立，可以求解，通解为

$$\begin{cases}y_1=A_1\cos\sqrt{\frac{k}{m}}t+A_2\sin\sqrt{\frac{k}{m}}t\\ y_2=e^{-\frac{b}{m}t}(B_1\cos\omega_1t+B_2\sin\omega_1t)\end{cases}$$

其中 $\omega_1=\sqrt{\frac{3k}{m}-\frac{b^2}{m^2}}$

(3)、初始条件：$y_1(0)=y_2(0)=0$，$\dot{y}_1(0)=\dot{y}_2(0)=v_0$

得 $A_1=0$，$A_2=v_0\sqrt{\frac{m}{k}}$，$B_1=0$，$B_2=\frac{v_0}{\omega_1}$，

经过足够长时间后，$y_2=x_1-x_2=0$，故 $x_1=x_2=\frac{1}{2}y_1=\frac{1}{2}v_0\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t$

满足 $x_1=x_2=\frac{v_0}{2\omega}\sin\omega t$ 形式，其中 $\omega=\sqrt{\frac{k}{m}}$

8-35

解：$\omega_1=\sqrt{\frac{k_1}{m_1}}$，$\omega_2=\sqrt{\frac{k_2}{m_2}}$ 代入数据得：

$\omega_1=\omega_2=1\ rad\cdot s^{-1}=\omega$

令 $A_0=10cm$，则振子运动方程：$x_1=A_0\cos\omega t$

设 $x_2=A_0\cos(\omega t+\varphi)$

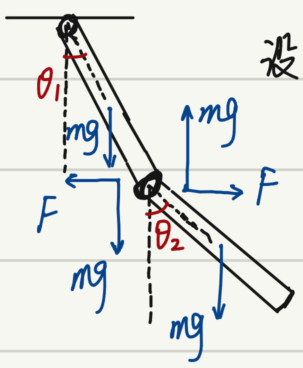
则 $x_1-x_2=2A_0\sin\frac{\varphi}{2}\sin(\omega t+\frac{\varphi}{2})$

取 $2A_0\sin\frac{\varphi}{2}=\pm 5cm=\pm\frac{1}{2}A_0$，

得 $\sin\frac{\varphi}{2}=\pm\frac{1}{4}$，$\varphi=2k\pi\pm 2\sin^{-1}\frac{1}{4}$，$k=0,1,2,\cdots$

$t=\frac{\varphi}{\omega}$，故 $t=(6.28k\pm 0.51)s$，$k=0,1,2,\cdots$

8-40.



设连接处有图示作用力 F，竖直力近似为下杆重力 mg

上杆（转动定律）：$\frac{1}{3}ml^2\ddot{\theta}_1=-\frac{1}{2}mgl\theta_1-Fl-mgl\theta_1$ ①

下杆（动力学方程）：$m(\frac{1}{2}l\ddot{\theta}_2+l\ddot{\theta}_1)=+F$ ②

下杆（转动定律）：$\frac{1}{12}ml^2\ddot{\theta}_2=-\frac{1}{2}Fl-\frac{1}{2}mgl\theta_2$ ③

联立①②③消掉 F，并整理：

$$\begin{cases}2\ddot{\theta}_1-\ddot{\theta}_2+\frac{9g}{l}\theta_1-\frac{6g}{l}\theta_2=0 & ①'\\ 6\ddot{\theta}_1+4\ddot{\theta}_2+\frac{6g}{l}\theta_2=0 & ②'\end{cases}$$

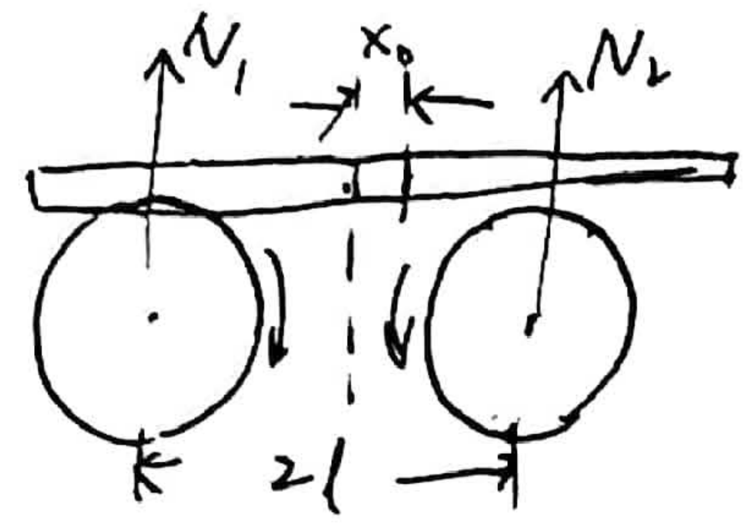
代入 $\theta_1=\tilde{A}_1e^{i\omega t}$，$\theta_2=\tilde{A}_2e^{i\omega t}$，并令 $\omega_0^2=\frac{g}{l}$，得：

$$\begin{cases}(9\omega_0^2-2\omega^2)\tilde{A}_1+(\omega^2-6\omega_0^2)\tilde{A}_2=0\\ -6\omega^2\tilde{A}_1+(6\omega_0^2-4\omega^2)\tilde{A}_2=0\end{cases}$$

使解非平凡，则使 $\begin{vmatrix}9\omega_0^2-2\omega^2 & \omega^2-6\omega_0^2\\ -6\omega^2 & 6\omega_0^2-4\omega^2\end{vmatrix}=0 \Rightarrow \omega^4-6\omega_0^2\omega^2+\frac{27}{7}\omega_0^4=0$

解得 $\omega=\sqrt{(3\pm\frac{6}{\sqrt{7}})\frac{g}{l}}$

8-6.



(1) 对过质心且垂直于纸面的轴，有:

$$N_1(l+x_0)=N_2(l-x_0) \qquad (1)$$

杆子在竖直方向并无加速度，有:

$$N_1+N_2=mg \qquad (2)$$

设向右的力为正，向右建立坐标 x，以中点为原点

$$m\ddot{x}=\mu N_1-\mu N_2 \qquad (3)$$

联立，得:

$$\ddot{x}+\frac{\mu g}{l}x=0$$

此为一简谐振动，振动角频率 $\omega_0=\sqrt{\frac{\mu g}{l}}$

(2). 若轮沿反方向转动 有:

$$\ddot{x}=\frac{\mu g x}{l}$$

由此给出:

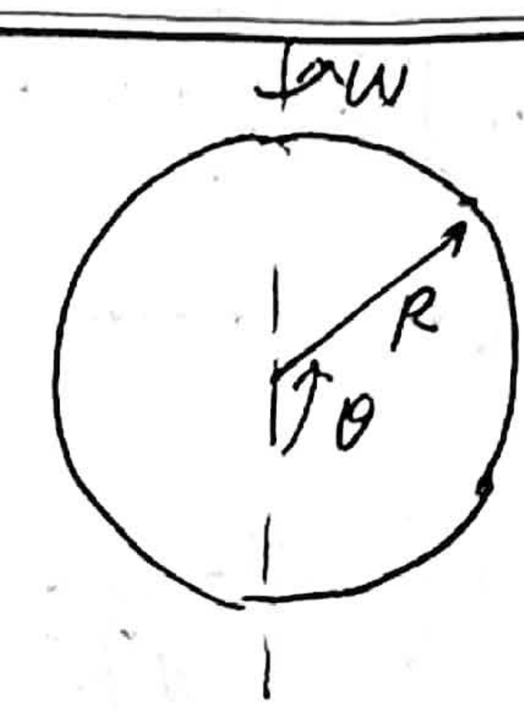
$$x=C_1 e^{\sqrt{\frac{\mu g}{l}}t}+C_2 e^{-\sqrt{\frac{\mu g}{l}}t}$$

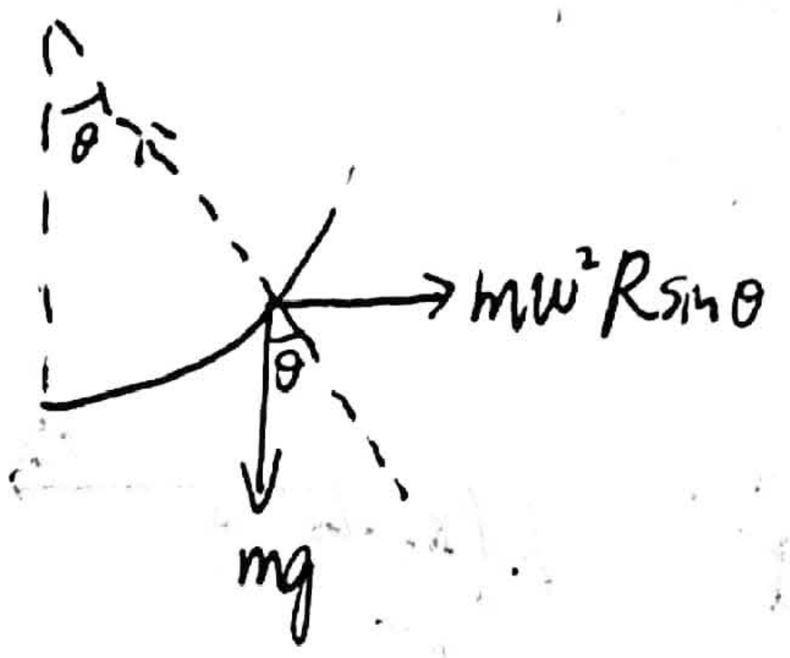
有 $x|_{t=0}=x_0$

$\dot{x}|_{t=0}=0$

$$\therefore x=\frac{1}{2}x_0\left(e^{\sqrt{\frac{\mu g}{l}}t}+e^{-\sqrt{\frac{\mu g}{l}}t}\right)$$

8-8.





(1) 在圆环的转动参考系中: 有

$$mg\tan\theta=m\omega^2 R\sin\theta$$

有: $\sin\theta=0$ or $\cos\theta=\frac{g}{\omega^2 R}$

即 $\theta=0,\pi$, or $\arccos\frac{g}{\omega^2 R}$ (if $g<\omega^2 R$)

(2). 在转动系中，考虑质点的势能

$$E_p=-mg\cos\theta\cdot R-\frac{1}{2}m\omega^2(R\sin\theta)^2$$

质点动能为

$$E_k=\frac{1}{2}m\cdot(\dot{\theta}R)^2$$

$E=E_k+E_p$ 有: $\frac{dE}{dt}=0$

得 $\ddot{\theta}=\sin\theta\left(\omega^2\cos\theta-\frac{g}{R}\right)$

$\theta=0$ 时，if $\omega^2<\frac{g}{R}$，则

$$\ddot{\theta}\approx-\theta\left(\frac{g}{R}-\omega^2\right)$$

角频率 $\omega_1=\sqrt{\frac{g}{R}-\omega^2}$

$\theta = \pi$ 为不稳定平衡点，不用

$\theta_0 = \arccos\frac{g}{\omega^2 R}$ 时. if when $\omega^2 R > g$

将 $\ddot{\theta}$ 在 θ_0 处 taylor 展开

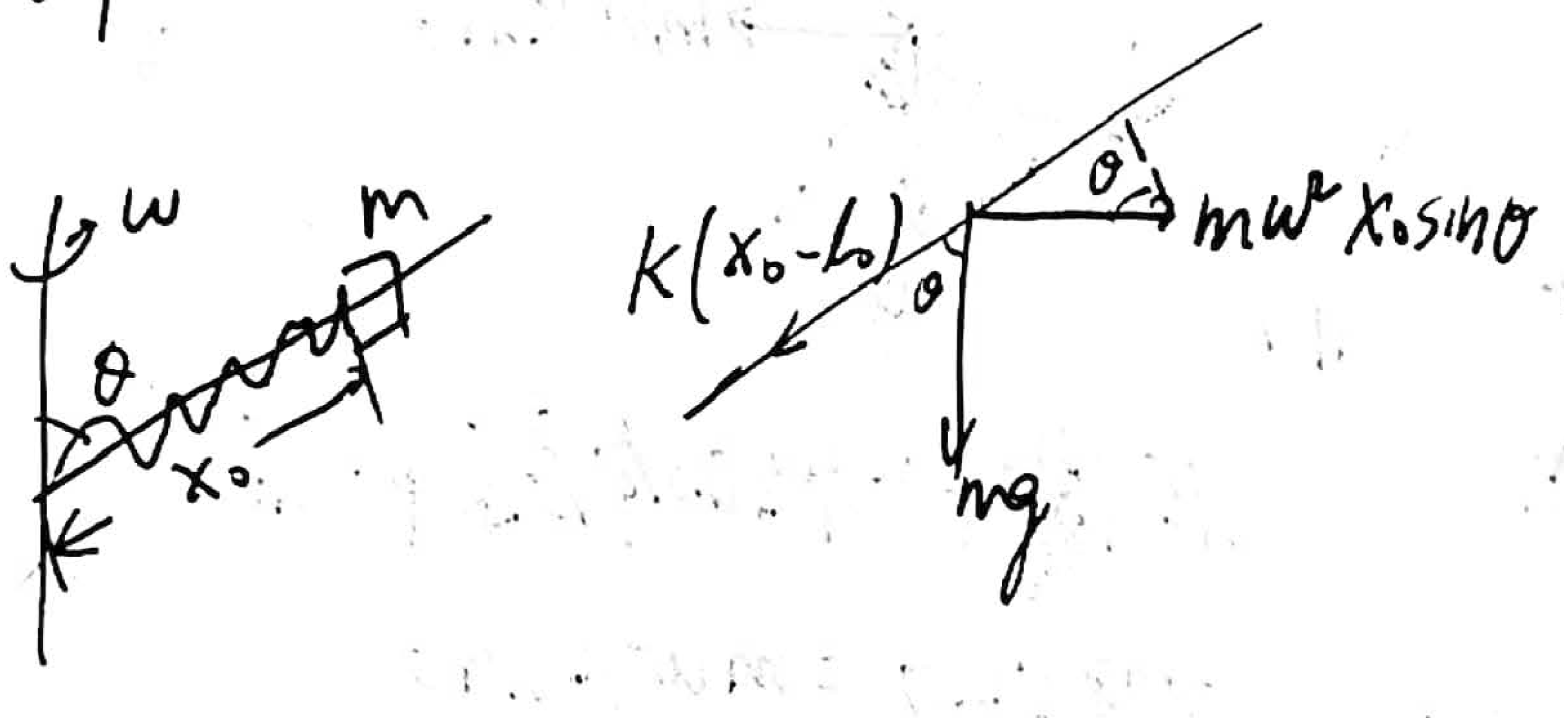
$$\ddot{\theta} = \omega^2\cos 2\theta_0(\theta-\theta_0) - \frac{g}{R}\cos\theta_0\cdot(\theta-\theta_0)$$

$$= (\omega^2\cos 2\theta_0 - \frac{g}{R}\cos\theta_0)(\theta-\theta_0)$$

$$\ddot{\theta} = -\frac{g^2}{\omega^2 R^2}(\theta-\theta_0)\cdot\left(\omega^2 - \frac{g^2}{\omega^2 R^2}\right)$$

$$\omega_2 = \sqrt{-\frac{g^2}{\omega^2 R^2} + \omega^2}$$

8-9.



(1). $(m\omega^2 x_0\sin\theta)\cdot\sin\theta = k(x_0-l_0) + mg\cos\theta$

$$x_0 = \frac{mg\cos\theta - kl_0}{m\omega^2\sin^2\theta - k}$$

(2). 设 m 坐标为 x.

有: $m\ddot{x} = -k(x-l_0) - mg\cos\theta + m[(2\omega)^2 x\sin\theta]\cdot\sin\theta$

$$\ddot{x} = \frac{kl_0 - mg\cos\theta}{m} - \left(\frac{k}{m} - 4\omega^2\sin^2\theta\right)\cdot x$$

if $\frac{k}{m} > 4\omega^2\sin^2\theta$, 则 m 将作简谐振动

$$\omega_0 = \sqrt{\frac{k}{m} - 4\omega^2\sin^2\theta}$$

$$t = \frac{\pi}{\omega_0} = \pi\cdot\frac{1}{\sqrt{\frac{k}{m} - 4\omega^2\sin^2\theta}}$$

8-10.

(1). $m_1\frac{v_0^2}{l_0}=m_2g$

$\therefore l_0=\frac{m_1v_0^2}{m_2g}$

(2). 写出径向动力学方程:

$m_1\dot{\theta}^2r-T=m_1\ddot{r}$

对m_2, $T-m_2g=m_2\ddot{r}$

给出角动量守恒式

$L=m_1v_0l_0=m_1r^2\dot{\theta}$

$\ddot{r}=\frac{v_0^2l_0^2}{r^3}-\frac{m_2}{m_1}g$

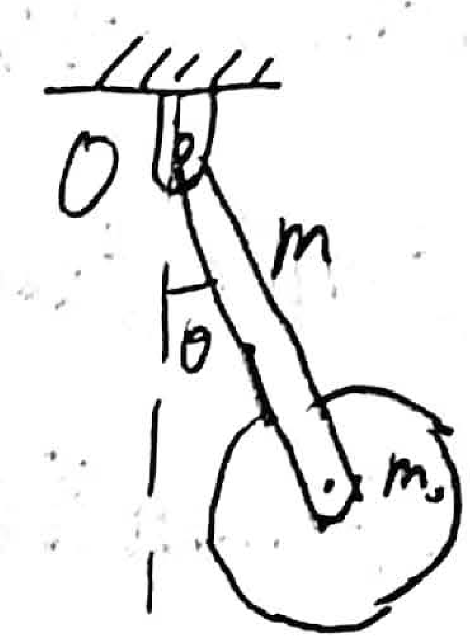
在 $r=l_0$ 处作 taylor展开

$\ddot{r}=-\frac{3v_0^2}{l_0^2}(r-l_0)-\frac{m_2}{m_1}g+\frac{v_0^2}{l_0}$

$=-\frac{3v_0^2}{m_1^2v_0^4}m_2^2g^2(r-l_0)=-\frac{3m_2^2g^2}{m_1^2v_0^2}(r-l_0)$

$\therefore \omega_0=\sqrt{\frac{3m_2^2g^2}{m_1^2v_0^2}}=\sqrt{3}\frac{m_2g}{m_1v_0}$

8-13



(1). 圆盘绕O转动惯量为

$I_1=\frac{1}{2}m_0R^2+m_0l^2$

杆绕O转动惯量

$I_2=\frac{1}{3}ml^2$

由转动定律:

$(I_1+I_2)\ddot{\theta}=-m_0gl\sin\theta-mg(\frac{1}{2}l\sin\theta)$

在θ很小时,有 $\sin\theta\approx\theta$

即: $\ddot{\theta}=-\frac{m_0+\frac{1}{2}m}{\frac{1}{2}m_0R^2+(m_0+\frac{1}{3}m)l^2}\cdot gl\theta$

$T=2\pi\sqrt{\frac{\frac{1}{2}m_0R^2+(m_0+\frac{1}{3}m)l^2}{m_0+\frac{1}{2}m}\cdot\frac{1}{gl}}$

$L=\frac{\frac{1}{2}m_0R^2+(m_0+\frac{1}{3}m)l^2}{(m_0+\frac{1}{2}m)l}$

(2). 轴光滑,相当于圆盘绕O转动惯量 $I'=m_0l^2$

同样地,可得 $(I'+I_2)\ddot{\theta}=-m_0gl\sin\theta-mg(\frac{1}{2}l\sin\theta)$

得: $\ddot{\theta}=-\frac{m_0+\frac{1}{2}m}{m_0+\frac{1}{3}m}\frac{g}{l}\theta$ 即 $T=2\pi\sqrt{\frac{l}{g}\frac{m_0+\frac{1}{3}m}{m_0+\frac{1}{2}m}}$

$L=l\cdot\frac{m_0+\frac{1}{3}m}{m_0+\frac{1}{2}m}$

8-16.

两根杆绕 O 的转动惯量：

$$I=\frac{1}{3}ml^2+\frac{1}{3}(2m)(2l)^2=3ml^2$$

在小角度下 系统受到的对 O 总力矩：

$$M=-2mg(l\sin\theta)+mg\frac{1}{2}l(1-\cos\theta)-kl\cos\theta\sin\theta$$

$$\approx-2mgl\theta-kl\cdot\theta$$

$$M=I\ddot{\theta}\Rightarrow\cancel{\ddot{\theta}=-\frac{2}{3}\frac{g}{l}\theta}\quad\ddot{\theta}=-\frac{2mg+kl}{3ml}\theta$$

$$T=2\pi\cancel{\sqrt{\frac{3l}{2g}}}\sqrt{\frac{3ml}{2mg+kl}}$$

8-17.

(1). 设杆进入水的长度为 x

$F_{浮}=-\rho_0 g\cdot(Sx)$ ……（$x\le L$ 时）

有：$m\ddot{x}=-\rho_0 g(Sx)+mg$

$m=\rho SL$

$$\ddot{x}=g-\frac{\rho_0}{\rho}\frac{g}{L}x=\frac{\rho_0}{\rho}\frac{g}{L}\left(x-\frac{\rho}{\rho_0}L\right)$$

可得振幅 $A=\frac{\rho}{\rho_0}L$. 有：$L=2A$

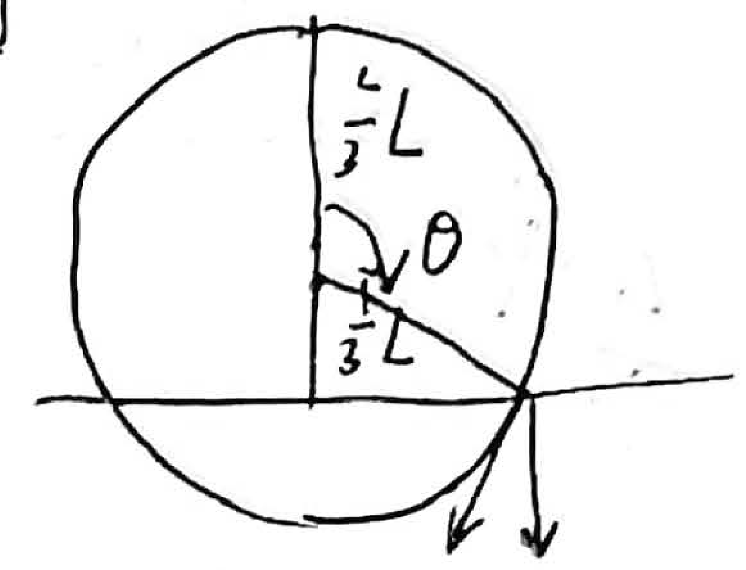
即 $\rho=\frac{1}{2}\rho_0$. ~~A=~~

(2). $A=\frac{1}{2}L$. $T=2\pi\sqrt{\frac{\rho L}{\rho_0 g}}=2\pi\sqrt{\frac{2L}{g}}$

(3). if $\rho=\frac{4}{3}\rho_1=\frac{2}{3}\rho_0$

有：$A=\frac{2}{3}L$，$w=\sqrt{\frac{\rho_0}{\rho}\frac{g}{L}}=\sqrt{\frac{3}{2}\frac{g}{L}}$

画出简谐图



可得：此时对应转过相位

$\theta=\frac{2}{3}\pi$，此段时间 $t_1=\frac{\theta}{w}$

此时运动速度

$$V=w\cdot A\cdot\cos30^\circ=\sqrt{\frac{g}{2L}}L=\sqrt{\frac{1}{2}gL}$$

而后管受恒力运动到底端

$$m\ddot{x}=mg-\rho_0 gSL$$

$$\ddot{x}=g-\frac{\rho_0}{\rho}g=-\frac{1}{2}g$$

此段时间 $t_2=\frac{V}{\frac{1}{2}g}=\sqrt{\frac{2L}{g}}$

$$t=t_1+t_2=\sqrt{\frac{2L}{g}}+\frac{2\pi}{3}\sqrt{\frac{2L}{3g}}$$

8-27.

(1). 在小角摆动下: 有

$$T = mg$$

$$m\ddot{x}_1 = -T\cancel{\theta}\sin\theta - C\dot{x}_1$$

有: $\sin\theta = \frac{x_1 - x_2}{l}$

代入可化简得:

$$\ddot{x}_1 + 2\delta\dot{x}_1 + \frac{g}{l}x_1 = g\frac{x_2}{l}$$

代令 $x_1 = B\cos(\omega t - \varphi)$

解得 $B = \dfrac{\frac{g}{l}A}{\sqrt{4\delta^2\omega^2 + (\frac{g}{l} - \omega^2)^2}}$

$$\tan\varphi = \frac{\cancel{2\delta}\ 2\delta\omega}{\frac{g}{l} - \omega^2}$$

(2). $\cancel{A = e^{-\delta t}}$

通过数给条件可求得 δ

单摆阻尼运动时,

$$x = x_0 e^{-\delta t}\cdot\cos(\omega_f t + \varphi_1)$$

可知: $e^{-\frac{2\pi}{\omega_f}\delta\cdot 50} = e^{-1}$

即: $\delta = \frac{\omega_f}{100\pi}$

即 $\cancel{\omega}\ \delta \ll \omega_f \approx \omega_0 = \sqrt{\frac{g}{l}}$

共振频率取 $\omega = \omega_0$

有: $B = \dfrac{\frac{g}{l}A}{2\delta\omega_0} = \dfrac{\omega_0}{2\delta}A = 50\pi\cdot A$

$\therefore B = 15.7\,cm$

(3). 取 $(\frac{g}{l} - \omega^2)^2 = \cancel{4\delta^2}\ 12\delta^2\omega_0^2$

得 $\omega_1 = \sqrt{\frac{g}{l} + 2\sqrt{3}\delta\omega_0}$

$\omega_2 = \sqrt{\frac{g}{l} - 2\sqrt{3}\delta\omega_0}$

$\omega_1 = 3.148\ rad/s$

$\omega_2 = 3.113\ rad/s$

$-\omega_2 + \omega_1 = 0.035\ rad/s$

8-32.

由 $m\ddot{x} = -C\dot{x} - kx + F_0\sin\omega t$

$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\sin\omega t$ ，$\omega_0^2 = \frac{k}{m}$

令 $x = B\sin(\omega t - \varphi)$ $\quad B = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}}$，$\tan\varphi = \frac{2\delta\omega}{\omega_0^2-\omega^2}$

驱动力做功功率 $P_F = F\cdot\dot{x} = F_0\,\omega B\cos\omega t\sin(\omega t-\varphi)$

$$\int_0^{\frac{2\pi}{\omega}} P_F\,dt = +\int_0^{\frac{2\pi}{\omega}} F_0\omega B\cos^2\omega t\cdot\sin\varphi\cdot dt = \frac{\pi}{\omega}F_0\omega B\sin\varphi$$

得平均功率

$$\bar{P} = \frac{\int_0^{\frac{2\pi}{\omega}} P_F dt}{\frac{2\pi}{\omega}} = \frac{F_0^2}{m}\cdot\frac{\delta\omega^2}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}$$

$\bar{P} = \frac{F_0^2}{m}\cdot\frac{\delta}{(\frac{\omega_0^2}{\omega}-\omega)^2+4\delta^2}$ ，当 $\omega=\omega_0$ 时 $\bar{P}$ 取到最大值 $\bar{P}_{max}$

$\omega_0 = 40\,rad/s$

$(\frac{\omega_0^2}{\omega}-\omega)^2 = 4\delta^2$ 时，$\bar{P} = \frac{1}{2}\bar{P}_{max}$

化简得到 $\omega_2-\omega_1 = 2\delta$，（$\omega_1,\omega_2$ 为方程的两个根）

从图中读得 $\omega_2-\omega_1 = 2.6\,rad/s$

$\because \frac{\omega_2-\omega_1}{\omega_0} \ll 1$

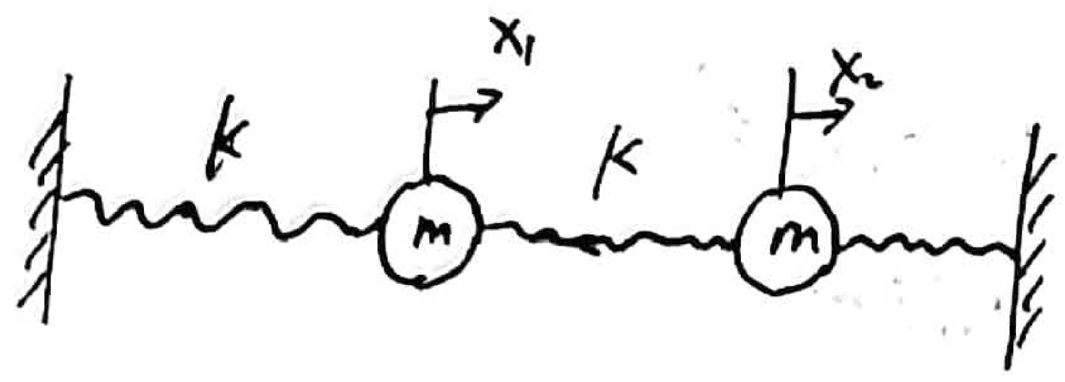
$Q \approx \frac{\omega_0}{2\delta} = \frac{\omega_0}{\omega_2-\omega_1} = 15.4$

(2).

$E = E_0 e^{-2\delta t}$ $\qquad t = \frac{5}{2}\frac{1}{\delta}$ 时 $\frac{E}{E_0} = e^{-5}$

经过 $\frac{t}{\frac{2\pi}{\omega_0}} = \frac{5}{2}\cdot\frac{1}{\delta}\cdot\frac{\omega_0}{2\pi} = \frac{12.2}{\pi}$ 周

8-33.



(1) $m\ddot{x}_1 = -kx_1 + k(x_2-x_1) - b(\dot{x}_1-\dot{x}_2)$ ...(1)

$m\ddot{x}_2 = -kx_2 - k(x_2-x_1) + b(\dot{x}_1-\dot{x}_2)$ (2)

(2).

(1)+(2)式

$m(\ddot{x}_1+\ddot{x}_2) = -k(x_1+x_2)$

即: $m\ddot{y}_1 = -ky_1$ ...... (3)

(1)-(2)式:

$m(\ddot{x}_1-\ddot{x}_2) = -k(x_1-x_2)+2k(x_2-x_1)-2b(\dot{x}_1-\dot{x}_2)$

$m\ddot{y}_2 = -3ky_2 - 2b\dot{y}_2$ ---- (4)

这是两个二阶齐次线性方程，可以求解

(3). 可以给出: $y_1 = A_1\cos(\omega_1 t+\varphi_1)$

$y_2 = A_2 e^{-\delta t}\cos(\omega_2 t+\varphi_2)$, $\omega_1=\sqrt{\frac{k}{m}}$

~~其中: y_2为一随时间的衰减项，经长时间后~~

~~$y_2=0$ 即 $x_1-x_2=0$，$\dot{x}_1-\dot{x}_2=0$~~ ~~$x_1-x_2=0$~~

~~x_1+x_2~~

~~$x_1=\frac{1}{2}($~~

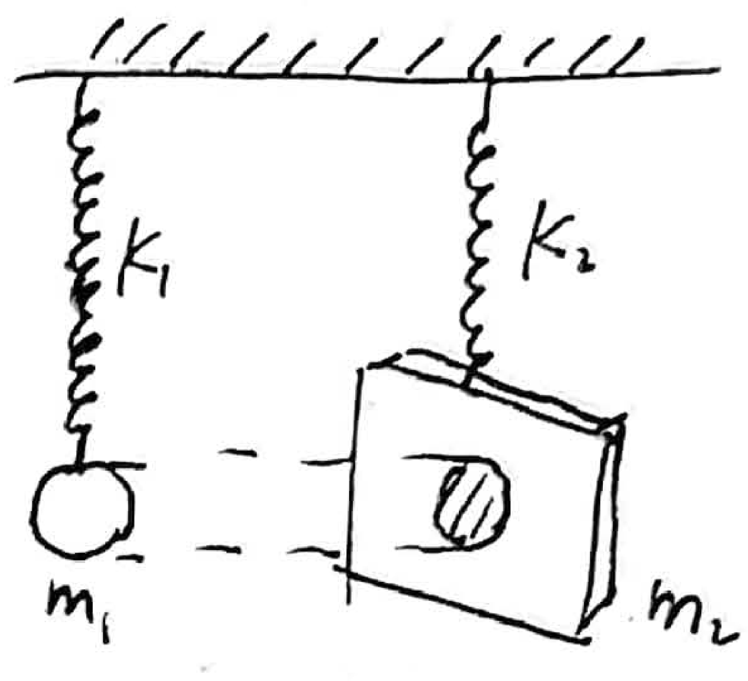
y_2经长时间后衰减为0，即 $\begin{cases} \dot{x}_1-\dot{x}_2=0 \\ x_1-x_2=0 \end{cases}$

而 $t=0$ 时　$y_1=0$　$\dot{y}_1 =$ ~~$-\omega_1\sin(\omega_1 t+\varphi_1)$~~ v_0

可得: $y = \frac{v_0}{\omega}\sin\omega t$

$\therefore$ 长时间后 $x_1 = x_2 = \frac{v_0}{2\omega}\sin\omega t$

8-35.



若同时释放，给出 m_1, m_2 的运动方程

记 m_1, m_2 相对其平衡位置 x_1, x_2

$-kx_1 = m_1\ddot{x}_1$ ~~得 $x_1=$~~

$-k_2x_2 = m_2\ddot{x}_2$

~~并~~利用 $\begin{cases} \dot{x}_1|_{t=0}=0 \\ x_1|_{t=0}=A_0 \end{cases}$ $\begin{cases} \dot{x}_2|_{t=0}=0 \\ x_2|_{t=0}=A_0 \end{cases}$

得 $x_1 = A_0\cos\omega t$

$x_2 = A_0\cos\omega t$

$\omega=\sqrt{\frac{k_1}{m_1}}=\sqrt{\frac{k_2}{m_2}}=1$ rad/s

设 m_2 晚 t_0 秒释放

则 $x_2 = A_0\cos[\omega(t-t_0)]$

影子在屏上运动

$x = x_1-x_2 = A_0[\cos\omega t - \cos[\omega(t-t_0)]]$

$= -2A_0\sin\frac{2\omega t-\omega t_0}{2}\sin(\frac{t_0}{2}\omega)$

$|2A_0\sin\frac{t_0}{2}| = \frac{1}{2}A_0$

~~$t_0=(0.505+6.283n)$~~ s　or

or ~~$t_0=(0.505+6.283n)$~~ s

~~3.14~~

6.283

得 $t_0=-0.505+6.283n$ (s)

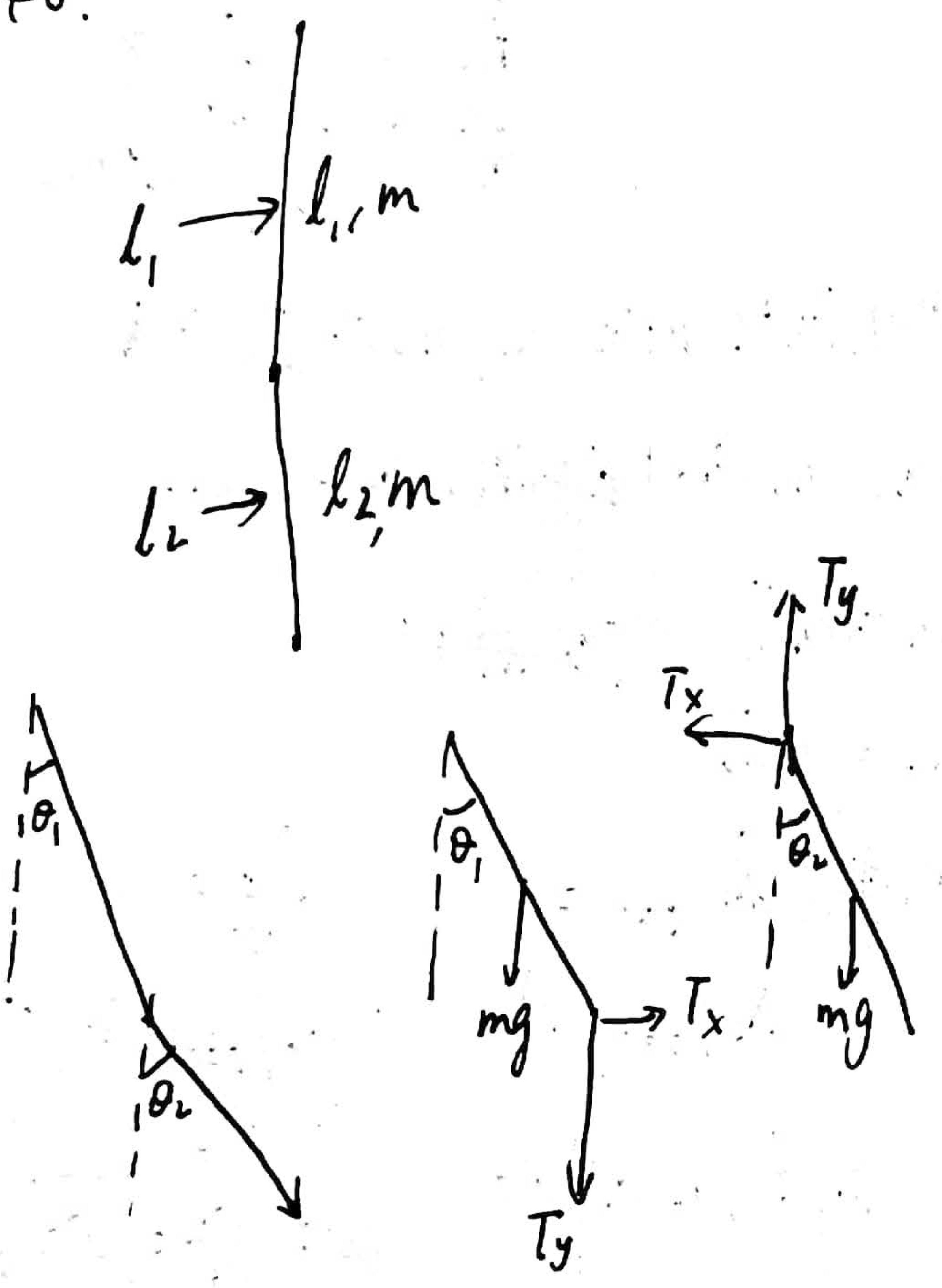
或 $t_0=0.505+6.283n$ (s)

对应振动方程

$$x=5\times(-1)^n\times\sin(t-0.250)\quad(\mathrm{cm})$$

或 $x=5\times(-1)^{n-1}\times\sin(t-0.250)$ (cm)

8-40.



对 l_1 杆:

$$T_x\cdot l\cos\theta_1-T_y l\sin\theta_1-\frac{1}{2}mgl\sin\theta_1=\frac{1}{3}ml^2\ddot{\theta}_1\quad(1)$$

对 l_2 杆

$$T_x\cdot\frac{1}{2}l\cos\theta_2-T_y\frac{1}{2}l\sin\theta_2=\frac{1}{12}ml^2\ddot{\theta}_2\quad(2)$$

$$-T_x=ma_{cx}$$

$$mg-T_y=ma_{cy}$$

其中 a_{cx}, a_{cy} 为 l_2 质心 C 的加速度

又有: $x_c=l\sin\theta_1+\frac{1}{2}l\sin\theta_2$

$y_c=l\cos\theta_1+\frac{1}{2}l\cos\theta_2$

利用 θ_1, θ_2 为小角条件

取: $\sin\theta_1\approx\theta_1$, $\sin\theta_2\approx\theta_2$

$\cos\theta_1\approx 1$ $\cos\theta_2\approx 1$

可得 $T_y\approx mg$ (3)

$$T_x=-ml\left(\ddot{\theta}_1+\frac{1}{2}\ddot{\theta}_2\right)\quad(4)$$

再利用 (1)~(4) 式

进行化简可以得到

$$\frac{4}{3}\ddot{\theta}_1+\frac{1}{2}\ddot{\theta}_2=-\frac{3}{2}\frac{g}{l}\theta_1\quad(5)$$

$$\ddot{\theta}_1+\frac{2}{3}\ddot{\theta}_2=-\frac{g}{l}\theta_2\quad(6)$$

给出力学量 $p=\theta_1+\alpha\theta_2$

有 $\theta_1=p-\alpha\theta_2$ (7)

$$\ddot{\theta}_1=\ddot{p}-\alpha\ddot{\theta}_2\quad(8)$$

利用 (5)~(8) 式化简得

$$\left(\frac{4}{3}+\frac{3}{2}\alpha\right)\ddot{p}+\frac{3}{2}\frac{g}{l}p=\ddot{\theta}_2\left(\frac{3}{2}\alpha^2+\frac{1}{3}\alpha-\frac{1}{2}\right)$$

令 $\frac{3}{2}\alpha^2+\frac{1}{3}\alpha-\frac{1}{2}=0$

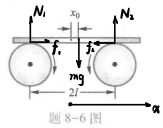
解得: $\alpha=-\frac{1}{9}\pm\frac{2\sqrt{7}}{9}$

可以发现, 此时的 p 恰好为该系统的简正坐标。

对应的, $\omega=\sqrt{\dfrac{\frac{3}{2}\frac{g}{l}}{\frac{4}{3}+\frac{3}{2}\alpha}}$

得到 $\omega=\sqrt{\left(3\pm\frac{6\sqrt{7}}{7}\right)\frac{g}{l}}$

8-6



题 8-6 图

(1) 竖直方向 $N_1+N_2=mg$

对质心 $N_1(l+x)-N_2(l-x)=0$ $\Rightarrow \begin{cases} N_1=\dfrac{l-x}{2l}mg \\ N_2=\dfrac{l+x}{2l}mg \end{cases}$

水平方向 $f_1-f_2=m\cdot\ddot{x}$ 其中 $f_1=\mu N_1$, $f_2=\mu N_2$ (高速转动)

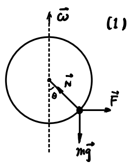
由上述各式得 $\ddot{x}+\dfrac{\mu g}{l}x=0$

故杆做简谐运动 $\omega=\sqrt{\dfrac{\mu g}{l}}$

(2) 修改为 $f_2-f_1=m\ddot{x}$ $\Rightarrow$ $\ddot{x}-\dfrac{\mu g}{l}x=0$ $\Rightarrow$ $\dfrac{dv^2}{2dx}=\dfrac{\mu g}{l}x$ $\Rightarrow$ $v^2=\dfrac{\mu g}{l}(x^2-x_0^2)$

故 杆向右运动 $v=\sqrt{\dfrac{\mu g}{l}(x^2-x_0^2)}$

8-8



(1) 在转动系中 $F=mR\omega^2\sin\theta$

横向 $F\cos\theta-mg\sin\theta=mR\ddot{\theta}$

得 $\ddot{\theta}+\left(\dfrac{g}{R}-\omega^2\cos\theta\right)\sin\theta=0$ ①

可见 $\omega^2\leq\dfrac{g}{R}$ 时 有平衡位置 $\theta_{0_1}=0$, $\theta_{0_2}=\pi$

$\omega^2>\dfrac{g}{R}$ 时 有平衡位置 $\theta_{0_1}=0$, $\theta_{0_2}=\pi$, $\theta_{0_3}=\arccos\dfrac{g}{R\omega^2}$

(2) 由①, 记 $\delta=\theta-\theta_0$ 为小量, 有

$$\ddot{\delta}+\left(\frac{g}{R}\cos\theta_0-\omega^2\cos2\theta_0\right)\delta=0$$

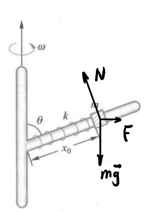
故 $\theta_{0_1}=0$ $\left(\omega^2<\dfrac{g}{R}\right)$

$\theta_{0_3}=\arccos\dfrac{g}{R\omega^2}$ $\left(\omega^2\geq\dfrac{g}{R}\right)$ 为稳定平衡

① $\theta_0=0$ 时 $\ddot{\delta}+\left(\dfrac{g}{R}-\omega^2\right)\delta=0$ $\Omega_1=\sqrt{\dfrac{g}{R}-\omega^2}$

③ $\theta_0=\arccos\dfrac{g}{R\omega^2}$ 时 $\ddot{\delta}+\left(\omega^2-\dfrac{g^2}{R^2\omega^2}\right)\delta=0$ $\Omega_3=\sqrt{\omega^2-\dfrac{g^2}{R^2\omega^2}}$

8-9



题 8-9 图

(1) 转动系中 $F=mx_0\omega^2\sin\theta$

沿杆方向 $F\sin\theta-mg\cos\theta+k(l_0-x)=m\ddot{x}$

得 $\ddot{x}+x\left(\dfrac{k}{m}-\omega^2\sin^2\theta\right)-\dfrac{kl_0}{m}+g\cos\theta=0$

$\ddot{x}=0$ 时 平衡位置 $x_0=\dfrac{mg\cos\theta-kl_0}{m\omega^2\sin^2\theta-k}$

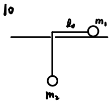
(2) 2ω 时 $\ddot{x}+x\left(\dfrac{k}{m}-4\omega^2\sin^2\theta\right)-\dfrac{kl_0}{m}+g\cos\theta=0$

平衡时 $\ddot{x}=0$，得平衡位置 $x_0' = \dfrac{mg\cos\theta - kl_0}{4m\omega^2\sin^2\theta - k}$

记 $x - x_0' = \delta$，有 $\ddot{\delta} + \left(\frac{k}{m} - 4\omega^2\sin^2\theta\right)\delta = 0$ ， $\Omega = \sqrt{-4\omega^2\sin^2\theta + \frac{k}{m}}$

故 $t = \dfrac{\pi}{\Omega} = \dfrac{\pi}{\sqrt{\frac{k}{m} - 4\omega^2\sin^2\theta}}$

8-10



(1) 绳中张力 T.

$$\begin{cases} T = m_1 \dfrac{v_0^2}{l_0} \\ T = m_2 g \end{cases} \Rightarrow l_0 = \frac{m_1 v_0^2}{m_2 g}$$

(2) 对 m_1　$T = m_1 r\dot{\theta}^2 - m_1\ddot{r}$

角动量守恒　$m_1 l_0 v_0 = m_1 r^2\dot{\theta} = L_0$

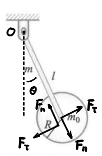
对 m_2　$T - m_2 g = m_2\ddot{r}$

由上述各式得　$(m_1+m_2)\ddot{r} - \dfrac{L_0^2}{m_1 r^3} + m_2 g = 0$

记 $r - l_0 = \delta \ll l_0$　有 $\dfrac{1}{r^3} = \dfrac{1}{l_0^3}\left(1 - \dfrac{3\delta}{l_0}\right)$

得 $\ddot{\delta} + \dfrac{3m_2^2 g^2}{(m_1+m_2)m_1 v_0^2}\delta = 0$ ， $\omega_0 = \sqrt{\dfrac{3m_2^2 g^2}{(m_1+m_2)m_1 v_0^2}} = \dfrac{m_2 g}{m_1 v_0}\sqrt{\dfrac{3m_1}{m_1+m_2}}$

8-13



题 8-13 图

(1) 固连，整体对 O 的转动惯量（默认杆均匀）　$I = \frac{1}{3}ml^2 + \frac{1}{2}m_0R^2 + m_0l^2$

对整体　$m_0 g l\sin\theta + \frac{1}{2}mgl\sin\theta = -I\ddot{\theta}$

$\theta \ll 1$ 时有　$\ddot{\theta} + \dfrac{(m_0 + \frac{1}{2}m)gl}{\frac{1}{3}ml^2 + \frac{1}{2}m_0R^2 + m_0l^2}\theta = 0$　　等效摆长对应 $\ddot{\theta} + \dfrac{g}{l_1}\theta = 0$

故周期　$T_1 = 2\pi\sqrt{\dfrac{\frac{1}{3}ml^2 + \frac{1}{2}m_0R^2 + m_0l^2}{(m_0 + \frac{1}{2}m)gl}}$

等效摆长　$l_1 = \dfrac{\frac{1}{3}ml^2 + \frac{1}{2}m_0R^2 + m_0l^2}{(m_0 + \frac{1}{2}m)l}$

(2) 光滑轴，对 m_0　$m_0 g\cos\theta - F_n = 0$

$m_0 g\sin\theta + F_\tau = -m_0 l\ddot{\theta}$

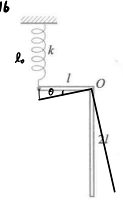
对 m　$F_\tau l - \frac{1}{2}mgl\sin\theta = I_0\ddot{\theta}$　其中 $I_0 = \frac{1}{3}ml^2$

近似 $\sin\theta = \theta$，由上述各式得　$\ddot{\theta} + \dfrac{\frac{1}{2}m + m_0}{\frac{1}{3}m + m_0}\dfrac{g}{l}\theta = 0$　　等效摆长对应 $\ddot{\theta} + \dfrac{g}{l_2}\theta = 0$

故周期 $T_1 = 2\pi\sqrt{\dfrac{\frac{1}{3}m + m_0}{\frac{1}{2}m + m_0}\dfrac{l}{g}}$

等效摆长 $l_1 = \dfrac{\frac{1}{3}m + m_0}{\frac{1}{2}m + m_0} l$

8-16



题 8-16 图

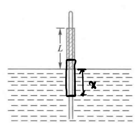
平衡时对 O $\quad k l_0 l = \frac{1}{2} m g l$

微小振动，$\theta \ll 1$ 近似有 $F = k(l_0 + l\theta)$，$\cos\theta = 1$，$\sin\theta = \theta$

对杆 $\frac{1}{2} m g l \cos\theta - m g l \sin\theta - F l = I\ddot{\theta}$ 其中 $I = \frac{1}{3} m l^2 + \frac{1}{3} 2m (2l)^2 = 3ml^2$

得 $\ddot{\theta} + \dfrac{kl + 2mg}{3ml}\theta = 0$，$T = 2\pi\sqrt{\dfrac{3ml}{kl + 2mg}}$

8-17



题 8-17 图

(1) 不考虑能量的耗散，将水对杆的作用视为保守的

$x \le l$ 时 对杆 $\rho_1 S g L - \rho_0 S g x = \rho_1 S L \ddot{x}$

得平衡位置 $x_0 = \dfrac{\rho_1}{\rho_0} L$

有 $\ddot{x} + \dfrac{\rho_0 g}{\rho_1 L}(x - x_0) = 0$

初态 $x = 0$，$\dot{x} = 0$ 得 $x = \dfrac{\rho_1}{\rho_0} L \left(1 - \cos\sqrt{\dfrac{\rho_0 g}{\rho_1 L}}\, t\right)$

故 $\dfrac{2\rho_1}{\rho_0} L = L \Rightarrow \rho_1 = \dfrac{\rho_0}{2}$

(2) 由(1)知其为简谐运动 振幅 $A = \dfrac{\rho_1}{\rho_0} L = \dfrac{L}{2}$

周期 $T = 2\pi\sqrt{\dfrac{\rho_1 L}{\rho_0 g}} = 2\pi\sqrt{\dfrac{L}{2g}}$

(3) $\frac{4}{3}\rho_1$ 时，同理，在 $x \le L$ 时有 $\ddot{x} + \dfrac{3g}{2L}(x - x_0) = 0$

$x = \dfrac{2}{3} L \left(1 - \cos\sqrt{\dfrac{3g}{2L}}\, t\right)$

令 $x = L$ 得 $t_1 = \dfrac{2\pi}{3}\sqrt{\dfrac{2L}{3g}}$ 此时 $v = \sqrt{\frac{1}{2} g L}$

$x > L$ 时 $\rho_1 S g L - \rho_0 S g L = \rho_1 S L a$ 匀减速 $t_2 = \dfrac{v_1}{-a} = \sqrt{\dfrac{2L}{g}}$

故 $t = t_1 + t_2 = \dfrac{2\pi}{3}\sqrt{\dfrac{2L}{3g}} + \sqrt{\dfrac{2L}{g}}$

8-27 (1) 阻尼力 $f = -2\delta m \dfrac{dx_1}{dt}$ 在悬点平动系下 $x = x_1 - x_2$ 惯性力 $F = -m\dfrac{d^2 x_2}{dt^2}$

故 $m\dfrac{d^2 x}{dt^2} = -mg\sin\theta + f + F$ 其中 $\theta \ll 1$，$\sin\theta = \dfrac{x}{l}$

代入得 $m\dfrac{d^2(x_1-x_2)}{dt^2} = -m\dfrac{g}{l}(x_1-x_2) - 2\delta m\dfrac{dx_1}{dt} - m\dfrac{d^2x_2}{dt^2}$

故 $\dfrac{d^2x_1}{dt^2} + 2\delta\dfrac{dx_1}{dt} + \dfrac{g}{l}x_1 = \dfrac{g}{l}x_2$

(2) 齐次通解: $x = \begin{cases} C_1 e^{(-\delta+\sqrt{\delta^2-\frac{g}{l}})t} + C_2 e^{(-\delta+\sqrt{\delta^2-\frac{g}{l}})t} & \delta > \sqrt{\frac{g}{l}} \\ (C_1+C_2t)e^{-\delta t} & \delta = \sqrt{\frac{g}{l}} \\ e^{-\delta t}\left(C_1\cos\sqrt{\frac{g}{l}}t + C_2\sin\sqrt{\frac{g}{l}}t\right) & \delta < \sqrt{\frac{g}{l}} \end{cases}$

结合条件知 $\delta T = \delta\cdot 2\pi\sqrt{\dfrac{l}{g}} = \dfrac{1}{50}$

得 $\delta = \dfrac{1}{100\pi}\sqrt{\dfrac{g}{l}} \approx 9.96\times10^{-3}\ s^{-1}$, $x = e^{-\delta t}\left(C_1\cos\sqrt{\dfrac{g}{l}}t + C_2\cos\sqrt{\dfrac{g}{l}}t\right)$

非齐次方程对应特解:

$$x = \frac{A\left[(\omega^2-\frac{g}{l})\cos\omega t + (2\delta\omega)\sin\omega t\right]}{(\omega^2-\frac{g}{l})^2 + (2\delta\omega)^2}\cdot\frac{g}{l}$$

注意到通解部分在稳态($t\to\infty$)时趋于0, 故稳态解即为特解:

$$x = \frac{\frac{g}{l}\cdot A}{\sqrt{(\omega^2-\frac{g}{l})^2+(2\delta\omega)^2}}\cos(\omega t-\varphi)$$

其中 $A = 1mm$, $\delta = 9.96\times10^{-3}\ s^{-1}$, $\dfrac{g}{l} = 9.8\ s^{-1}$

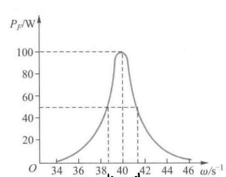
其中 $\varphi = \arctan\dfrac{2\delta\omega}{\omega^2-\frac{g}{l}}$

(2) 振幅 $A_r = \dfrac{\frac{g}{l}\cdot A}{\sqrt{(\omega^2-\frac{g}{l})^2+(2\delta\omega)^2}} = \dfrac{\frac{g}{l}\cdot A}{\sqrt{\omega^4-2(\frac{g}{l}-2\delta^2)\omega^2+\frac{g^2}{l^2}}}$ 在 $\omega = \sqrt{\dfrac{g}{l}-2\delta^2}$ 时最大

故共振频率 $\omega = \sqrt{\dfrac{g}{l}-2\delta^2} \approx 3.13\ rad/s$, 振幅 $A_r = \dfrac{\frac{g}{l}A}{2\delta\sqrt{\frac{g}{l}-\delta^2}} \approx 0.157\ m$

(3) 振幅 $A' = \dfrac{\frac{g}{l}\cdot A}{\sqrt{(\omega^2-\frac{g}{l})^2+(2\delta\omega)^2}} = \dfrac{1}{2}A_r$ 解得 $\omega_1 = 3.1131562\ rad\cdot s^{-1}$, $\omega_2 = 3.1476755\ rad\cdot s^{-1}$

8-32



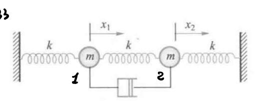
题 8-32 图

(1) 由图 $\omega_0 = 40\ rad\cdot s^{-1}$

记 ω_+, ω_- 为半功率点的角频率, 则 $Q = \dfrac{\omega_0}{|\omega_+-\omega_-|} = \dfrac{40}{2.5} = 16$

(2) $Q = 2\pi\dfrac{E}{\Delta E} = \dfrac{\omega}{2\beta}$, $e^{-2\beta nT} = e^{-5}$ $n = \dfrac{5}{2\beta T} = \dfrac{5Q}{2\pi} \approx 13$ (次)

8-33



题 8-33 图

(1) 1. $-2kx_1 + kx_2 + b(\dot{x}_2-\dot{x}_1) = m\ddot{x}_1$ ①

2. $kx_1 - 2kx_2 - b(\dot{x}_2-\dot{x}_1) = m\ddot{x}_2$ ②

(2) ①+②, ②−① 得:

$$\ddot{x}_1+\ddot{x}_2+\frac{k}{m}(x_1+x_2) = 0$$

$$\ddot{x}_2-\ddot{x}_1+\frac{2b}{m}(\dot{x}_2-\dot{x}_1)+\frac{3k}{m}(x_2-x_1) = 0$$

令 $y_1 = x_1+x_2$, $y_2 = x_1-x_2$

有 $\begin{cases} \ddot{y}_1+\frac{k}{m}y_1=0 \\ \ddot{y}_2+\frac{2b}{m}\dot{y}_2+\frac{3k}{m}y_2=0 \end{cases}$

(3) 通解： $y_1=C_1\cos\sqrt{\frac{k}{m}}t+C_2\sin\sqrt{\frac{k}{m}}t$

y_2 为阻尼振动方程的解，稳态（即 $t\to+\infty$）时趋于 0．即 $y_2=x_1-x_2=0$

初态： $t=0$ 时 $y_1=x_1+x_2=0$ $\quad \dot{y}_1=\dot{x}_1+\dot{x}_2=v_0$

得 $\begin{cases} y_1=v_0\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t \\ y_2=0 \end{cases}$ 从而 $x_1=x_2=\frac{v_0}{2}\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t$ ， $\omega=\sqrt{\frac{k}{m}}$

8-35 对 m_1 $-kx_1=m\ddot{x}_1 \Rightarrow \ddot{x}_1+\frac{k_1}{m_1}x_1=0$ 由初态 $x_1=A=10cm$ 和 $\dot{x}_1=0$ 得 $x_1=-A\cos\sqrt{\frac{k_1}{m_1}}t$

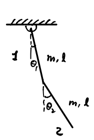
对 m_2 同理 $x_2=-A\cos\sqrt{\frac{k_2}{m_2}}(t-t_0)$ 其中 t_0 为屏落后球的时间， 记 $\sqrt{\frac{k_1}{m_1}}=\sqrt{\frac{k_2}{m_2}}=\omega=1\ rad\cdot s^{-1}$

则： $x_3=x_1-x_2=A\cos\omega t-A\cos\omega(t-t_0)=-2A\sin\frac{\omega t_0}{2}\sin\omega\left(t-\frac{t_0}{2}\right)$

振幅 $\left|2A\sin\frac{\omega t_0}{2}\right|=5cm \Rightarrow t_0=\left[2n\pi\pm2\arcsin\frac{1}{4}\right]s$ $(n=0,1,2,\cdots)$

此时 $x_3=-10\sin\frac{\omega t_0}{2}\sin\left(t-\frac{t_0}{2}\right)cm=0.05\cos\left[t\pm\arccos\frac{1}{4}\right]m$ （t,t_0 单位 s）

8-40



动能 $T=\frac{1}{2}\cdot\frac{1}{3}ml^2\cdot\dot{\theta}_1^2+\frac{1}{2}mv_c^2+\frac{1}{2}\cdot\frac{1}{12}ml^2\cdot\dot{\theta}_2^2$

其中 $v_c^2=l^2\dot{\theta}_1^2+\frac{l^2}{4}\dot{\theta}_2^2+2\frac{l^2}{2}\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)$

故 $T=\frac{1}{6}ml^2\left(4\dot{\theta}_1^2+3\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)+\dot{\theta}_2^2\right)$

势能 $V=mg\frac{l}{2}(1-\cos\theta_1)+mg\left[l(1-\cos\theta_1)+\frac{l}{2}(1-\cos\theta_2)\right]$

拉格朗日函数 $L=T-V$ ， 直接小角近似得 $L=\frac{1}{6}ml^2\left(4\dot{\theta}_1^2+3\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^2\right)-\frac{1}{4}mgl\left(3\theta_1^2+\theta_2^2\right)$

代入拉格朗日方程 $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_i}\right)-\frac{\partial L}{\partial\theta_i}=0$ $(i=1,2)$ 得

$$\begin{cases} \frac{1}{6}ml^2\left(8\ddot{\theta}_1+3\ddot{\theta}_2\right)+\frac{3}{2}mgl\,\theta_1=0 \\ \frac{1}{6}ml^2\left(3\ddot{\theta}_1+2\ddot{\theta}_2\right)+\frac{1}{2}mgl\,\theta_2=0 \end{cases} \xRightarrow{\text{令}\ \omega_0^2=\frac{g}{l}} \begin{cases} 8\ddot{\theta}_1+3\ddot{\theta}_2+9\omega_0^2\theta_1=0 \\ 3\ddot{\theta}_1+2\ddot{\theta}_2+3\omega_0^2\theta_2=0 \end{cases}$$

要求 $\begin{vmatrix} 9\omega_0^2-8\omega^2 & -3\omega^2 \\ -3\omega^2 & 3\omega_0^2-2\omega^2 \end{vmatrix}=7\omega^4-42\omega^2\omega_0^2+27\omega_0^4=0$

解得 $\omega_1=\sqrt{3+\frac{6}{\sqrt{7}}}\ s^{-1}$ $\quad \omega_2=\sqrt{3-\frac{6}{\sqrt{7}}}\ s^{-1}$